# Notes for the support class of the course MA4L2

March 20, 2017

**Exercise 2.1.** Let $P_\epsilon$ be the projector onto the subspace of $\mathcal{H}_\Lambda$ spanned by configurations $\omega$ such that $\frac{1}{|\Lambda|}\sum_x \omega_x \notin (-\epsilon, \epsilon)$. Suppose that $\left[H_{\Lambda,\beta,0}, \sum_{x\in\Lambda} S_x^3\right] = 0$. Show that for any sequence $\Lambda_n \Uparrow \mathbb{Z}^d$, if $\psi(\beta, h)$ is differentiable in $h$ at $(\beta, 0)$ , then for $\epsilon > 0$, and for large enough $n$:

$$\langle P_\epsilon \rangle_{\Lambda_n,\beta,0} \leq \mathrm{e}^{-c|\Lambda_n|} \tag{1}$$

for some $c > 0$ uniform in $\Lambda_n$. (Hint: Show that a Chernov inequality holds.)

*Proof.* Firstly, we can rewrite the projector in the following way:

$$P_\epsilon = \mathcal{P}_{+\epsilon} + \mathcal{P}_{-\epsilon}, \tag{2}$$

where the two projectors on the right hand side project over the subspaces of $\mathcal{H}_{\Lambda_n}$ spanned by configurations $\omega$ such that $\frac{1}{|\Lambda_n|}\sum_{x\in\Lambda_n} \omega_x \geq \epsilon$ (or respectively $\leq -\epsilon$). Let $\mathcal{H}_{\Lambda_n}^{+\epsilon}$ (respectively $\mathcal{H}_{\Lambda_n}^{-\epsilon}$) denote such spaces. We are going to show only that the result holds for $\langle \mathcal{P}_{+\epsilon}\rangle_{\Lambda_n,\beta,0}$ - the result for the projector on $\mathcal{H}_{\Lambda_n}^{-\epsilon}$ follows in a similar way. Notice that for any $h > 0$, $\mathcal{P}_{+\epsilon} \leq \mathrm{e}^{-\frac{\epsilon}{2}h|\Lambda_n|}\mathrm{e}^{h\sum_x S_x^3}$. Indeed, we have:

$$\langle\omega|\mathrm{e}^{-\frac{\epsilon}{2}|\Lambda_n|}\mathrm{e}^{h\sum_x S_x^3}|\omega\rangle = \mathrm{e}^{-\frac{\epsilon}{2}|\Lambda_n|h+\frac{h}{2}\sum_x \omega_x} \begin{cases} \geq 1 & \text{if } \omega \in \mathcal{H}_{\Lambda_n}^{\epsilon} \\ \in (0,1) & \text{otherwise.} \end{cases} \tag{3}$$

Then it follows that

$$\langle \mathcal{P}_{+\epsilon}\rangle_{\Lambda_n,\beta,0} \leq \langle \mathrm{e}^{-\frac{\epsilon}{2}h|\Lambda_n|}\mathrm{e}^{h\sum_x S_x^3}\rangle_{\Lambda_n,\beta,0} = \mathrm{e}^{-\frac{\epsilon}{2}h|\Lambda_n|}\frac{Z_{\Lambda_n,\beta,h}}{Z_{\Lambda_n,\beta,0}} = \mathrm{e}^{-\frac{\epsilon}{2}h|\Lambda_n|}\mathrm{e}^{|\Lambda_n|(\psi_{\Lambda_n}(\beta,h)-\psi_{\Lambda_n}(\beta,0))} \tag{4}$$

In order to prove the result, it is sufficient to show that:

$$\limsup_{n\to\infty} \frac{1}{|\Lambda_n|} \log\langle \mathcal{P}_{+\epsilon}\rangle_{\Lambda_n,\beta,0} \leq -C, \tag{5}$$

where $C$ is some positive constant. Notice that:

$$\begin{aligned} \limsup_{n\to\infty} \frac{1}{|\Lambda_n|} \log\langle \mathcal{P}_{+\epsilon}\rangle &\leq -\lim_{n\to\infty} \left(\frac{\epsilon h}{2} - (\psi_{\Lambda_n}(\beta,h) - \psi_{\Lambda_n}(\beta,0))\right) \\ &= -\left(\frac{\epsilon}{2} - (\psi(\beta,h) - \psi(\beta,0))\right) \end{aligned} \tag{6}$$

By a Taylor expansion we have:

$$\frac{\epsilon}{2} - (\psi(\beta,h) - \psi(\beta,0)) = h\frac{\epsilon}{2}\left(1 - 2\frac{\partial}{\partial h}\psi(\beta,\hat{h})\right), \tag{7}$$

with $\hat{h} \in (0,h)$. Remember that $h$ is arbitrary – so we can then fix a value of $h$ such that the right hand side of the equation above is positive. Such a value exists because $\frac{\partial}{\partial h}\psi(\beta,0) = 0$ and $\psi(\beta,h)$ is convex. □

**Exercise 3.1.** Let $h > 0$. Show that $\langle \prod_{x\in X} S^3_x\rangle_{\Lambda,\beta,h} \geq 0$ for all $X \subset \Lambda$, where $\langle\cdot\rangle_{\Lambda,\beta,h}$ is the Gibbs state related to the Heisenberg hamiltonian.

*Proof.* Recall the explicit formulation of the Heisenberg hamiltonian:

$$H^{heis}_{\Lambda,\beta,h} = -\beta \sum_{\{x,y\}\in\mathcal{E}_\Lambda} S^1_x S^1_y + S^2_x S^2_y + S^3_x S^3_y - h\sum_{x\in\Lambda} S^3_x. \tag{8}$$

Let us define the following operator:

$$U = \prod_{x\in\Lambda} \mathrm{e}^{\frac{i\pi}{2}S^2_x}. \tag{9}$$

Notice that by the result in Exercise 1.3 of the lecture notes

$$\mathrm{e}^{-\frac{i\pi}{2}S^2} S^1 \mathrm{e}^{\frac{i\pi}{2}S^2} = -S^3, \tag{10}$$

$$\mathrm{e}^{-\frac{i\pi}{2}S^2} S^2 \mathrm{e}^{\frac{i\pi}{2}S^2} = S^2, \tag{11}$$

$$\mathrm{e}^{-\frac{i\pi}{2}S^2} S^3 \mathrm{e}^{\frac{i\pi}{2}S^3} = S^1. \tag{12}$$

We can apply this transformation to the Heisenberg hamiltonian and find:

$$H'_{\Lambda,\beta,h} = U^{-1} H^{heis}_{\Lambda,\beta,h} U = H^{heis}_{\Lambda,\beta,0} - h\sum_{x\in\Lambda} S^1_x, \tag{13}$$

i.e. the magnetic field is now aligned with the first direction of spin. Since the partition function is positive by definition, it is sufficient to study the sign of $\mathrm{Tr}\,\prod_{x\in X} S_x^3 \mathrm{e}^{-H^{heis}_{\Lambda,\beta,h}}$. We use the cyclicity of the trace and the equations above:

$$\begin{aligned}\mathrm{Tr}\prod_{x\in X} S_x^3\,\mathrm{e}^{-H^{heis}_{\Lambda,\beta,0}} &= \mathrm{Tr}\,U^{-1}\prod_{x\in X} S_x^3 U\,U^{-1}\mathrm{e}^{-H^{heis}_{\Lambda,\beta,0}} \\ &= \mathrm{Tr}\prod_{x\in X} S_x^1 \mathrm{e}^{-U^{-1}H^{heis}_{\Lambda,\beta,0}U} = \mathrm{Tr}\prod_{x\in X} S_x^1\,\mathrm{e}^{-H'_{\Lambda,\beta,0}}.\end{aligned} \tag{14}$$

Since $S^1$ has non negative matrix elements, $\mathrm{Tr}\,\prod_{x\in A} S_x^1\,\mathrm{e}^{-H'_{\Lambda,\beta,0}}$ is nonnegative because it is the trace of an operator with nonnegative elements. The result is thus proved. □

**Exercise 4.2.** Let $H_{\Lambda,\beta,0} = -\beta\sum_{\{x,y\}\in\mathcal{E}_\Lambda}\left(S_x^1S_y^1 + uS_x^2S_y^2 + S_x^3S_y^3\right)$ where $u\in[0,1]$ is a fixed parameter. Find operators $b_{xy}$ that satisfy Eq. (4.3) of the lecture notes and such that the Gibbs state $\langle\cdot\rangle'_{\Lambda,\beta}$ with hamiltonian $H'_{\Lambda,\beta} = -\sum_{\{x,y\}\in\mathcal{E}_{\Lambda,\beta}} b_{xy}$ has identical spin correlations i.e. $\langle S_x^3S_y^3\rangle_{\Lambda,\beta,0} = \langle S_x^3S_y^3\rangle'_{\Lambda,\beta}$.

*Proof.* Notice that:

$$\begin{aligned}H_{\Lambda,\beta,0} &= -\beta\sum_{\{x,y\}\in\mathcal{E}_\Lambda} u\left(S_x^1S_y^1 + S_x^2S_y^2\right) + (1-u)S_x^1S_y^1 + S_x^3S_y^3 \\ &= -\beta\sum_{\{x,y\}\in\mathcal{E}_\Lambda}\frac{u}{2}\left(S_x^+S_y^- + S_x^-S_y^+\right) + (1-u)S_x^1S_y^1 + S_x^3S_y^3,\end{aligned} \tag{15}$$

where $S^\pm = S^1 \pm iS^2$. Define

$$b_{xy} = \beta\left(\frac{u}{2}(S_x^+S_y^- + S_x^-S_y^+) + (1-u)S_x^1S_y^1 + S_x^3S_y^3 + \frac{1}{4}\mathbb{1}\right) \tag{16}$$

Notice that $b_{xy}$ applied to a state $\omega$ acts only on the spins at sites $x$ and $y$. Moreover, since $S_x^\pm$, $S_x^1$ and $S_x^3S_y^3 + \frac{1}{4}\mathbb{1}$ have nonnegative matrix elements for any $x,y\in\Lambda$. The conditions in Eq. (4.3) of the lecture notes are thus satisfied. Notice that $H'_{\Lambda,\beta}$ so defined differs from $H_{\Lambda,\beta,0}$ only by a constant, which does not affect the Gibbs state. The result is thus proved. □

**Exercise A.1 (i).** Show that $\lim_{p\to\infty}\|A\|_p = \|A\|_\infty$

*Proof.* Recall the singular value decomposition: for any square $n$x$n$ matrix $A$ there exist $V$ and $W$ unitary matrices such that $A = VDW^*$ with $D = \mathrm{diag}\,(\sigma_1(A), \sigma_2(A), \dots \sigma_n(A))$, with $\sigma_1(A) \geq \sigma_2(A) \geq \cdots \geq \sigma_n(A) \geq 0$. The $\sigma_i(A)$ are called "singular values" – in the case A is hermitian they are the absolute values of eigenvalues. Notice that

$$\begin{aligned} \|A\|_\infty = \sup_{x:\, \|x\|=1} (x, A^* A x)^{\frac{1}{2}} = \\ \sup_{x:\, \|x\|=1} \left(x, V^* D^2 V x\right)^{\frac{1}{2}} = \sup_{x:\, \|x\|=1} \left(Vx, D^2 V x\right)^{\frac{1}{2}} = \sigma_1(A), \end{aligned} \tag{17}$$

due to the unitarity of $V$ and $W$. Moreover, by the ciclicity of trace:

$$\|A\|_p = \left(\sum_i \sigma_i(A)^p\right)^{\frac{1}{p}}. \tag{18}$$

Notice that

$$\lim_{p\to\infty} A_p = \sigma_1(A)\left(1 + \sum_{i\geq 2}\left(\frac{\sigma_i(A)}{\sigma_1(A)}\right)^p\right)^{\frac{1}{p}} = \sigma_1(A), \tag{19}$$

since $\frac{\sigma_i(A)}{\sigma_1(A)} < 1$ for all $i \geq 2$. □

**Exercise A.1 (ii).** Show that $\|A\|_p$ is a norm for all $1 \leq p \leq \infty$.

*Proof.* To prove this statement, we use extensively Eq. (18). The properties we need to show are:

1. $\|A\|_p \geq 0$ for any $A$ square matrix,
2. $\|A\|_p = 0$ if and only if $A = 0$,
3. $\|\alpha A\|_p = |\alpha|\|A\|_p$ for all complex $\alpha$,
4. $\|A + B\|_p \leq \|A\|_p + \|B\|_p$.

Properties 1. and 2. are straightforward from Eq. (18). Property 3. can be proved explicitely:

$$\|\alpha A\|_p = (\text{Tr } ((\alpha A)^*(\alpha A))^{\frac{p}{2}})^{\frac{1}{p}} = |\alpha|\|A\|_p \tag{20}$$

for any $\alpha \in \mathbb{C}$ and any square matrix $A$. To prove the triangular inequality, we need the following Lemma (its proof can be found, for instance, in *Topics in Matrix Analysis* by R.A. Horn and C.R. Johnson, Cambridge University Press (1991), Lemma 3.3.8):

**Lemma 1.** *Let $x_1 \geq x_2 \geq \cdots \geq x_n$ and $y_1 \geq y_2 \geq \cdots \geq y_n$ , with $x_i$ , $y_i \in \mathbb{R}$ and $n \in \mathbb{N}$ such that:*

$$\sum_{i=1}^{k} x_i \leq \sum_{i=1}^{k} y_i, \qquad k \in \{1, \dots, n\}.$$

*Let $f$ be an increasing and convex function real valued function. Then, $f(x_1) \geq \cdots \geq f(x_n)$, $f(y_1) \geq \cdots \geq f(y_n)$ and*

$$\sum_{i=1}^{k} f(x_i) \leq \sum_{i=1}^{k} f(y_i), \qquad k \in \{1, \dots, n\}.$$

Firstly, we prove the triangular inequality for $\| \cdot \|_1$ . Notice that for $A$ with singular value decomposition $A = VDW^*$:

$$\sup_{C:\|C\|_\infty = 1} \text{Tr } AC = \sup_{C:\|C\|_\infty = 1} \text{Tr } DW^*CV. \tag{21}$$

Due to the unitarity of $W$ and $V$ , $\|W^*CV\|_\infty = \|C\|_\infty$ , so:

$$\sup_{C:\|C\|_\infty = 1} \text{Tr } AC = \sup_{C:\|C\|_\infty = 1} \text{Tr } DC = \|A\|_1. \tag{22}$$

Then:

$$\begin{aligned} \|A + B\|_1 = \sup_{C:\|C\|_\infty = 1} \text{Tr } (A + B)C &\leq \sup_{C:\|C\|_\infty = 1} \text{Tr } AC + \sup_{C:\|C\|_\infty = 1} \text{Tr } BC \\ &= \|A\|_1 + \|B\|_1. \end{aligned} \tag{23}$$

Let us now extend this result to any value of $p$. Notice that for $\| \cdot \|_1$ the triangular inequality is equivalent to

$$\sum_i \sigma_i(A + B) \leq \sum_i \sigma_i(A) + \sigma_i(B). \tag{24}$$

By Eq. (24) and the Lemma above:

$$\begin{aligned} \|A + B\|_p = \left( \sum_i \sigma_i(A + B)^p \right)^{\frac{1}{p}} &\leq \left( \sum_i (\sigma_i(A) + \sigma_i(B))^p \right)^{\frac{1}{p}} \\ \leq \left( \sum_i \sigma_i(A)^p \right)^{\frac{1}{p}} &+ \left( \sum_i \sigma_i(B)^p \right)^{\frac{1}{p}} = \|A\|_p + \|B\|_p. \end{aligned} \tag{25}$$

The last line of the expression above comes from Minkowski inequality: for any $(x_1, \dots, x_n)$, $(y_1, \dots, y_n) \in \mathbb{R}^n$ and $p \geq 1$

$$\left( \sum_{i=1}^{n} |x_i + y_i|^p \right)^{\frac{1}{p}} \leq \left( \sum_{i=1}^{n} |x_i|^p \right)^{\frac{1}{p}} + \left( \sum_{i=1}^{n} |y_i|^p \right)^{\frac{1}{p}}. \tag{26}$$

□

**Exercise A.1 (iii).** Use Hölder inequality to show that $\|A\|_p$ is submultiplicative, that is, $\|AB\|_p \leq \|A\|_p \|B\|_p$.

*Proof.* Firstly, let us show that $\|A\|_p$ is nonincreasing in $p$ for any square matrix $A$. Since $\|A\|_p$ is just the $\ell_p$ -norm of the vector of singular values of $A$, it is sufficient to prove that the $\ell_p$-norm is nonincreasing in $p$. Let $q \geq p$ and $x = (x_1, \ldots, x_n)$ with $x_i \geq 0$. Assume without loss of generality that $\|x\|_q = 1$ (and by consequence $x_i \leq 1$). It is sufficient to prove

$$\log \|x\|_p \geq \log \|x\|_q = 0. \tag{27}$$

By the concavity of the logarithm:

$$\begin{aligned} \log \|x\|_p &= \frac{1}{p} \log \sum_i x_i^p = \frac{1}{p} \log \sum_i x_i^q x_i^{-(q-p)} \\ &\geq \frac{1}{p} \sum_i x_i^q \log x_i^{-(q-p)} = -\frac{q-p}{p} \sum_i x_i^q \log x_i \geq 0. \end{aligned} \tag{28}$$

Now, let $p \geq 1$ and $r, r' \geq 1$ such that $\frac{1}{r} + \frac{1}{r'} = \frac{1}{p}$. It follows that $p = \frac{rr'}{r+r'}$, so $p \leq r, r'$. Then, by Hölder inequality and by monotonicity of the $p$-norm:

$$\|AB\|_p \leq \|A\|_r \|B\|_{r'} \leq \|A\|_p \|B\|_p. \tag{29}$$

□