CHAPTER 5

The Ising model

1. Ferromagnetism

The discovery of magnetic materials predates the invention of writing. Humans were fascinated by the attractive or repulsive forces between magnets and they assigned magical and esoteric values to these objects. Later the Chinese discovered the Earth's magnetic field and use magnets as compasses. Starting with the XVIIth Century, electric and magnetic phenomena were scientifically investigated. The corresponding chemical elements were identified and such properties as the dependence of magnetization on temperature were measured.

When placed in an external magnetic field, some materials create a magnetic field of their own. It points either in the same direction ("paramagnetism") or in the opposite direction ("diamagnetism"). **Ferromagnetism** is the ability of a paramagnetic material to retain **spontaneous magnetization** as the external magnetic field is removed. The only ferromagnetic elements are iron (Fe), cobalt (Co), nickel (Ni), gadolinium (Gd), and dysprosium (Dy). In addition, there are composite substances. The current understanding is far from satisfactory. It is clear, however, that ferromagnetism involves the spins of electrons of the outer layers.

Element	Atomic Nr	Electronic structure	Curie temp. [K]	β
Fe	26	[Ar] $3d^6$ $4s^2$	1043	0.33–0.37
Co	27	[Ar] $3d^7$ $4s^2$	1388	0.33–0.37
Ni	28	[Ar] $3d^8$ $4s^2$	627	0.33–0.37
Gd	64	[Xe] $4f^7$ $5d^1$ $6s^2$	293	0.33–0.37
Dy	66	[Xe] $4f^{10}$ $5d^0$ $6s^2$	85	0.33–0.37

Table 1. Ferromagnetic elements with some of their properties. The electronic structure of argon is $1s^2$ $2s^2$ $2p^6$ $3s^2$ $3p^6$, and that of xenon is [Ar] $3d^{10}$ $4s^2$ $4p^6$ $4d^{10}$ $5s^2$ $5p^6$. The last column gives the critical exponent β for the magnetization.

Spontaneous magnetization always depends on the temperature; the typical graph of $M(T)$ is depicted in Fig. 5.1. The critical temperature is called the **Curie temperature** and varies wildly from a material to another. As $T \nearrow T_{\rm c}$, the magnetization goes to 0 following a power law, $M(T) \approx (T_{\rm c} - T)^\beta$, where β is called a **critical exponent**. There are other critical exponents, for instance for the magnetic susceptibility. Contrary to the Curie temperature, critical exponents are nearly identical in all ferromagnetic materials. It is believed that they depend on such general characteristics as the spatial dimension, the broken microscopic symmetries, etc..., but not on specific characteristics such as the actual type of the lattice or the form of the interactions (nearest-neighbor only, or with longer range). This phenomenon is called **universality**.

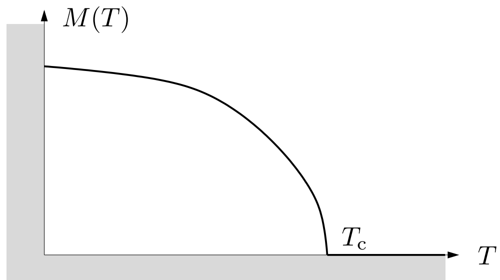

FIGURE 5.1. Spontaneous magnetization as function of the temperature in a typical ferromagnetic material.

2. Definition of the Ising model

The Ising model is a crude model for ferromagnetism. It was invented by Lenz who proposed it to his student Ernst Ising, whose PhD thesis appeared in 1925. It can be derived from quantum mechanical considerations through several educated guesses and rough simplifications.

It is an extremely interesting model despite its (apparent!) simplicity. There are several reasons for the great attention that it has received from both physicists and mathematicians:

- It is the simplest model of statistical mechanics where phase transitions can be rigorously established.
- Ferromagnetic phase transitions are "universal", in the sense that critical exponents appear to be identical in several different situations. Thus, studying one model allows to infer properties of other models.
- The Ising model has a probabilistic interpretation. The magnetization can be viewed as a sum of Bernoulli random variables that are identically distributed, but not independent. The law of large numbers and the central limit theorem take a subtle form that is best understood using physical intuition.

The Ising system describes spins on a finite lattice $D \subset \mathbb{Z}^d$. Here, we will always consider D to be a cubic box centered at the origin. By $|D|$ we denote the number of sites in D. Spins are little magnetic moments that can take two possible values, $+1$ or -1; they are often refered to as "spin up" and "spin down". A *spin configuration* ω is an assignment $\omega(x) = \pm 1$ to each $x \in D$. The state space Ω is the set of all possible configurations, i.e.

$$\Omega = \{-1, 1\}^D.$$

We define the total magnetization M to be

$$M(\omega) = \sum_{x \in D} \omega(x).$$

The energy of a configuration is given by the Hamiltonian function

$$H(\omega) = - \sum_{\substack{\{x,y\} \subset D \\ |x-y|=1}} \omega(x)\omega(y).$$

The Hamiltonian is thus given by a sum over nearest-neighbors, the value depending on whether the corresponding spins are aligned or anti-aligned. Its origin lies in quantum tunneling effects between electrons on a lattice of atoms ("condensed matter system"). The absence of dynamical variables may come as a surprise, in a model that is ultimately related to heat phenomena. But this is similar to the classical gas; after integration of the momenta, only static variables remain.

The most energetically favorable configurations are the constant configurations $\omega(x) \equiv 1$ and $\omega(x) \equiv -1$, that corresponds to full magnetization (in the "up" or "down" direction). These are the "ground state configurations", and the "ground state energy" is equal to $-|D|d$, up to an irrelevant boundary correction.

Actually, there is another interpretation of the Ising model that is worth mentioning. Namely, it represents a gas of "lattice particles" whose positions are restricted to the sites of a lattice, with no more than one particle at each site. Thus $\omega(x) = 1$ if the site x is occupied by a particle, and $\omega(x) = 0$ if x is empty. Magnetization and number of particles are related by $M = 2N - |D|$, and $N = 0, 1, \dots, |D|$.

The microcanonical partition function again measures the number of available states for given energy and number of particles (magnetization in the magnetic interpretation). Namely, we define

$$X(U, D, M) = \#\Big\{\omega \in \Omega : M(\omega) = M \text{ and } H(\omega) = U\Big\}. \tag{5.1}$$

We will always suppose that M takes values $-|D|, -|D|+2, \dots, |D|$, and that U is an integer between $-d|D|$ and $d|D|$. The finite volume Boltzmann entropy per site is then

$$s_D(u, m) = \log X(|D|u, D, |D|m). \tag{5.2}$$

A major difference with the classical gas is that $X(U, D, M)$ is *decreasing* with respect to U when U is large, and it is zero when $U > d|D|$, for any M. Consequently temperatures are *negative* for U large. This is clearly unphysical, a result of the lattice which puts a bound on the maximum energy of the system. However, this regime is worth considering, because it amounts to studying the negagtive of the Hamiltonian, $-H(\omega)$, which is known as the *Ising antiferromagnet*. The ground state configurations for $-H(\omega)$ are the two "chessboard configurations". The Ising antiferromagnet differs from the ferromagnet in many aspects.

The canonical ensemble involves variables β and M. It is not much used, so we do not introduce it. The grand-canonical ensemble, on the other hand, is very convenient. The partition function is

$$Z(\beta, D, h) = \sum_{\omega \in \Omega} \mathrm{e}^{-\beta[H(\omega) - hM(\omega)]}\,. \tag{5.3}$$

Here, β is the inverse temperature, and h is the external magnetic field. In the lattice gas interpretation we write $\mu N(\omega)$ instead of $hM(\omega)$. The logarithm of the grand-canonical partition function normally gives the pressure. But people usually considers the negative of the pressure:

$$q_D(\beta, h) = -\frac{1}{\beta|D|} \log Z(\beta, D, h). \tag{5.4}$$

It is always called "free energy", although the free energy really should be the logarithm of the canonical partition function with parameters β, m.

The existence of the thermodynamic limit, and the equivalence of ensembles, can be proved in a similar way as for the classical gas in the continuum — the lattice case is actually simpler. So we state the result without proof. Here, the limit $D \nearrow \mathbb{Z}^d$ ss in the sense of Fisher, see the definition in above chapters; it has straightforward generalization to the case of the lattice.

Proposition 5.1. *Consider $D \nearrow \mathbb{Z}^d$ in the sense of Fisher. Let $u_D \to u$ such that $|D|u \in \mathbb{Z}$, and $m_D \to m$ such that $|D|m_D$ is of the form $-|D| + 2k$. If $-d < u < d$ and $-1 < m < 1$, the function $s_D(u_D, m_D)$ converges pointwise to $s(u,m)$ (which takes finite values). If $|u| > d$ or $|m| > 1$, $s_D(u_D, m_D)$ goes to $-\infty$. The function $s(u,m)$ is concave in (u,m).*

The function $q_D(\beta,h)$ converges to $q(\beta,h)$, which is finite for any β, h. The function $\beta q(\beta,h)$ is concave in (β,h).

Finally, entropy and free energy are related as follows:

$$q(\beta,h) = \inf_{u,m}\bigl(u - hm - \tfrac{1}{\beta}s(u,m)\bigr); \tag{5.5}$$

$$s(u,m) = \inf_{\beta,h}\beta\bigl(u - hm - q(\beta,h)\bigr). \tag{5.6}$$

The model possesses several *symmetries*. The most important is the "spin-flip" symmetry: Let $\overline{\omega}$ be the configuration where all spins have been reversed, i.e. $\overline{\omega}(x) = -\omega(x)$. Then $H(\overline{\omega}) = H(\omega)$ and $M(\overline{\omega}) = -M(\omega)$. It follows that

$$s_D(u,-m) = s_D(u,m), \qquad \text{and} \qquad q_D(\beta,-h) = q_D(\beta,h). \tag{5.7}$$

These properties clearly extend to the infinite volume functions $s(u,m)$ and $q(\beta,h)$. Another symmetry is obtained by flipping the spins on a sublattice. Namely, let $\widehat{\omega}(x) = -\omega(x)$ if x belongs to a "white" site, and $\widehat{\omega}(x) = \omega(x)$ if x belongs to a "black" site. Then $H(\widehat{\omega}) = -H(\omega)$. There is no simple relation for $M(\omega)$, but we consider $h = 0$ when it plays no rôle. This symmetry implies that

$$Z(-\beta,D,0) = Z(\beta,D,0) \qquad \text{and} \qquad q_D(-\beta,0) = -q_D(\beta,0). \tag{5.8}$$

In the grand-canonical ensemble, one easily finds that

$$-\frac{\partial}{\partial h}q_D(\beta,h)\Big|_\beta = \frac{\sum_{\omega\in\Omega}\frac{M(\omega)}{|D|}\,\mathrm{e}^{-\beta[H(\omega)-hM(\omega)]}}{\sum_{\omega\in\Omega}\mathrm{e}^{-\beta[H(\omega)-hM(\omega)]}} \equiv \Bigl\langle\frac{M(\omega)}{|D|}\Bigr\rangle. \tag{5.9}$$

It is therefore natural to understand the derivative of q with respect to h as the magnetization per site. Rather than the finite volume expression above, we would like to understand the properties of

$$m(\beta,h) = -\frac{\partial q}{\partial h}\Big|_\beta \tag{5.10}$$

Notice that q is not necessarily differentiable; $m(\beta,h)$ may be discontinuous.

3. One dimension — Exact computation

We can derive a close-form expression for the grand-canonical potential using the method of "transfer matrices". Here, we take for D the set $\{1,2,\dots,L\}$ with periodic boundary conditions. With $\omega,\omega' = \pm1$, we define

$$A_{\omega,\omega'} = \exp\Bigl\{\beta\omega\omega' + \beta h\tfrac{\omega+\omega'}{2}\Bigr\}.$$

It is convenient to view A as a 2×2 matrix, namely

$$A = \begin{pmatrix} e^{\beta+\beta h} & e^{-\beta} \\ e^{-\beta} & e^{\beta-\beta h} \end{pmatrix}.$$

For any configuration $\omega \in \{-1,1\}^L$, we then have

$$e^{-\beta[H(\omega)-hM(\omega)]} = A_{\omega(1),\omega(2)} A_{\omega(2),\omega(3)} \dots A_{\omega(L-1),\omega(L)} A_{\omega(L),\omega(1)}.$$

Summing over all $\omega(2), \dots, \omega(L)$ yields $(A^L)_{\omega(1),\omega(1)}$. If we now sum over $\omega(1)$, we get the trace of A^L, so that

$$Z(\beta, D, h) = \operatorname{Tr} A^L.$$

One easily finds the eigenvalues of A,

$$\lambda_\pm = e^\beta \cosh(\beta h) \pm \sqrt{e^{2\beta} \sinh^2(\beta h) + e^{-2\beta}}.$$

As $L \to \infty$, $\frac{1}{L} \log \operatorname{Tr} A^L$ converges to the logarithm of the largest eigenvalue λ_+ (Why?). We thus find an expression for the thermodynamic potential of the one-dimensional Ising model, namely

$$q(\beta, h) = -\tfrac{1}{\beta} \log\Bigl\{ e^\beta \cosh(\beta h) + \sqrt{e^{2\beta} \sinh^2(\beta h) + e^{-2\beta}} \Bigr\}.$$

The magnetization can be found by differentiating with respect to h, and we get

$$m(\beta, h) = \frac{\sinh(\beta h)}{\sqrt{\sinh^2(\beta h) + e^{-4\beta}}}.$$

Both q and m are smooth functions of β and h. This shows the absence of phase transition in one dimension. This was found by Ising, who then argued that no phase transition should ever occur in higher dimensions. As we will see, Ising was completely wrong!

4. Two dimensions — A phase transition!

Contrary to the conclusions drawn by Ising in his PhD thesis, the two-dimensional model does exhibit a phase transition. Indeed, Peierls showed in 1936 that the magnetization is discontinuous as a function of the magnetic field, provided that the temperature is small enough. This is a fundamental result, but it failed to be appreciated at the time. It reached mathematical rigor in the 60's thanks to Griffiths and Dobrushin. Onsager suceeded in 1944 in deriving a close-form expression for $q(\beta, 0)$; it is non-analytic at $\beta_c = \frac{1}{2} \log(1 + \sqrt{2})$. Later, Yang obtained the spontaneous magnetization, and found in particular that the critical exponent β is equal to $1/8$.

Here we state and prove a theorem based on Peierls ideas. It actually holds for all dimensions greater or equal to 2. It states that $q(\beta, h)$ is not differentiable at $h = 0$, provided β is large enough.

Theorem I. *Let $d = 2$. There exists $\beta_0 < \infty$ such that if $\beta > \beta_0$, the magnetization $m(\beta, h)$ has a jump at $h = 0$:*

$$D_{-h}\, q(\beta, 0) > 0 > D_{+h}\, q(\beta, 0).$$

The proof of this theorem involves three results, stated below as lemmas.

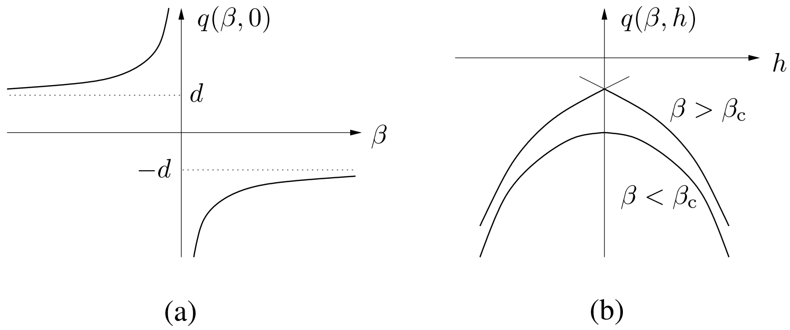

FIGURE 5.2. (a) $q(\beta, 0)$ is odd with respect to β. (b) $q(\beta, h)$ is smooth with respect to h when β is small, but it exhibits cusp for β large.

- $q(\beta, h)$ is equal to the thermodynamic limit of a finite volume expression involving "+ boundary conditions", that we denote $q_{D,+}(\beta, h)$.
- Derivatives of $q(\beta, h)$ can be estimated using derivatives of $q_{D,+}(\beta, h)$.
- The derivative of $q_{D,+}(\beta, h)$ involves the expectation of a spin at a given location. We use *Peierls argument* to find a suitable lower bound.

Let us introduce the Hamiltonian H_+ with + boundary conditions. In addition to summing over all nearest neighbors inside D, we add interactions between spins inside D with their nearest neighbors outside D, where the spins are taken to be $+1$. Mathematically, the definition of H_+ is

$$H_+(\omega) = -\sum_{\substack{\{x,y\}\subset D \\ |x-y|=1}} \omega(x)\omega(y) - \sum_{\substack{x\in D, y\in D^c \\ |x-y|=1}} \omega(x).$$

The spin flip symmetry has been broken, and the only ground state configuration is $\omega(x) \equiv 1$. We can define $Z_+(\beta, D, h)$ to be as $Z(\beta, D, h)$ in (5.3), but with H_+ instead of H. Next we define the finite volume thermodynamic potential $q_{D,+}$ by

$$q_{D,+}(\beta, h) = -\frac{1}{\beta|D|} \log Z_+(\beta, D, h).$$

LEMMA 5.2. *As $D \nearrow \mathbb{Z}^d$, $q_{D,+}(\beta, h)$ converges to $q(\beta, h)$.*

PROOF. We prove it for general dimension d. Hamiltonians H and H_+ differ just by a boundary term, so that for any configuration ω,

$$H(\omega) - 2d|D|^{\frac{d-1}{d}} \leqslant H_+(\omega) \leqslant H(\omega) + 2d|D|^{\frac{d-1}{d}}.$$

Using this inequality in the partition functions Z and Z_+, we obtain

$$Z(\beta, D, h)\, e^{2d\beta|D|^{\frac{d-1}{d}}} \geqslant Z_+(\beta, D, h) \geqslant Z(\beta, D, h)\, e^{-2d\beta|D|^{\frac{d-1}{d}}}.$$

Then

$$-\frac{1}{\beta|D|} \log Z(\beta, D, h) - 2d|D|^{-\frac{1}{d}} \leqslant q_{D,+}(\beta, h) \leqslant -\frac{1}{\beta|D|} \log Z(\beta, D, h) + 2d|D|^{-\frac{1}{d}}.$$

Both the left and the right sides converge to $q(\beta, h)$. So does the middle side, too. ∎

The next lemma about sequences of concave functions allows to compare q and $q_{D,+}(\beta,h)$.

LEMMA 5.3. *Suppose that f_n are concave, C^1 functions $\mathbb{R} \to \mathbb{R} \cup \{-\infty\}$, that converge pointwise to $f : \mathbb{R} \to \mathbb{R} \cup \{-\infty\}$. Then f is concave, and*

$$\liminf_{n\to\infty} f_n'(x) \geqslant \tfrac{f(x+h)-f(x)}{h}$$

for any $h > 0$

PROOF. Concavity of f is trivial. Since f_n is concave, we have

$$f_n'(x) = \sup_{h>0} \tfrac{f_n(x+h)-f_n(x)}{h}.$$

Recall the general property

$$\liminf_i \bigl(\sup_j a_{ij}\bigr) \geqslant \sup_j \bigl(\liminf_i a_{ij}\bigr).$$

As a consequence,

$$\liminf_{n\to\infty} f_n'(x) \geqslant \sup_{h>0} \tfrac{f(x+h)-f(x)}{h} \geqslant \tfrac{f(x+h)-f(x)}{h}$$

for any $h > 0$. □

We use the lemma with $f = q$ and $f_n = q_{D,+}$. The derivative of $q_{D,+}$ with respect to h is equal to

$$\frac{\partial}{\partial h} q_{D,+}(\beta,h)\Big|_{h=0} = -\frac{1}{|D|} \sum_{x\in D} \langle \omega(x) \rangle_+,$$

where the expectation value of the "random variable" $\omega(x)$ is given by

$$\langle \omega(x) \rangle_+ = \frac{\sum_\omega \omega(x)\, \mathrm{e}^{-\beta H_+(\omega)}}{\sum_\omega \mathrm{e}^{-\beta H_+(\omega)}}.$$

Notice that $\langle \omega(x) \rangle_+$ depends on β and D, although the notation does not show it. We use now the "Peierls argument" to obtain a lower bound on $\langle \omega(x) \rangle_+$ that is uniform in D and x.

LEMMA 5.4 (Peierls argument). *Let $d = 2$. For any D and any $x \in D$, we have the lower bound*

$$\langle \omega(x) \rangle_+ \geqslant 1 - \frac{1}{9} \sum_{n=4,6,8,\dots} n\, 3^n\, \mathrm{e}^{-2\beta n}.$$

The result makes sense if β is large enough, so that the bound is strictly positive.

PROOF. Following Peierls, we introduce *contours* separating regions of $+$ and $-$ spins, as shown in Fig. 5.3. A configuration ω uniquely determines a set of non-intersecting contours $\Gamma(\omega) = \{\gamma_1, \dots, \gamma_n\}$, that consist of bonds in the dual lattice. Conversely, a set of contours determines a configuration. The Hamiltonian $H_+(\omega)$ can be expressed in terms of contours as

$$\begin{aligned} H_+(\omega) &\sum_{\substack{\{x,y\}\\ |x-y|=1}} \Bigl[\bigl(1 - \omega(x)\omega(y)\bigr) - 1\Bigr] \\ &= 2\sum_{i=1}^n |\gamma_i| - C_D. \end{aligned} \tag{5.11}$$

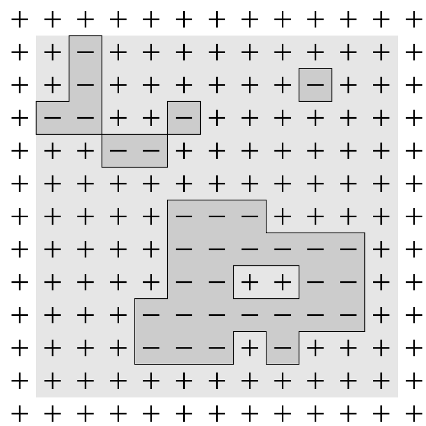

FIGURE 5.3. A configuration with + boundary conditions, and its four contours.

Here, $|\gamma|$ denotes the length of the contour γ; $C_D \approx 2|D|$ is the number of the bonds of D, including the bonds between D and D^c. It does not depend on ω.

Consider $1 - \omega(x)$. It takes value 0 if the number of contours enclosing x is even, and value 2 if this number is odd. Its expectation value is then bounded by

$$\langle 1 - \omega(x) \rangle_+ < 2 \frac{\sum_{\omega : \exists \gamma \circlearrowleft x} \mathrm{e}^{-\beta H_+(\omega)}}{\sum_\omega \mathrm{e}^{-\beta H_+(\omega)}} < 2 \sum_{\gamma \circlearrowleft x} \frac{\sum_{\omega : \Gamma(\omega) \ni \gamma} \mathrm{e}^{-\beta H_+(\omega)}}{\sum_\omega \mathrm{e}^{-\beta H_+(\omega)}}.$$

The last inequality is strict because of configurations with n contours enclosing x, that are counted n times in the right side.

Given a contour γ and a configuration ω with $\Gamma(\omega) \ni \gamma$, we define the configuration $\bar{\omega}$ where all spins inside γ have been flipped. This operation erases the contour γ, i.e. $\Gamma(\bar{\omega}) = \Gamma(\omega) \setminus \{\gamma\}$, and

$$H_+(\omega) = 2|\gamma| + H_+(\bar{\omega}).$$

Let Ω' be the set of configurations of the type $\bar{\omega}$; precisely,

$$\Omega' = \{\omega \in \Omega : \omega(x) = \omega(y) \ \forall \{x, y\} \in \gamma\}.$$

Here, the notation $\{x, y\} \in \gamma$ means that the nearest neighbors x, y lie on opposite sides of γ. Because $\Omega' \subset \Omega$, we obtain

$$\langle 1 - \omega(x) \rangle_+ < 2 \sum_{\gamma \circlearrowleft x} \mathrm{e}^{-2\beta|\gamma|} \frac{\sum_{\omega \in \Omega'} \mathrm{e}^{-\beta H_+(\omega)}}{\sum_{\omega \in \Omega} \mathrm{e}^{-\beta H_+(\omega)}} < 2 \sum_{\gamma \circlearrowleft x} \mathrm{e}^{-2\beta|\gamma|}.$$

The sum over contours enclosing a given site can be estimated as follows. We sum over its length $n = 4, 6, 8, \dots$. The contour necessarily contains one of the $\frac{n}{2} - 1$ vertical bonds situated to the right of x. The number of contours containing a given bond is less than 3^{n-2}. We then obtain the expression stated in the lemma. $\square$

The proof of Theorem I then follows: By Lemmas 5.3 and 5.4, we have

$$\frac{q(h) - q(0)}{h} \leqslant \liminf_{D \nearrow \mathbb{Z}^2} -\frac{1}{|D|} \sum_{x \in D} \langle \omega(x) \rangle_+ \leqslant -1 + \frac{1}{9} \sum_{n = 4,6,8,\dots} n 3^n \mathrm{e}^{-2\beta n}.$$

For β large, the right side is strictly negative.

5. Duality at two dimensions

Kramers and Wannier found in 1941 a kind of symmetry that relates high and low temperatures. It holds in the two-dimensional model in absence of external magnetic field. It should not be confused with the genuine symmetry $q(\beta,0) = -q(-\beta,0)$ that involve negative temperatures. Here, given $\beta \in (0,\infty)$, we introduce the dual inverse temperature β^*,

$$\beta^* = -\tfrac12 \log\tanh\beta \qquad \Longleftrightarrow \qquad \sinh(2\beta)\sinh(2\beta^*) = 1. \tag{5.12}$$

The latter relation shows that the symmetry of the definition, and shows also that β^* is small when β is large, and conversely.

PROPOSITION 5.5 (Kramers-Wannier duality). *Let $d = 2$, and β and β^* be related as above. Then*

$$\beta q(\beta,0) + 2\log\cosh\beta + \log 2 = \beta^* q(\beta^*,0) + 2\beta^*.$$

Among the consequence of this duality, we get exact value of the critical temperature, *assuming* there is exactly one critical temperature — a fact that follows from Onsager's exact solution. If $q(\beta,0)$ fails to be analytic at exactly one value of β, then we must have $\beta_{\rm c} = \beta_{\rm c}^*$. It then easily follows from (5.12) that $\beta_{\rm c} = \frac12\log(1+\sqrt2)$.

PROOF. The proof is based on a high temperature expansion of the partition function, that turns out to be identical to the low temperature expansion in terms of Peierls contours. We learned from the Peierls argument that the partition function can be written as a sum over sets Γ of closed contours in the dual lattice:

$$Z_+(\beta,D,0) = \mathrm{e}^{\beta C_D} \sum_\Gamma \mathrm{e}^{-2\beta|\Gamma|}. \tag{5.13}$$

This can be viewed as an expansion in terms of the parameter $\mathrm{e}^{-2\beta}$, which is small when the temperature is low.

Let $\mathcal{B}(D)$ denote the set of "bonds" in D, i.e. the set of pairs of nearest neighbours $\{x,y\} \subset D$ with $|x-y| = 1$. The high temperature expansion that we describe now starts with the identity $\mathrm{e}^{\pm\beta} = \cosh\beta \pm \sinh\beta$. Then, since $\omega(x)\omega(y) = \pm1$,

$$\begin{aligned} Z(\beta,D,0) &= \sum_{\omega\in\Omega} \prod_{\{x,y\}\in\mathcal{B}(D)} \underbrace{\mathrm{e}^{\beta\omega(x)\omega(y)}}_{\cosh\beta + \omega(x)\omega(y)\sinh\beta} \\ &= (\cosh\beta)^{|\mathcal{B}(D)|} \sum_{\omega\in\Omega} \sum_{B\subset\mathcal{B}(D)} (\tanh\beta)^{|B|} \prod_{\{x,y\}\in B} \omega(x)\omega(y). \end{aligned} \tag{5.14}$$

Given $B \subset \mathcal{B}(D)$, the expression above involves

$$\sum_{\omega\in\Omega} \prod_{\{x,y\}\in B} \omega(x)\omega(y) = \sum_{\omega\in\Omega} \prod_{x\in D} [\omega(x)]^{b_x(B)}, \tag{5.15}$$

where $b_x(B)$ is the number of bonds in B containing x. This term is zero if there is an $x \in D$ with $b_x(B) = 1,3$; it is equal to $2^{|D|}$ if $b_x(B) = 0,2,4$ for each $x \in D$. This puts a restriction on B, namely that B needs to be a set of closed contours on D. One obtains an expression that closely resembles (5.13), but with $\tanh\beta$ instead of $\mathrm{e}^{-2\beta}$. We therefore define β^* such that $\mathrm{e}^{-2\beta^*} = \tanh\beta$ (this relation is

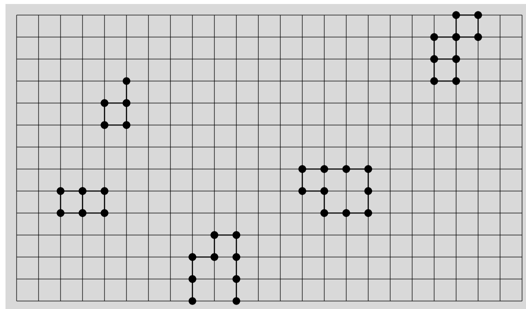

FIGURE 5.4. A set B of bonds yields a set of closed and open contours. Only closed contours remain after summing over spin configurations. The leftmost three contours are open (one seems closed, but it involves two sites with three bonds), the rightmost two are closed.

equivalent to (5.12)). We then obtain

$$\begin{aligned} Z(\beta, D, 0) &= (\cosh\beta)^{|\mathcal{B}(D)|} 2^{|D|} \sum_{B \,:\, b_x(B) \text{ is even } \forall x} \mathrm{e}^{-2\beta^*|B|} \\ &= (\cosh\beta)^{|\mathcal{B}(D)|} 2^{|D|} \, \mathrm{e}^{-\beta^* C_{D'}} \, Z_+(\beta^*, D', 0). \end{aligned} \tag{5.16}$$

We used (5.13). Here, the domain D' is a cube of length $L' = L - 1$. We now take the logarithm of both sides, divide by $|D|$, and send D to $\mathbb{Z}^d$. Using

$$\tfrac{|D'|}{|D|} \to 1, \qquad \tfrac{C_D}{|D|} \to 2, \tag{5.17}$$

we obtain the duality as stated in the proposition. □

Exercise 5.1. Ising model in the canonical ensemble. Define the partition function

$$Y(\beta, D, M) = \sum_{\omega \in \Omega : M(\omega) = M} \mathrm{e}^{-\beta H(\omega)} .$$

Let $f_D(\beta, m) = -\frac{1}{\beta|D|} \log Y(\beta, D, |D|m)$, where we suppose that $|D|m$ is of the form $-|D| + 2k$. Show that, with $\beta = 0$,

$$\lim_{D \nearrow \mathbb{Z}^d} f_D(0, m) = -\tfrac{1+m}{2} \log \tfrac{1+m}{2} - \tfrac{1-m}{2} \log \tfrac{1-m}{2}.$$

Exercise 5.2. Use cluster expansions to prove that

$$\lim_{\beta \to \infty} q(\beta, h) = -d - |h|.$$

Idea: Let $h > 0$; the sum over configurations can be viewed as a sum over subsets of D, that represent the positions of the $-$ spins. Decomposing these sets into connected subsets, the partition function becomes

$$Z(\beta, D, h) = \mathrm{e}^{\beta|D|(d+h)} \sum_{k \geqslant 0} \frac{1}{k!} \sum_{A_1, \dots, A_k} \prod_{i=1}^{k} \mathrm{e}^{-\beta(|\partial A_i| - 2h|A_i|)} ,$$

where the A_i's are connected and mutually disjoint: $\text{dist}(A_i, A_j) \geqslant 1$ if $i \neq j$. Here, $|\partial A|$ is the number of nearest neighbours $\{x, y\}$ with $x \in A$ and $y \notin A$. This form of the partition function allows to apply the theorem on cluster expansions.

Exercise 5.3. Same exercise as above, but in the limit $\beta \to -\infty$. Show that

$$\lim_{\beta \to -\infty} q(\beta, h) = \begin{cases} -d + |h| & \text{if } |h| \geqslant 2d \\ d & \text{if } -2d \leqslant h \leqslant 2d. \end{cases}$$

Exercise 5.4. Find the behaviour of $q(\beta, h)$ for small β (high temperature), using an analogous expansion to that of the classical gas.

Exercise 5.5. The Blume-Capel model describes "spin 1" particles. The state space for the one-dimensional model is $\Omega = \{-1, 0, 1\}^L$, and its Hamiltonian is

$$H(\omega) = -\sum_{i=1}^{L} \omega(i)\omega(i+1).$$

(We use periodic boundary conditions, where ω_{L+1} is identified with $\omega(i)$.) Compute the grand-canonical free energy using the transfer matrix method.

Exercise 5.6. Suppose that the convex function f is even, i.e. $f(-x) = f(x)$. Is the Legendre transform also even?