Introduction to quantum spin systems

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June 3, 2017

Contents

1 Setting

Let $\Lambda \Subset \mathbb{Z}^d$ denote the domain. Recall that $\Omega_\Lambda = \{-1,1\}^\Lambda$ is the set of classical spin configurations of the Ising model. We consider here the complex Hilbert space

$$\mathcal{H}_\Lambda = \operatorname{span}\Omega_\Lambda, \tag{1.1}$$

that is, $\mathcal{H}_\Lambda$ consists of all linear combinations of elements of Ω_Λ; $\dim\mathcal{H}_\Lambda = 2^{|\Lambda|}$. Equivalently, $\mathcal{H}_\Lambda = \ell^2(\Omega_\Lambda)$. We use Dirac's notation: If $\omega \in \Omega_\Lambda$, the corresponding element in $\mathcal{H}_\Lambda$ is written $|\omega\rangle$. The Hilbert space $\mathcal{H}_\Lambda$ is the state space of "spin $\frac12$ systems".

The counterparts to the local functions $\{\sigma_i\}_{i\in\Lambda}$ are the spin operators $\{S^i_x\}^{i=1,2,3}_{x\in\Lambda}$. Their action on the basis elements $\{|\omega\rangle\}_{\omega\in\Omega_\Lambda}$ proceeds as follows:

$$\begin{aligned} S^1_x|\omega\rangle &= \tfrac12|\omega^{(x)}\rangle, \\ S^2_x|\omega\rangle &= \tfrac12\mathrm{i}\omega_x|\omega^{(x)}\rangle, \\ S^3_x|\omega\rangle &= \tfrac12\omega_x|\omega\rangle. \end{aligned} \tag{1.2}$$

We state important properties in the form of exercises, which we encourage the readers to do.

Exercise 1.1. *Show that $\{S^i_x\}$ satisfy the usual spin commutation relations:*

$$\begin{aligned} &[S^1_x, S^2_y] = \mathrm{i}\,\delta_{x,y}\,S^3_x \qquad + \textit{cyclic permutations of (1,2,3)}, \\ &(S^1_x)^2 + (S^2_x)^2 + (S^3_x)^2 = \tfrac34\mathbb{1}. \end{aligned}$$

Exercise 1.2. *Consider $\Lambda = \{x\}$, so that $\mathcal{H}_\Lambda \simeq \mathbb{C}^2$ with basis $\{|-1\rangle, |+1\rangle\}$. Show that the spin operators are given by (one half of) the Pauli matrices, namely*

$$S^1 = \tfrac12\begin{pmatrix}0&1\\1&0\end{pmatrix}, \qquad S^2 = \tfrac12\begin{pmatrix}0&-\mathrm{i}\\ \mathrm{i}&0\end{pmatrix}, \qquad S^3 = \tfrac12\begin{pmatrix}1&0\\0&-1\end{pmatrix}.$$

Exercise 1.3. *Spin operators on $\mathbb{C}^2$ and rotations in $\mathbb{R}^3$. For $\vec a \in \mathbb{R}^3$, let*

$$S^{\vec a} = \vec a \cdot \vec S = a_1 S^1 + a_2 S^2 + a_3 S^3,$$

where S_i is the Pauli matrix of the previous exercise. Show that

(i) $[S^{\vec a}, S^{\vec b}] = \mathrm{i}S^{\vec a\times\vec b}$.

(ii) Let $R_{\vec a}\vec b$ denote the vector $\vec b$ rotated around $\vec a$ by the angle $\|\vec a\|$. Then

$$\mathrm{e}^{-\mathrm{i}S^{\vec a}}\, S^{\vec b}\, \mathrm{e}^{\mathrm{i}S^{\vec a}} = S^{R_{\vec a}\vec b}.$$

The energy of the system is given by a self-adjoint operator on $\mathcal{H}_\Lambda$. It involves interactions between nearest-neighbours and interactions with an external magnetic field. We list here five important hamiltonians; recall that $\mathcal{E}_\Lambda$ denote the set of edges (unordered pairs of nearest-neighbours) of Λ.

- **Ising**: $H^{\text{Ising}}_{\Lambda,\beta,h} = -\beta \sum_{\{x,y\}\in\mathcal{E}_\Lambda} S^3_x S^3_y - h\sum_{x\in\Lambda} S^3_x$.
- **Quantum Ising**: $H^{\text{qu.Ising}}_{\Lambda,\beta,h} = -\beta \sum_{\{x,y\}\in\mathcal{E}_\Lambda} S^1_x S^1_y - h\sum_{x\in\Lambda} S^3_x$.
- **Quantum XY**: $H^{\text{XY}}_{\Lambda,\beta,h} = -\beta \sum_{\{x,y\}\in\mathcal{E}_\Lambda} \big(S^1_x S^1_y + S^3_x S^3_y\big) - h\sum_{x\in\Lambda} S^3_x$.
- **Heis. ferromagnet**: $H^{\text{Heis.F}}_{\Lambda,\beta,h} = -\beta \sum_{\{x,y\}\in\mathcal{E}_\Lambda} \big(S^1_x S^1_y + S^2_x S^2_y + S^3_x S^3_y\big) - h\sum_{x\in\Lambda} S^3_x$.
- **Heisenberg antiferromagnet**: $H^{\text{Heis.AF}}_{\Lambda,\beta,h} = -H^{\text{Heis.F}}_{\Lambda,\beta,h}$.

The **partition function** for the hamiltonian $H_{\Lambda,\beta,h}$ is

$$Z_{\Lambda,\beta,h} = \mathrm{Tr}_{\mathcal{H}_\Lambda} \, \mathrm{e}^{-H_{\Lambda,\beta,h}}. \tag{1.3}$$

We only consider free boundary conditions (or perhaps also periodic boundary conditions); unlike in the classical situation, one cannot play with boundary conditions here. In the case of the Ising hamiltonian, all operators are diagonal, and

$$Z^{\rm Ising}_{\Lambda,\beta,h} = \mathrm{Tr}\, \mathrm{e}^{-H^{\rm Ising}_{\Lambda,\beta,h}} = \sum_{\omega\in\Omega_\Lambda} \mathrm{e}^{\frac14\beta\sum_{\{x,y\}}\omega_x\omega_y + \frac12 h\sum_x \omega_x}, \tag{1.4}$$

and we recognise the partition function of the usual Ising model with inverse temperature $\frac14\beta$ and magnetic field $\frac12 h$.

2 Pressure and magnetisation

The pressure is defined as the logarithm of the partition function divided by the volume, as in the classical case. It is also convex in β and h; the proof is more difficult and it involves the Hölder and Golden-Thompson inequalities; to prove the latter, we need the Trotter product formula.

The **pressure** is

$$\psi_\Lambda(\beta,h) = \frac{1}{|\Lambda|}\log Z_{\Lambda,\beta,h}. \tag{2.1}$$

Theorem 2.1. *The pressure $\psi_\Lambda(\beta,h)$ is jointly convex in (β,h) and even in h.*

There is no "spin flip" symmetry in the quantum case, but we can use spin rotations. Let $U_\Lambda = \prod_{x\in\Lambda} \mathrm{e}^{\mathrm{i}\pi S^1_x}$ be the unitary that rotates the spin operators around the first axis by angle π; it sends (S^1_x, S^2_x, S^3_x) to $(S^1_x, -S^2_x, -S^3_x)$. All five hamiltonians listed above satisfy $U_\Lambda H_{\Lambda,\beta,h} U^*_\Lambda = H_{\Lambda,\beta,-h}$. Then

$$Z_{\Lambda,\beta,h} = \mathrm{Tr}\, \mathrm{e}^{-H_{\Lambda,\beta,h}} = \mathrm{Tr}\, U_\Lambda \, \mathrm{e}^{-H_{\Lambda,\beta,h}} U^*_\Lambda = \mathrm{Tr}\, \mathrm{e}^{-H_{\Lambda,\beta,-h}} = Z_{\Lambda,\beta,-h}. \tag{2.2}$$

Then $\psi_\Lambda(\beta,h) = \psi_\Lambda(\beta,-h)$. Regarding convexity, it is more elegant to prove the following, much more general statement.

Theorem 2.2 (Convexity of the abstract pressure)**.** *The function*

$$f(A) = \log \mathrm{Tr}\, \mathrm{e}^A$$

is a convex function on the space of hermitian matrices.

Proof. We use the Golden-Thompson inequality (Proposition A.6) and then the Hölder inequality (Proposition A.1).

$$\begin{aligned} f(sA+(1-s)B) &= \log \mathrm{Tr}\, \mathrm{e}^{sA+(1-s)B} \\ &\leq \log \mathrm{Tr}\, \mathrm{e}^{sA}\, \mathrm{e}^{(1-s)B} \\ &\leq \log\Big[\Big(\mathrm{Tr}\, \big(\mathrm{e}^{sA}\big)^{\frac1s}\Big)^s \Big(\mathrm{Tr}\, \big(\mathrm{e}^{(1-s)B}\big)^{\frac{1}{1-s}}\Big)^{1-s} \Big] \\ &= sf(A) + (1-s)f(B). \end{aligned} \tag{2.3}$$

□

We now turn to the thermodynamic limit (i.e. infinite-volume limit) of the pressure. Recall that $\Lambda \Uparrow \mathbb{Z}^d$ denote the limit in the sense of van Hove.

Theorem 2.3. *There exists a function $\psi(\beta,h)$, that is convex in (β,h) and even in h, such that*

$$\psi(\beta,h) = \lim_{n\to\infty} \psi_{\Lambda_n}(\beta,h)$$

along all sequences of domains such that $\Lambda_n \Uparrow \mathbb{Z}^d$.

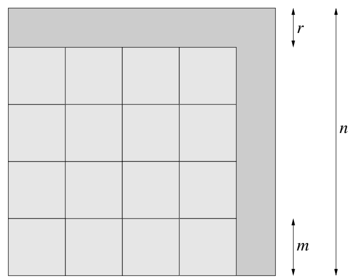

Figure 2.1: The large box of size n is decomposed in k^d boxes of size m; there are no more than drn^{d-1} remaining sites in the darker area.

Partial proof. We consider the hamiltonian $H^{\rm Heis.F}_{\Lambda,\beta,h}$, but the modifications for the other models are straightforward. We only consider the sequence (Λ_n) of boxes of size n. We use a subadditive argument. Notice that the inequality $\mathrm{Tr}\, \mathrm{e}^{A+B} \geq \mathrm{Tr}\, \mathrm{e}^A$ holds for all self-adjoint operators A, B with $B \geq 0$. (This follows e.g. from the minimax principle, or from Klein's inequality, Proposition A.7.) We rewrite the Hamiltonian so as to have only positive definite terms. Namely, let

$$h_{x,y} = \vec{S}_x \cdot \vec{S}_y + \tfrac34 \mathbb{1}. \tag{2.4}$$

Then

$$Z_{\Lambda_n,\beta,h} = \mathrm{e}^{-\frac{3}{4}\beta|\mathcal{E}_{\Lambda_n}|}\,\mathrm{Tr}\,\exp\Big(\beta \sum_{\{x,y\}\in\mathcal{E}_{\Lambda_n}} h_{x,y} + h\sum_{x\in\Lambda_n} S_x^3\Big). \tag{2.5}$$

Let m, n, k, r be integers such that $n = km + r$ and $0 \le r < m$. The box Λ_n is the disjoint union of k^d boxes of size m, and of some remaining sites (fewer than drn^{d-1}); see Figure 2.1 for an illustration. We get an inequality for the partition function in Λ_n by dismissing all $h_{x,y}$ where $\{x, y\}$ are not inside a single box of size m. The boxes Λ_m become independent, and

$$\begin{aligned} Z_{\Lambda_n,\beta,h} &\ge \mathrm{e}^{-\frac{3}{4}\beta|\mathcal{E}_{\Lambda_n}|}\Big[\mathrm{Tr}_{\mathcal{H}_{(\Lambda_m)}}\exp\Big(\beta\sum_{\{x,y\}\in\mathcal{E}_{\Lambda_m}} h_{x,y} + h\sum_{x\in\Lambda_m} S_x^{(3)}\Big)\Big]^{k^d} \\ &= [Z_{\Lambda_m,\beta,h}]^{k^d}\,\mathrm{e}^{-\frac{3}{4}\beta|\mathcal{E}_{\Lambda_n}|}\,\mathrm{e}^{k^d\frac{3}{4}\beta|\mathcal{E}_{\Lambda_m}|}. \end{aligned} \tag{2.6}$$

We have neglected the contribution of $\mathrm{e}^{hS_x^3}$ for x outside the boxes Λ_m, which is possible because their traces are greater than 1. It is not hard to check that

$$|\mathcal{E}_{\Lambda_n}| \le k^d|\mathcal{E}_{\Lambda_m}| + k^d dm^{d-1} + d^2rn^{d-1}. \tag{2.7}$$

We then obtain a subbaditive relation for the free energy, up to error terms that will soon disappear:

$$\psi_{\Lambda_n}(\beta,h) \ge \frac{(km)^d}{n^d}\psi_{\Lambda_m}(\beta,h) - \frac{3\beta k^d dm^{d-1}}{4n^d} - \frac{3\beta d^2 r}{4n}. \tag{2.8}$$

Then, since $\frac{km}{n} \to 1$ as $n\to\infty$,

$$\liminf_{n\to\infty}\psi_{\Lambda_n}(\beta,h) \ge \psi_{\Lambda_m}(\beta,h) - \frac{3\beta d}{4m}. \tag{2.9}$$

Taking the $\limsup$ over m in the right side, we see that it is smaller or equal to the $\liminf$, and so the limit necessarily exists. □

Corollary 2.4 (Thermodynamic limit with periodic boundary conditions)**.** *Let $(\Lambda_n^{\mathrm{per}})$ be the sequence of cubes in $\mathbb{Z}^d$ of size n with periodic boundary conditions and nearest-neighbor edges. Then $(\psi_{\Lambda_n^{\mathrm{per}}}(\beta,h))_{n\ge1}$ converges pointwise to the same function $\psi(\beta,h)$ as in Theorem 2.3, uniformly on compact sets.*

This follows from $|\psi_{\Lambda_n^{\mathrm{per}}}(\beta,h) - \psi_{\Lambda_n}(\beta,h)| \le \frac{3\beta d}{4n}$, which is not too hard to prove, and Theorem 2.3.

A finite-volume **Gibbs state** is a positive, normalised, linear map on the space of operators on $\mathcal{H}_\Lambda$ of the form

$$\langle A\rangle_{\Lambda,\beta,h} = \frac{1}{Z_{\Lambda,\beta,h}}\mathrm{Tr}_{\mathcal{H}_\Lambda} A\,\mathrm{e}^{-H_{\Lambda,\beta,h}}. \tag{2.10}$$

The finite-volume **magnetisation** $m_\Lambda(\beta,h)$ is defined as

$$m_\Lambda(\beta,h) = \frac{\partial}{\partial h}\psi_\Lambda(\beta,h) = \Big\langle \frac{1}{|\Lambda|}\sum_{x\in\Lambda} S_x^3\Big\rangle_{\Lambda,\beta,h}. \tag{2.11}$$

Theorem 2.5. *First, assume that $\psi(\beta,h)$ is differentiable in h at (β,h); then*

(a) The limit $m(\beta,h) = \lim_{\Lambda\Uparrow\mathbb{Z}^d} m_\Lambda(\beta,h)$ exists.

(b) We have $m(\beta,h) = \frac{\partial}{\partial h}\psi(\beta,h)$.

Second, without assuming that $\psi(\beta,h)$ is differentiable in h at (β,h), we have

(c) $\frac{\partial}{\partial h^+}\psi(\beta,h) = \lim_{h'\to h^+}\liminf_{\Lambda\Uparrow\mathbb{Z}^d}\Big\langle\frac{1}{|\Lambda|}\sum_{x\in\Lambda}S_x^3\Big\rangle_{\Lambda,\beta,h'}$.

Part (c) also holds with $\limsup$ instead of $\liminf$. We have proved parts (a) and (b) of this theorem in the case of the Ising model. The proof actually extends to the quantum case without modifications.

Proof. We use the fact that

$$\begin{aligned} \limsup_i\big(\inf_j a_{ij}\big) &\le \inf_j\big(\limsup_i a_{ij}\big), \\ \liminf_i\big(\sup_j\big) &\ge \sup_j\big(\liminf_i a_{ij}\big), \end{aligned} \tag{2.12}$$

and the following expressions for left- and right-derivatives of convex functions:

$$\begin{aligned} \frac{\mathrm{d}f}{\mathrm{d}h^-}(h) &= \sup_{s>0}\frac{f(h)-f(h-s)}{s}, \\ \frac{\mathrm{d}f}{\mathrm{d}h^+}(h) &= \inf_{s>0}\frac{f(h+s)-f(h)}{s}. \end{aligned} \tag{2.13}$$

Since ψ is convex, we have

$$\begin{aligned}\frac{\partial\psi}{\partial h^-}(\beta,h) &= \sup_{s>0}\liminf_{\Lambda\Uparrow\mathbb{Z}^d}\frac{\psi_\Lambda(\beta,h)-\psi_\Lambda(\beta,h-s)}{s}\\ &\le \liminf_{\Lambda\Uparrow\mathbb{Z}^d}\frac{\partial\psi_\Lambda}{\partial h}(\beta,h)\\ &\le \limsup_{\Lambda\Uparrow\mathbb{Z}^d}\frac{\partial\psi_\Lambda}{\partial h}(\beta,h)\\ &= \limsup_{\Lambda\Uparrow\mathbb{Z}^d}\inf_{s>0}\frac{\psi_\Lambda(\beta,h+s)-\psi_\Lambda(\beta,h)}{s}\\ &= \frac{\partial\psi}{\partial h^+}(\beta,h).\end{aligned} \tag{2.14}$$

But ψ is differentiable, so that inequalities are identities and we get

$$\frac{\partial\psi}{\partial h}(\beta,h) = \liminf_{\Lambda\Uparrow\mathbb{Z}^d}\frac{\partial\psi_\Lambda}{\partial h}(\beta,h) = \limsup_{\Lambda\Uparrow\mathbb{Z}^d}\frac{\partial\psi_\Lambda}{\partial h}(\beta,h). \tag{2.15}$$

Since $\frac{\partial\psi_\Lambda}{\partial h}(\beta,h) = m_\Lambda(\beta,h)$, this shows that the infinite volume limit of the latter exists, and is equal to the derivative of ψ. This proves (a) and (b).

For (c), let $h_n \to h^+$ such that ψ is differentiable in h at (β,h_n). We have just proved that $\frac{\partial}{\partial h^+}\psi(\beta,h_n) = \lim_\Lambda m_\Lambda(\beta,h_n)$. Since right-derivatives of convex functions are right-continuous, and since m_Λ is nondecreasing in h, we get the result. □

Exercise 2.1. *Let P_ε be the projector onto the subspace of $\mathcal{H}_\Lambda$ spanned by configurations ω such that $\frac{1}{|\Lambda|}\sum_x \omega_x \notin (-\varepsilon,\varepsilon)$. Show that if $\psi(\beta,h)$ is differentiable in h at $(\beta,0)$, then*

$$\langle P_\varepsilon\rangle_{\Lambda,\beta,0} \le \mathrm{e}^{-c|\Lambda|}$$

for some $c>0$ that is uniform in Λ. (Hint: Show that a Chernov inequality holds.)

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3 Correlation functions

Correlation functions give useful information about the state of the system; they usually characterise phases. The most natural *truncated correlation function* deals with 3rd components of spins and is given by

$$\kappa_{\Lambda,\beta,h}(x,y) = \langle S_x^3 S_y^3\rangle_{\Lambda,\beta,h} - \langle S_x^3\rangle_{\Lambda,\beta,h}\langle S_y^3\rangle_{\Lambda,\beta,h}. \tag{3.1}$$

It is a general fact that, when the Gibbs state is unique, truncated correlations decrease to zero as $\|x-y\|$ becomes large. When truncated correlations do not

vanish, one talks about "long-range order" and this indeed implies the existence of several Gibbs states.

We focus in this section on the Heisenberg ferromagnet. We expect the Gibbs state to be unique when $h \neq 0$, or when β is small. The most interesting case is then when $h=0$, in which case

$$\kappa_{\Lambda,\beta,0}(x,y) = \langle S_x^3 S_y^3\rangle_{\Lambda,\beta,0}. \tag{3.2}$$

In order to relate correlations to magnetisation, let $M_\Lambda = \sum_{x\in\Lambda} S_x^3$ denote the operator for the 3rd component of the total spin.

Proposition 3.1. *For all $\beta \ge 0$, we have*

(a) $\displaystyle \frac{\partial}{\partial h^+}\psi(\beta,0) \ge \tfrac12 \liminf_{\Lambda\Uparrow\mathbb{Z}^d}\Big\langle \frac{|M_\Lambda|}{|\Lambda|}\Big\rangle_{\Lambda,\beta,0}.$

(b) $\displaystyle \Big\langle \frac{|M_\Lambda|}{|\Lambda|}\Big\rangle_{\Lambda,\beta,0}^2 \le \Big\langle\Big(\frac{M_\Lambda}{|\Lambda|}\Big)^2\Big\rangle_{\Lambda,\beta,0} \le \tfrac12\Big\langle \frac{|M_\Lambda|}{|\Lambda|}\Big\rangle_{\Lambda,\beta,0}.$

Proof. For (a), we start with Theorem 2.5 (c) to get

$$\frac{\partial}{\partial h^+}\psi(\beta,0) = \lim_{h\to0^+}\liminf_{\Lambda\Uparrow\mathbb{Z}^d}\frac{1}{|\Lambda|}\langle M_\Lambda\rangle_{\Lambda,\beta,h}. \tag{3.3}$$

Let $\{\varphi_j\}$ be an orthonormal basis of eigenvectors of $H_{\Lambda,\beta,0}$ and M_Λ with eigenvalues e_j and m_j, respectively. Since $H_{\Lambda,\beta,h} = H_{\Lambda,\beta,0} - hM_\Lambda$ and $[H_{\Lambda,\beta,0}, M_\Lambda] = 0$, the eigenvalues of $H_{\Lambda,\beta,h}$ are $e_j - hm_j$. Because of symmetry (rotation around 1st direction of spin by angle π), the set $\{m_j\}$ is symmetric around 0 (with multiplicity). Let $h>0$. We have

$$\langle M_\Lambda\rangle_{\Lambda,\beta,h} = \frac{\sum_{j:m_j>0} m_j\,\mathrm{e}^{-e_j}\,(\mathrm{e}^{hm_j} - \mathrm{e}^{-hm_j})}{\sum_{j:m_j>0}\mathrm{e}^{-e_j}\,(\mathrm{e}^{hm_j}+\mathrm{e}^{-hm_j}) + \sum_{j:m_j=0}\mathrm{e}^{-e_j}}. \tag{3.4}$$

Let $\langle\cdot\rangle'_{\Lambda,\beta,h}$ be the Gibbs state with hamiltonian $H_{\Lambda,\beta,0} - h|M_\Lambda|$. We have

$$\langle |M_\Lambda|\rangle'_{\Lambda,\beta,h} = \frac{2\sum_{j:m_j>0} m_j\,\mathrm{e}^{-e_j+hm_j}}{2\sum_{j:m_j>0}\mathrm{e}^{-e_j+hm_j} + \sum_{j:m_j=0}\mathrm{e}^{-e_j}}. \tag{3.5}$$

Then

$$\langle M_\Lambda\rangle_{\Lambda,\beta,h} \ge \tfrac12\langle|M_\Lambda|\rangle'_{\Lambda,\beta,h} - \frac{\sum_{j:m_j>0} m_j\,\mathrm{e}^{-e_j-hm_j}}{\sum_{j:m_j>0}\mathrm{e}^{-e_j+hm_j}}. \tag{3.6}$$

The last term goes to 0 in the limit $\Lambda \Uparrow \mathbb{Z}^d$. Indeed, the sum of terms with $m_j \le \sqrt{|\Lambda|}$ contribute less than $1/\sqrt{|\Lambda|}$, and the sum of terms with $m_j > \sqrt{|\Lambda|}$ contribute less than $|\Lambda| \, \mathrm{e}^{-h\sqrt{|\Lambda|}}$. By convexity of the pressure of the model with hamiltonian $H_{\Lambda,\beta,0} - h|M_\Lambda|$ (Theorem 2.2), we get

$$\langle |M_\Lambda| \rangle'_{\Lambda,\beta,h} = \frac{\partial}{\partial h} \log \mathrm{Tr} \, \mathrm{e}^{-H_{\Lambda,\beta,0}+h|M_\Lambda|} \geq \langle |M_\Lambda| \rangle_{\Lambda,\beta,0}. \tag{3.7}$$

For the first inequality in (b), we can use $|M_\Lambda| = |M_\Lambda| \mathbb{1}$ and the following Cauchy-Schwarz inequality:

$$|\langle A^* B \rangle|^2 \leq \langle A^* A \rangle \, \langle B^* B \rangle, \tag{3.8}$$

where $\langle \cdot \rangle = \mathrm{Tr} \cdot \mathrm{e}^{-H}$ denotes a Gibbs state, and A, B are any operators that commute with H. For the second inequality in (b), observe that $|M_\Lambda| \leq \frac12 |\Lambda| \mathbb{1}$ implies that $M_\Lambda^2 \leq \frac12 |\Lambda| |M_\Lambda|$, and use the fact that the Gibbs state is a positive linear functional. □

Exercise 3.1. *Let $h \geq 0$. Show that $\Big\langle \prod_{x \in X} S_x^3 \Big\rangle_{\Lambda,\beta,h} \geq 0$ for all $X \subset \Lambda$.*

We now prove a variant of the Mermin-Wagner theorem; in broad terms, it states that continuous symmetries cannot be broken in two dimensions. More precisely, we prove that spin correlations have (at least) power-law decay in the quantum Heisenberg model. This result goes back to McBryan and Spencer (1977) in the case of the classical Heisenberg model, and to Koma and Tasaki (1992) in the quantum case. We only consider the Heisenberg model, but the method and the result apply to general models with U(1) symmetry.

The decay of correlations is measured by the following expression:

$$\xi_\beta(x) = \sup_{\substack{(\phi_y) \in \mathbb{R}^\Lambda \\ \phi_x = 0}} \Big[\phi_0 - \tfrac12 \beta \sum_{\{y,z\} \in \mathcal{E}_\Lambda} \big(\cosh(\phi_y - \phi_z) - 1 \big) \Big]. \tag{3.9}$$

The solution of this variational problem is essentially a discrete harmonic function. We can estimate it explicitly in the case of "2D-like" graphs with nearest-neighbor couplings. Let Λ denote a graph, i.e. a finite set of vertices and a set of edges, and let $d(x,y)$ denote the graph distance, i.e. the length of the shortest path that connects x and y.

Lemma 3.2. *Assume that there exists a constant K such that, for any $\ell \in \mathbb{N}$,*

$$\# \big\{ \{x,y\} \subset \Lambda : d(0,x) = \ell \text{ and } d(0,y) = \ell + 1 \big\} \leq K\ell.$$

Then there exists $C = C(\beta, K)$, which does not depend on x, such that

$$\xi_\beta(x) \geq \frac{1}{2\beta K} \log\big(d(0,x) + 1 \big) - C.$$

Proof. With c to be chosen later, let

$$\phi_y = \begin{cases} c \log \frac{d(0,x)+1}{d(0,y)+1} & \text{if } d(0,y) \leq d(0,x), \\ 0 & \text{otherwise.} \end{cases} \tag{3.10}$$

Then

$$\xi_\beta(x) \geq c \log(d(0,x)+1) - \tfrac12 \beta K \sum_{\ell=0}^{d(0,x)-1} \big(\cosh(c \log \tfrac{\ell+2}{\ell+1}) - 1 \big) \ell. \tag{3.11}$$

From Taylor expansions of the logarithm and of the hyperbolic cosine, there exist C, C' such that

$$\begin{aligned} \xi_\beta(x) &\geq c \log(d(0,x)+1) - \tfrac12 \beta K c^2 \sum_{\ell=1}^{d(0,x)} \frac{1}{\ell} - C' \\ &\geq \big[c - \tfrac12 \beta K c^2 \big] \log(d(0,x)+1) - C. \end{aligned} \tag{3.12}$$

The optimal choice is $c = (\beta K)^{-1}$. □

Theorem 3.3. *For $i = 1,2,3$, we have*

$$\big| \langle S_0^i S_x^i \rangle_{\Lambda,\beta,0} \big| \leq \tfrac12 \, \mathrm{e}^{-\xi_\beta(x)}.$$

In the case of 2D-like graphs, we can use Lemma 3.2 and we obtain algebraic decay with a power greater than $(2\beta K)^{-1}$.

Proof of Theorem 3.3. We use the method of complex rotations. Let

$$S_y^\pm = S_y^1 \pm \mathrm{i} S_y^2. \tag{3.13}$$

One can check that for any $a \in \mathbb{C}$, we have

$$\mathrm{e}^{a S_y^3} \, S_y^\pm \, \mathrm{e}^{-a S_y^3} = \mathrm{e}^{\pm a} \, S_y^\pm. \tag{3.14}$$

The Heisenberg hamiltonian can be rewritten as

$$H_{\Lambda,\beta,0} = - \sum_{\{y,z\} \in \mathcal{E}_\Lambda} \big(S_y^+ S_z^- + S_y^- S_z^+ + S_y^3 S_z^3 \big) \tag{3.15}$$

Given numbers ϕ_y, let

$$A = \prod_{y \in \Lambda} \mathrm{e}^{\phi_y S_y^3}. \tag{3.16}$$

Then

$$\mathrm{Tr} \, S_0^+ S_x^- \, \mathrm{e}^{-H_{\Lambda,\beta,0}} = \mathrm{Tr} \, A S_0^+ S_x^- A^{-1} \, \mathrm{e}^{-A H_{\Lambda,\beta,0} A^{-1}}. \tag{3.17}$$

We now compute the rotated hamiltonian.

$$\begin{aligned} A H_{\Lambda,\beta,0} A^{-1} &= - \sum_{\{y,z\}\in\mathcal{E}_\Lambda} \big(e^{\phi_y-\phi_z} S_y^+ S_z^- + e^{\phi_z-\phi_y} S_y^- S_z^+ + S_y^3 S_z^3 \big) \\ &= H_{\Lambda,\beta,0} - \sum_{\{y,z\}\in\mathcal{E}_\Lambda} \big(\cosh(\phi_y-\phi_z)-1\big)\,(S_y^+S_z^- + S_y^-S_z^+) \\ &\qquad - \sum_{\{y,z\}\in\mathcal{E}_\Lambda} \sinh(\phi_y-\phi_z)\,(S_y^+S_z^- - S_y^-S_z^+) \\ &\equiv H_{\Lambda,\beta,0} + B + C. \end{aligned} \tag{3.18}$$

Notice that $B^* = B$ and $C^* = -C$. We obtain

$$\operatorname{Tr} S_0^+ S_x^- \, e^{-H_{\Lambda,\beta,0}} = e^{\phi_0-\phi_x} \operatorname{Tr} S_0^+ S_x^- \, e^{-H_{\Lambda,\beta,0}-B-C}. \tag{3.19}$$

We now estimate the trace in the right side using the Trotter product formula and the Hölder inequality for traces. Recall that $\|B\|_s = (\operatorname{Tr}|B|^s)^{1/s}$, with $\|B\|_\infty = \|B\|$ being the usual operator norm.

$$\begin{aligned} \operatorname{Tr} S_0^+ S_x^- \, e^{-H_{\Lambda,\beta,0}-\beta B-\beta C} &= \lim_{N\to\infty} \operatorname{Tr} S_0^+ S_x^- \Big(e^{-\frac{\beta}{N} H_\Lambda} \, e^{-\frac{\beta}{N} B} \, e^{-\frac{\beta}{N} C} \Big)^N \\ &\leq \lim_{N\to\infty} \|S_0^+ S_x^-\|_\infty \, \| e^{-\frac1N H_{\Lambda,\beta,0}} \|_N^N \, \| e^{-\frac1N B} \|_\infty^N \, \| e^{-\frac1N C} \|_\infty^N. \end{aligned} \tag{3.20}$$

Observe now that $\| e^{-\frac1N H_\Lambda} \|_N^N = Z_\Lambda$, $\| e^{-\frac1N B} \|^N \leq e^{\|B\|}$, and $\| e^{-\frac1N C} \| = 1$. We also have $\|S_0^+ S_x^-\| = 2S^2$ (check it!). The theorem then follows from

$$\|B\| \leq \tfrac12 \sum_{\{y,z\}\in\mathcal{E}_\Lambda} \big(\cosh(\phi_y-\phi_z)-1\big). \tag{3.21}$$

□

4 High temperature expansions

Let us recall the method in the case of the Ising model, since it can be essentially generalised in the quantum case. We started by writing $e^{\beta\omega_x\omega_y} = \cosh\beta(1 + \omega_x\omega_y \tanh\beta)$, and then expanded the Gibbs factor so as to get a sum of sets of edges with nonnegative weights. This allowed to get

$$\langle\sigma_x\sigma_y\rangle_{\Lambda,\beta,0}^{\varnothing} \leq \sum_{\substack{E\subset\mathcal{E}_\Lambda \\ \text{connected}, x,y\in E}} (\tanh\beta)^{|E|}; \tag{4.1}$$

the sum is restricted over sets E are such that the graphs with vertices $\cup_{\{u,v\}\in E}\{u,v\}$, and edges E, is connected and it contains the sites x, y.

Lemma 4.1. *The number of connected sets $E \subset \mathcal{E}_\Lambda$, with $|E| = k$, and that contain the origin, is less than $(4d-1)^{2k}$.*

Proof. Given any connected graph with k edges, and any vertex, there exists a sequence of edges $(e_1, \dots, e_{2k})$ such that e_1 contains this vertex, $e_{i+1} \cap e_i \neq \emptyset$, and each edge of the graph appears exactly twice. (This can be easily proved by induction, adding edges or vertices.) The number of connected sets is then less than the number of such sequences. Given an edge, there are $4d-1$ overlapping edges, hence the bound. □

One then gets exponential decay for small β. Indeed, with $d(x,y)$ being the graph distance between $x, y \in \Lambda$, we have

$$\begin{aligned} e^{cd(x,y)} \langle\sigma_x\sigma_y\rangle_{\Lambda,\beta,0}^{\varnothing} &\leq \sum_{\substack{E\subset\mathcal{E}_\Lambda \\ \text{connected}, x,y\in E}} \big(e^c \tanh\beta \big)^{|E|} \\ &\leq \sum_{k\geq0} \Big[(4d-1)^2 e^c \tanh\beta \Big]^k. \end{aligned} \tag{4.2}$$

The right side is bounded if $(4d-1)^2 \tanh\beta < 1$, and if c is small enough. Then $\langle\sigma_x\sigma_y\rangle_{\Lambda,\beta,0}^{\varnothing} \leq C\, e^{-cd(x,y)}$ with a constant C independent of Λ, x, y.

We now turn to quantum spin systems. We consider here a hamiltonian of the form $H'_\Lambda = -\beta \sum_{\{x,y\}\in\mathcal{E}_\Lambda} b_{x,y}$, where the operators $b_{x,y}$ satisfy

$$\langle\omega|b_{x,y}|\omega'\rangle = \langle-\omega|b_{x,y}|-\omega'\rangle \begin{cases} = 0 & \text{if } w_z \neq \omega'_z \text{ for some } z \neq x, y, \\ \geq 0 & \text{always}, \end{cases} \tag{4.3}$$

In words, we suppose that $b_{x,y}$ affects the sites x, y only, that it has nonnegative matrix elements, and that its elements are invariant under spin flips. Without loss of generality, we can suppose that the matrix elements of $b_{x,y}$ are less than 1 (we can rescale β otherwise).

We expand the exponential in Taylor series and get

$$\begin{aligned} \operatorname{Tr} S_x^3 S_y^3 \, e^{-H'_\Lambda} &= \sum_{n\geq0} \frac{\beta^n}{n!} \sum_{e_1,\dots,e_n\in\mathcal{E}_\Lambda} \operatorname{Tr} S_x^3 S_y^3 b_{e_1} \dots b_{e_n} \\ &= \sum_{n\geq0} \frac{\beta^n}{n!} \sum_{e_1,\dots,e_n\in\mathcal{E}_\Lambda} \sum_{\omega\in\Omega_\Lambda} \tfrac14 \omega_x\omega_y \langle\omega|b_{e_1}\dots b_{e_n}|\omega\rangle. \end{aligned} \tag{4.4}$$

In order for the sum over configuration to differ from zero, the sites x, y must be connected by a path of overlapping edges. Summing first over edges forming a connected set that contains x, y, and then over the remaining edges, we get

$$\begin{aligned}
\mathrm{Tr}\, S_x^3 S_y^3 \,\mathrm{e}^{-H'_\Lambda} &= \sum_{n\geq 0} \frac{\beta^n}{n!} \sum_{k=0}^{n} \binom{n}{k} \sum_{\substack{e_1,\dots,e_k \in \mathcal{E}_\Lambda \\ \text{connected}, x,y \in \cup_i e_i}} \sum_{\omega \in \Omega_{\cup_i e_i}} \tfrac14 \omega_x \omega_y \langle \omega | b_{e_1} \dots b_{e_k} | \omega \rangle \\
&\qquad \sum_{e'_1,\dots,e'_{n-k} \in \mathcal{E}_{\Lambda \setminus \cup_i e_i}} \sum_{\omega' \in \Omega_{\Lambda \setminus \cup_i e_i}} \langle \omega' | b_{e'_1} \dots b_{e'_{n-k}} | \omega \rangle \\
&= \sum_{k\geq 0} \frac{\beta^k}{k!} \sum_{\substack{e_1,\dots,e_k \in \mathcal{E}_\Lambda \\ \text{connected}, x,y \in \cup_i e_i}} \sum_{\omega \in \Omega_{\cup_i e_i}} \tfrac14 \omega_x \omega_y \langle \omega | b_{e_1} \dots b_{e_k} | \omega \rangle \mathrm{Tr}_{\mathcal{H}_{\Lambda \setminus \cup_i e_i}} \mathrm{e}^{-H'_{\Lambda \setminus \cup_i e_i, \beta}} .
\end{aligned} \tag{4.5}$$

Next we use

$$\begin{aligned}
&\sum_{\omega \in \Omega_{\cup_i e_i}} \tfrac14 \omega_x \omega_y \langle \omega | b_{e_1} \dots b_{e_k} | \omega \rangle \leq \tfrac14 2^{|\cup_i e_i|}, \\
&2^{|\cup_i e_i|} \mathrm{Tr}_{\mathcal{H}_{\Lambda \setminus \cup_i e_i}} \mathrm{e}^{-H'_{\Lambda \setminus \cup_i e_i, \beta}} \leq \mathrm{Tr}_{\mathcal{H}_\Lambda} \mathrm{e}^{-H'_{\Lambda,\beta}} .
\end{aligned} \tag{4.6}$$

We obtain

$$\langle S_x^3 S_y^3 \rangle'_{\Lambda,\beta} \leq \tfrac14 \sum_{k\geq 0} \frac{\beta^k}{k!} \sum_{\substack{e_1,\dots,e_k \in \mathcal{E}_\Lambda \\ \text{connected}, x,y \in \cup_i e_i}} 1. \tag{4.7}$$

By first summing over sets of edges, and then over their number of occurrences, we get

$$\begin{aligned}
\sum_{k\geq 0} \frac{\beta^k}{k!} \sum_{\substack{e_1,\dots,e_k \in \mathcal{E}_\Lambda \\ \text{connected}, x,y \in \cup_i e_i}} 1 &= \sum_{\substack{E \subset \mathcal{E}_\Lambda \\ \text{connected}, x,y \in E}} \sum_{n_1,\dots,n_{|E|} \geq 1} \binom{n_1 + \dots + n_{|E|}}{n_1 \dots n_{|E|}} \frac{\beta^{n_1 + \dots + n_{|E|}}}{(n_1 + \dots + n_{|E|})!} \\
&= \sum_{\substack{E \subset \mathcal{E}_\Lambda \\ \text{connected}, x,y \in E}} \big(\mathrm{e}^{\beta} - 1 \big)^{|E|}.
\end{aligned} \tag{4.8}$$

Finally, we can proceed as in the case of Ising, namely

$$\begin{aligned}
\mathrm{e}^{cd(x,y)} \langle S_x^3 S_y^3 \rangle'_{\Lambda,\beta} &\leq \tfrac14 \sum_{\substack{E \subset \mathcal{E}_\Lambda \\ \text{connected}, x,y \in E}} \big(\mathrm{e}^{c} (\mathrm{e}^{\beta} - 1) \big)^{|E|} \\
&\leq \sum_{k\geq 0} \Big[(4d-1)^2 \mathrm{e}^{c} (\mathrm{e}^{\beta} - 1) \Big]^k .
\end{aligned} \tag{4.9}$$

Let us summarise the results obtained in this section.

Theorem 4.2. *Let $\langle \cdot \rangle'_{\Lambda,\beta}$ be the Gibbs state with hamiltonian $H'_{\Lambda,\beta} = -\beta \sum_{\{x,y\} \in \mathcal{E}_\Lambda}$, where the operators $b_{x,y}$ satisfy the proprieties (4.3), and with matrix elements less than 1. Then, if $(4d-1)^2(\mathrm{e}^{\beta} - 1) < 1$, there exist constants C, c that are uniform in $\Lambda \Subset \mathbb{Z}^d$ and $x, y \in \Lambda$, such that*

$$\langle S_x^3 S_y^3 \rangle'_{\Lambda,\beta} \leq C \,\mathrm{e}^{-cd(x,y)} .$$

The method uses the fact that the hamiltonian can be written using matrices with nonnegative elements. There is a powerful method, called cluster expansions, that allows to obtain exponential decay more generally.

Exercise 4.1. *Let $T_{x,y}$ denote the transposition operator on sites x, y, namely*

$$T_{x,y} |\omega\rangle = |\omega'\rangle, \qquad \text{where } \omega'_z = \begin{cases} \omega_y & \text{if } z = x; \\ \omega_x & \text{if } z = y; \\ \omega_z & \text{if } z \neq x, y. \end{cases}$$

Show that $\vec{S}_x \cdot \vec{S}_y = \frac12 T_{x,y} - \frac14$.

Exercise 4.2. *Let $H_{\Lambda,\beta,0} = -\beta \sum_{\{x,y\} \in \mathcal{E}_\Lambda} \big(S_x^1 S_y^1 + u S_x^2 S_y^2 + S_x^3 S_y^3 \big)$, where $u \in [0,1]$ is a fixed parameter. Find operators $b_{x,y}$ that satisfy (4.3), and such that the Gibbs state $\langle \cdot \rangle'_{\Lambda,\beta}$ with hamiltonian $H'_{\Lambda,\beta} = -\beta \sum_{\{x,y\} \in \mathcal{E}_\Lambda} b_{x,y}$ has identical spin correlations, i.e. $\langle S_x^3 S_y^3 \rangle_{\Lambda,\beta,0} = \langle S_x^3 S_y^3 \rangle'_{\Lambda,\beta}$.*

A Matrix inequalities

We collect here a series of useful properties of square matrices. Recall that the "absolute value" of a matrix is $|A| = (A^*A)^{\frac{1}{2}}$, where the square root of a nonnegative hermitian matrix can be defined by diagonalising and taking the square root of the eigenvalues. The p-norm of a matrix is then defined as

$$\|A\|_p = (\operatorname{Tr}|A|^p)^{1/p}. \tag{A.1}$$

Exercise A.1.

(i) Show that $\|A\|_p$ is decreasing in p, and that $\lim_{p\to\infty}\|A\|_p = \|A\|$.

(ii) Prove that $\|A\|_p$ is a norm for all $1 \le p \le \infty$.

(iii) Use Hölder inequality to show that $\|A\|_p$ is submultiplicative, that is, $\|AB\|_p \le \|A\|_p\|B\|_p$.

Proposition A.1 (Hölder inequality for matrices)**.** *If $1 \le p, q, r \le \infty$ with $\frac{1}{p} + \frac{1}{q} = \frac{1}{r}$, we have*

$$\|AB\|_r \le \|A\|_p\|B\|_q.$$

It follows from a simple induction that

$$\Big\|\prod_{j=1}^{n} A_j\Big\|_r \le \prod_{j=1}^{n}\|A_j\|_{p_j} \tag{A.2}$$

whenever $1 \le r, p_1, \dots, p_n$ with $\sum_{j=1}^n \frac{1}{p_j} = \frac{1}{r}$. There are no short proofs of Hölder's inequality for matrices. The proof is due to Fröhlich [1978] and it uses chessboard estimates. The proof of Proposition A.1 can be found after Lemma that of A.4.

Lemma A.2 (Chessboard estimate)**.** *For any $n \in \mathbb{N}$ and any matrices $A_1, \dots, A_{2n}$, we have*

$$\big|\operatorname{Tr} A_1 \dots A_{2n}\big| \le \prod_{i=1}^{2n}\Big(\operatorname{Tr}(A_iA_i^*)^n\Big)^{1/2n}.$$

Proof. Since $(A, B) \mapsto \operatorname{Tr} A^*B$ is an inner product, the following inequality follows from Cauchy-Schwarz:

$$\big|\operatorname{Tr} A_1 \dots A_{2n}\big|^2 \le \operatorname{Tr}\big(A_1 \dots A_nA_n^* \dots A_1^*\big)\operatorname{Tr}\big(A_{2n}^* \dots A_{n+1}^*A_{n+1} \dots A_{2n}\big). \tag{A.3}$$

This allows to use a reflection positivity argument. It is enough to prove the inequality for matrices that satisfy $\operatorname{Tr}(A_iA_i^*)^n = 1$; the general result follows from scaling.

Let $A_1, \dots, A_{2n}$ be matrices that maximise $|\operatorname{Tr} A_1 \dots A_{2n}|$, with maximum number of matching neighbours $A_{i+1} = A_i^*$. Suppose there exists an index j such that $A_{j+1} \ne A_j^*$. Using cyclicity, we can assume that $j = n$. By the inequality (A.3), $A_1, \dots, A_n, A_n^*, \dots, A_1^*$ and $A_{2n}^*, \dots, A_{n+1}^*, A_{n+1}, \dots, A_{2n}$ are also maximisers. At least one has strictly more matching neighbours, hence a contradiction. The maximum is then $\operatorname{Tr}(AA^*)^n$ for some matrix A, which is equal to 1. □

Chessboard estimates allow to prove what is essentially the case $r = 1$ of Hölder's inequality.

Corollary A.3. *We have*

$$|\operatorname{Tr} A_1 \dots A_n| \le \prod_{i=1}^{n}\|A_i\|_{p_i}$$

for all n and all rational p_i's such that $\sum_{i=1}^n \frac{1}{p_i} = 1$.

Proof. Let ℓ be a positive integer such that $2\ell/p_i$ is integer for all i. Let $A_i = U_i|A_i|$ be the polar decomposition of A_i, and let

$$B_i = |A_i|^{p_i/2\ell}, \qquad \hat{B}_i = U_i|A_i|^{p_i/2\ell}. \tag{A.4}$$

Then $A_i = \hat{B}_iB_i^{(2\ell/p_i)-1}$, and we have

$$\begin{aligned}\big|\operatorname{Tr} A_1 \dots A_n\big| &= \big|\operatorname{Tr}\hat{B}_1\underbrace{B_1 \dots B_1}_{(2\ell/p_1)-1} \dots \hat{B}_n\underbrace{B_n \dots B_n}_{(2\ell/p_n)-1}\big| \\ &\le \prod_{i=1}^{n}(\operatorname{Tr}|A_i|^{p_i})^{1/p_i} \qquad (A.5)\\ &= \prod_{i=1}^{n}\|A_i\|_{p_i}.\end{aligned}$$

The inequality follows from Lemma A.2 and from the identities

$$\operatorname{Tr}(B_iB_i^*)^\ell = \operatorname{Tr}(\hat{B}_i\hat{B}_i^*)^\ell = \operatorname{Tr}|A_i|^{p_i}. \tag{A.6}$$

□

Lemma A.4. *Let $r, r' \in [1,\infty]$ such that $\frac{1}{r}+\frac{1}{r'}=1$. Then for any square matrix A, we have*
$$\|A\|_r = \sup_{\|C\|_{r'}=1} \operatorname{Tr} C^* A.$$

Proof. The right side is smaller by Corollary A.3:
$$\big|\operatorname{Tr} C^* A\big| \le \|C\|_{r'} \|A\|_r = \|A\|_r. \tag{A.7}$$
In order to check that this inequality is saturated, let $A = U|A|$ be the polar decomposition of A, and choose $C = \|A\|_r^{1-r} U |A|^{r-1}$. Then $\|C\|_{r'} = 1$ and $\operatorname{Tr} C^* A = \|A\|_r$. □

Proof of Proposition A.1. Starting with Lemma A.4 and then using Corollary A.3, we have
$$\begin{aligned} \|AB\|_r &= \sup_{\|C\|_{r'}=1} \operatorname{Tr} C^* AB \\ &\le \sup_{\|C\|_{r'}=1} \|C\|_{r'} \|A\|_p \|B\|_q. \end{aligned} \tag{A.8}$$
□

Proposition A.5 (Trotter formula)**.** *Let A, B be square matrices. Then*
$$\mathrm{e}^{A+B} = \lim_{n\to\infty} \Big(\mathrm{e}^{\frac{1}{n}A}\, \mathrm{e}^{\frac{1}{n}B} \Big)^n = \lim_{n\to\infty} \Big[\big(1 + \tfrac{1}{n}A\big)\, \mathrm{e}^{\frac{1}{n}B} \Big]^n.$$

Proof. We prove the second formula — the mild changes for the first formula are straightforward. Let K_n be the matrix such that
$$\big(1+\tfrac{1}{n}A\big)\,\mathrm{e}^{\frac{1}{n}B} = 1 + \tfrac{1}{n}(A+B) + K_n. \tag{A.9}$$
It is clear that $\|K_n\| = O(\frac{1}{n^2})$. We have
$$\Big[\big(1+\tfrac{1}{n}A\big)\,\mathrm{e}^{\frac{1}{n}B}\Big]^n = \Big(1+\tfrac{1}{n}(A+B)\Big)^n + R_n, \tag{A.10}$$
where R_n is a matrix whose norm satisfies
$$\|R_n\| \le \sum_{k=0}^{n-1} \tbinom{n}{k}\, \|1 + \tfrac{1}{n}(A+B)\|^k\, \|K_n\|^{n-k} = O(\tfrac{1}{n}). \tag{A.11}$$
The first term in the right side of (A.10) converges to e^{A+B}. □

Proposition A.6 (Golden-Thompson inequality)**.** *Let A, B be hermitian matrices. Then*
$$\operatorname{Tr}\big(\mathrm{e}^{A+B}\big) \le \operatorname{Tr}\big(\mathrm{e}^{A}\,\mathrm{e}^{B}\big).$$

Proof. Hölder's inequality, Proposition A.1, implies that $\operatorname{Tr}(AB)^n \le \|AB\|_n^n$. The latter is equal to $\operatorname{Tr}(A^2B^2)^{n/2}$ since A, B are hermitian. If n is a power of 2, we can iterate and we get
$$\operatorname{Tr}(AB)^n \le \operatorname{Tr} A^n B^n. \tag{A.12}$$
We use this inequality with $A \mapsto \mathrm{e}^{\frac{1}{n}A}$ and $B \mapsto \mathrm{e}^{\frac{1}{n}B}$, which gives
$$\operatorname{Tr}\Big(\mathrm{e}^{\frac{1}{n}A}\,\mathrm{e}^{\frac{1}{n}B}\Big)^n \le \operatorname{Tr} \mathrm{e}^{A}\,\mathrm{e}^{B}. \tag{A.13}$$
The left side converges to $\operatorname{Tr} \mathrm{e}^{A+B}$ as $n\to\infty$ by the Trotter formula. □

Proposition A.7 (Klein inequality)**.** *Let f be a convex differentiable function, and A, B be hermitian matrices with eigenvalues in the domain of f. Then*
$$\operatorname{Tr}\big[f(A) - f(B) - (A-B)f'(B)\big] \ge 0.$$

With $f(s) = \mathrm{e}^s$, exchanging A and B, we get
$$\operatorname{Tr}\big(\mathrm{e}^{A} - \mathrm{e}^{B}\big) \le \operatorname{Tr}(A-B)\,\mathrm{e}^{A}. \tag{A.14}$$

Proof. Let (ϕ_i) and (ψ_i) be orthonormal bases of eigenvectors of A and B, and let (a_i) and (b_i) the eigenvalues. Let $c_{ij} = (\phi_i, \psi_j)$. Then
$$\begin{aligned} \big(\phi_i, \big[f(A) - f(B) - (A-B)f'(B)\big]\phi_i\big) &= f(a_i) - \sum_j |c_{ij}|^2 f(b_j) - \sum_j |c_{ij}|^2 (a_i - b_j) f'(b_j) \\ &= \sum_j |c_{ij}|^2 \big[f(a_i) - f(b_j) - (a_i - b_j) f'(b_j)\big] \\ &\ge 0. \end{aligned} \tag{A.15}$$
□

Index