CHAPTER 8

Spin systems

This chapter will contain the XYZ model. Their symmetries. Correlation inequalities. Ward identities.

8.1. Spin operators

Let $S \in \frac{1}{2}\mathbb{N}$. On $\mathbb{C}^{2S+1}$, let $S^{(1)}, S^{(2)}, S^{(3)}$ be hermitian matrices that satisfy the following properties:

$$[S^{(1)}, S^{(2)}] = \mathrm{i}S^{(3)}, \quad [S^{(2)}, S^{(3)}] = \mathrm{i}S^{(1)}, \quad [S^{(3)}, S^{(1)}] = \mathrm{i}S^{(2)}, \tag{8.1}$$

$$[S^{(1)}]^{(2)} + [S^{(2)}]^{(2)} + [S^{(3)}]^{(2)} = S(S+1)\mathtt{Id}. \tag{8.2}$$

The existence of such matrices follows by construction: Let $|a\rangle$, $a \in \{-S, -S+1, \dots, S\}$ denote an orthonormal basis of $\mathbb{C}^{2S+1}$, and define $S^{(3)}|a\rangle = a|a\rangle$. Next, let $S^{(+)}, S^{(-)}$ be defined by

$$S^{(+)}|a\rangle = \sqrt{S(S+1) - a(a+1)}\;|a{+}1\rangle, \quad S^{(-)}|a\rangle = \sqrt{S(S+1) - (a-1)a}\;|a{-}1\rangle. \tag{8.3}$$

Then we set $S^{(1)} = \frac{1}{2}(S^{(+)} + S^{(-)})$ and $S^{(2)} = \frac{1}{2\mathrm{i}}(S^{(+)} - S^{(-)})$.

Lemma 8.1. *The operators $S^{(1)}, S^{(2)}, S^{(3)}$ constructed above satisfy the relations (8.1) and (8.2).*

Proof. One can check the following commutation relations:

$$[S^{(3)}, S^{(+)}] = S^{(+)}, \qquad [S^{(3)}, S^{(-)}] = -S^{(-)}, \qquad [S^{(+)}, S^{(-)}] = 2S^{(3)}. \tag{8.4}$$

The relations (8.1) follow. Finally,

$$[S^{(1)}]^2 + [S^{(2)}]^2 + [S^{(3)}]^2 = S^{(+)}S^{(-)} + [S^{(3)}]^2 - S^{(3)} = S(S+1)\mathtt{Id}. \tag{8.5}$$

□

For $S = \frac{1}{2}$, the choice above gives the Pauli matrices (multiplied by $\frac{1}{2}$):

$$S^{(1)} = \frac{1}{2}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \qquad S^{(2)} = \frac{1}{2}\begin{pmatrix} 0 & -\mathrm{i} \\ \mathrm{i} & 0 \end{pmatrix}, \qquad S^{(3)} = \frac{1}{2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \tag{8.6}$$

For $S = 1$, we get

$$S^{(1)} = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}, \quad S^{(2)} = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 & -\mathrm{i} & 0 \\ \mathrm{i} & 0 & -\mathrm{i} \\ 0 & \mathrm{i} & 0 \end{pmatrix}, \quad S^{(3)} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix}. \tag{8.7}$$

Notice that, for $S > 1$, the matrix of $S^{(1)}$ is not proportional to $\delta_{|i-j|,1}$. Spin operators are not unique, but their spectrum is uniquely determined by the commutation relations.

LEMMA 8.2. *Assume that $S^{(1)}, S^{(2)}, S^{(3)}$ are hermitian matrices in $\mathbb{C}^{2S+1}$ that satisfy the relations (8.1) and (8.2). Then each $S^{(i)}$ has eigenvalues $\{-S, -S+1, \dots, S\}$.*

PROOF. It is enough to prove the claim for $S^{(3)}$. Define $S^{(+)} = S^{(1)} + \mathrm{i}S^{(2)}$ and $S^{(-)} = S^{(1)} - \mathrm{i}S^{(2)}$. One can check that

$$\begin{aligned} S^{(+)}S^{(-)} &= S(S+1)\mathtt{Id} - [S^{(3)}]^2 + S^{(3)}, \\ S^{(-)}S^{(+)} &= S(S+1)\mathtt{Id} - [S^{(3)}]^2 - S^{(3)}. \end{aligned} \tag{8.8}$$

Let $|a\rangle$ be an eigenvector of $S^{(3)}$ with eigenvalue a. It follows from Eq. (8.8) that

$$\begin{aligned} \big\|S^{(+)}|a\rangle\big\|^2 &= \langle a|S^{(-)}S^{(+)}|a\rangle = S(S+1) - a^2 - a \geq 0, \\ \big\|S^{(-)}|a\rangle\big\|^2 &= \langle a|S^{(+)}S^{(-)}|a\rangle = S(S+1) - a^2 + a \geq 0. \end{aligned} \tag{8.9}$$

Then $|a| \leq S$, and $S^{(+)}|a\rangle \neq 0$ if $a \neq S$. Next, observe that $[S^{(3)}, S^{(+)}] = S^{(+)}$. Then

$$S^{(3)}S^{(+)}|a\rangle = (a+1)S^{(+)}|a\rangle. \tag{8.10}$$

Then if $a \neq S$ is an eigenvalue, $a+1$ is also an eigenvalue. There are similar relations with $S^{(-)}$, so that if $a \neq -S$ is an eigenvalue, $a-1$ is also an eigenvalue. It follows that $\{-S, -S+1, \dots S\}$ is the set of eigenvalues. □

Notice that the relations (8.3) always hold; this follows from (8.10) and (8.9). It follows from the parallelogram identity that $\|S^{\pm}\| = \sqrt{2}S$:

$$\begin{aligned} \|S^{(+)}\|^2 &= \tfrac14 (2\|S^{(+)}\|^2 + 2\|S^{(-)}\|^2) = \tfrac14 (\|S^{(+)} + S^{(-)}\|^2 + \|S^{(+)} - S^{(-)}\|^2) \\ &= \tfrac14 (4\|S^{(1)}\|^2 + 4\|S^{(2)}\|^2) = 2S^{(2)}. \end{aligned} \tag{8.11}$$

Spin operators are related to rotations in $\mathbb{R}^{(3)}$. Let $\vec{S} = (S^{(1)}, S^{(2)}, S^{(3)})$. Given $\vec{a} \in \mathbb{R}^{(3)}$, let

$$S^{\vec{a}} = \vec{a} \cdot \vec{S} = a_1 S^{(1)} + a_2 S^{(2)} + a_3 S^{(3)}. \tag{8.12}$$

By linearity, the commutation relations (8.1) generalize as

$$[S^{\vec{a}}, S^{\vec{b}}] = \mathrm{i}S^{\vec{a}\times\vec{b}}. \tag{8.13}$$

Finally, let $R_{\vec{a}}\vec{b}$ denote the vector $\vec{b}$ rotated around $\vec{a}$ by the angle $\|\vec{a}\|$.

LEMMA 8.3.
$$e^{-iS^{\vec a}}\,S^{\vec b}\,e^{iS^{\vec a}}=S^{R_{\vec a}\vec b}.$$

PROOF. We replace $\vec a$ by $s\vec a$, and we check that both sides of the identity satisfy the same differential equation. We find

$$\frac{d}{ds}\,e^{-iS^{s\vec a}}\,S^{\vec b}\,e^{iS^{s\vec a}}=-i[S^{\vec a},\,e^{-iS^{s\vec a}}\,S^{\vec b}\,e^{iS^{s\vec a}}\,], \tag{8.14}$$

and

$$\frac{d}{ds}S^{R_{s\vec a}\vec b}=\Big(\frac{d}{ds}R_{s\vec a}\vec b\Big)\cdot\vec S=\Big(\vec a\times R_{s\vec a}\vec b\Big)\cdot\vec S=-i[S^{\vec a},S^{R_{s\vec a}\vec b}]. \tag{8.15}$$

We used (8.13) for the last identity. □

It also follows from Lemmas 8.2 and 8.3 that any matrix $S^{\vec a}$, $\vec a\in\mathbb{R}^{(3)}$ with $\|\vec a\|=1$, has eigenvalues $\{-S,-S+1,\dots,S\}$.

COROLLARY 8.4. *Let $\psi_{\vec b,c}$ be the eigenvector of $S^{\vec b}$ with eigenvalue c. Then $e^{-iS^{\vec a}}\,\psi_{\vec b,c}$ is eigenvector of $S^{R_{\vec a}\vec b}$ with eigenvalue c.*

PROOF. Using Lemma 8.3,

$$S^{R_{\vec a}\vec b}\,e^{-iS^{\vec a}}\,\psi_{\vec b,c}=\;e^{-iS^{\vec a}}\,S^{\vec b}\psi_{\vec b,c}=c\,e^{-iS^{\vec a}}\,\psi_{\vec b,c}. \tag{8.16}$$

□

Finally, let us note the following useful relations:

$$\begin{aligned} e^{-iaS^{(3)}}\,S^{(+)}\,e^{iaS^{(3)}} &= e^{-ia}\,S^{(+)},\\ e^{-iaS^{(3)}}\,S^{(-)}\,e^{iaS^{(3)}} &= e^{ia}\,S^{(-)}. \end{aligned} \tag{8.17}$$

8.2. Hamiltonian and Gibbs states

We consider this rather general "XYZ" hamiltonian, with two-body interactions in each spin directions, in the finite domain $\Lambda\Subset\mathbb{Z}^d$. The coupling constants are assumed to be symmetric, that is, $J^{(i)}_{xy}=J^{(i)}_{yx}$ for all $x,y\in\Lambda$ and all $i=1,2,3$.

$$H_{\Lambda,h}=-\tfrac12\sum_{x,y\in\Lambda}\Big(J^{(1)}_{xy}S^{(1)}_xS^{(1)}_y+J^{(2)}_{xy}S^{(2)}_xS^{(2)}_y+J^{(3)}_{xy}S^{(3)}_xS^{(3)}_y\Big)-h\sum_{x\in\Lambda}S^{(3)}_x. \tag{8.18}$$

With $Z_{\Lambda,h}=\operatorname{Tr}\,e^{-\beta H_{\Lambda,h}}$ denoting the partition function, the correlation functions at inverse temperature β are given by

$$\langle S^{(i)}_0S^{(i)}_x\rangle_{\Lambda,h}=\frac{1}{Z_{\Lambda,h}}\operatorname{Tr}\,(S^{(i)}_0S^{(i)}_x\,e^{-\beta H_{\Lambda,h}}\,). \tag{8.19}$$

The case $J^{(1)}_{xy}=J^{(2)}_{xy}=0$, for all $x,y\in\Lambda$, corresponds to the Ising model, which is in fact a classical model. The case $J^{(3)}_{xy}=0$ and $J^{(1)}_{xy}=J^{(2)}_{xy}$, for all x,y, corresponds to the quantum XY model. And the symmetric case, $J^{(1)}_{xy}=$

$J^{(2)}_{xy} = J^{(3)}_{xy}$, corresponds to the isotropic Heisenberg model. Positive values of the couplings correspond to ferromagnetic order, while negative values of the couplings correspond to antiferromagnetism.

8.3. Correlation inequalities - arbitrary spins

THEOREM 8.5. *Assume that, for all $x, y \in \Lambda$, the couplings satisfy*

$$|J^{(2)}_{xy}| \leq J^{(1)}_{xy}.$$

Then we have that

$$\big|\langle S^{(2)}_0 S^{(2)}_x \rangle_{\Lambda,h}\big| \leq \langle S^{(1)}_0 S^{(1)}_x \rangle_{\Lambda,h},$$

for all $x \in \Lambda$. More generally, for all $x_1, \dots x_k \in \Lambda$ and $j_1, \dots, j_k \in \{1, 2\}$,

$$\big|\langle S^{(j_1)}_{x_1} \dots S^{(j_k)}_{x_k} \rangle_{\Lambda,h}\big| \leq \langle S^{(1)}_{x_1} \dots S^{(1)}_{x_k} \rangle_{\Lambda,h}.$$

Further inequalities can be generated using symmetries. Some inequalities hold for the staggered two-point function $(-1)^{|x|}\langle S^{(i)}_0 S^{(i)}_x \rangle_{\Lambda,h}$.

PROOF. Let $S \in \frac{1}{2}\mathbb{N}$ such that $2S + 1 = N$, and let $|a\rangle$, $a \in \{-S, \dots, S\}$ denote basis elements of $\mathbb{C}^{2S+1}$. Let the operators $S^{(\pm)}$ be defined by

$$S^{(+)}|a\rangle = \sqrt{S(S+1) - a(a+1)}\;|a{+}1\rangle, \quad S^{(-)}|a\rangle = \sqrt{S(S+1) - (a-1)a}\;|a{-}1\rangle, \tag{8.20}$$

with the understanding that $S^{(+)}|S\rangle = S^{(-)}|-S\rangle = 0$. Then let $S^{(1)} = \frac{1}{2}(S^{(+)} + S^{(-)})$, $S^{(2)} = \frac{1}{2\mathrm{i}}(S^{(+)} - S^{(-)})$, and $S^{(3)}|a\rangle = a|a\rangle$. It is well-known that these operators satisfy the spin commutation relations. Further, the matrix elements of $S^{(1)}, S^{(\pm)}$ are all nonnegative, and the matrix elements of $S^{(2)}$ are all less than or equal to those of $S^{(1)}$ in absolute values. Using the Trotter formula and multiple resolutions of the identity, we have

$$\begin{aligned}
\big|\operatorname{Tr} S^{(2)}_0 S^{(2)}_x \,\mathrm{e}^{-\beta H_{\Lambda,h}}\big| \leq \lim_{N\to\infty} \sum_{\sigma_0,\dots,\sigma_N \in \{-S,\dots,S\}^\Lambda} \Big| &\langle\sigma_0| S^{(2)}_0 S^{(2)}_x |\sigma_1\rangle \\
&\langle\sigma_1| \,\mathrm{e}^{\frac{\beta}{N}\sum J^{(3)}_{yz} S^{(3)}_y S^{(3)}_z + \frac{\beta h}{N}\sum S^{(3)}_x} |\sigma_1\rangle\langle\sigma_1| \Big(1 + \frac{\beta}{N}\sum_{y,z\in\Lambda} (J^{(1)}_{yz} S^{(1)}_y S^{(1)}_z + J^{(2)}_{yz} S^{(2)}_y S^{(2)}_z)\Big)|\sigma_2\rangle \\
&\dots\langle\sigma_N| \,\mathrm{e}^{\frac{\beta}{N}\sum J^{(3)}_{yz} S^{(3)}_y S^{(3)}_z + \frac{\beta h}{N}\sum S^{(3)}_x} |\sigma_N\rangle\langle\sigma_N| \Big(1 + \frac{\beta}{N}\sum_{y,z\in\Lambda} (J^{(1)}_{yz} S^{(1)}_y S^{(1)}_z + J^{(2)}_{yz} S^{(2)}_y S^{(2)}_z)\Big)|\sigma_0\rangle \Big|.
\end{aligned} \tag{8.21}$$

Observe that the matrix elements of all operators are nonnegative, except for $S^{(2)}_0 S^{(2)}_x$. Indeed, this follows from

$$\begin{aligned}
J^{(1)}_{yz} S^{(1)}_y S^{(1)}_z + J^{(2)}_{yz} S^{(2)}_y S^{(2)}_z =& \tfrac{1}{4}(J^{(1)}_{yz} - J^{(2)}_{yz})(S^{(+)}_y S^{(+)}_z + S^{(-)}_y S^{(-)}_z) \\
&+ \tfrac{1}{4}(J^{(1)}_{yz} + J^{(2)}_{yz})(S^{(+)}_y S^{(-)}_z + S^{(-)}_y S^{(+)}_z).
\end{aligned} \tag{8.22}$$

We get an upper bound for the right side of (8.21) by replacing $|\langle\sigma_0|S_0^{(2)}S_x^{(2)}|\sigma_1\rangle|$ with $\langle\sigma_0|S_0^{(1)}S_x^{(1)}|\sigma_1\rangle$. We have obtained

$$\Big|\operatorname{Tr} S_0^{(2)}S_x^{(2)}\,\mathrm{e}^{-\beta H_{\Lambda,h}}\Big| \leq \operatorname{Tr} S_0^{(1)}S_x^{(1)}\,\mathrm{e}^{-\beta H_{\Lambda,h}}\,, \tag{8.23}$$

which proves the first claim. The second claim can be proved exactly the same way. □

Corollary 8.6. *Assume that for all $x, y \in \Lambda$, the couplings satisfy*

$$J_{xy}^{(1)} = J_{xy}^{(2)} \geq 0.$$

Then we have for all $x, y, z, u \in \Lambda$

$$\frac{\partial}{\partial J_{xy}^{(1)}}\langle S_z^{(2)}S_u^{(2)}\rangle_{\Lambda,h} \leq \frac{\partial}{\partial J_{xy}^{(1)}}\langle S_z^{(1)}S_u^{(1)}\rangle_{\Lambda,h}.$$

Proof. For $i = 1, 2, 3$, we have

$$\frac{1}{\beta}\frac{\partial}{\partial J_{xy}^{(1)}}\langle S_z^{(i)}S_u^{(i)}\rangle_{\Lambda,h} = (S_x^{(1)}S_y^{(1)}, S_z^{(i)}S_u^{(i)}) - \langle S_x^{(1)}S_y^{(1)}\rangle_{\Lambda,h}\langle S_z^{(i)}S_u^{(i)}\rangle_{\Lambda,h}, \tag{8.24}$$

where (A, B) denotes the Duhamel two-point function,

$$(A, B) = \frac{1}{Z_{\Lambda,h}}\int_0^1 \operatorname{Tr} A\,\mathrm{e}^{-s\beta H_{\Lambda,h}}\,B\,\mathrm{e}^{-(1-s)\beta H_{\Lambda,h}}\,\mathrm{d}s. \tag{8.25}$$

It is not hard to extend the proof of Theorem 8.5 to the Duhamel function, so that

$$\big|(S_x^{(1)}S_y^{(1)}, S_z^{(2)}S_u^{(2)})\big| \leq (S_x^{(1)}S_y^{(1)}, S_z^{(1)}S_u^{(1)}). \tag{8.26}$$

Further, we have $\langle S_z^{(2)}S_u^{(2)}\rangle_{\Lambda,h} = \langle S_z^{(1)}S_u^{(1)}\rangle_{\Lambda,h}$ by symmetry. The result follows. □

8.4. Correlation inequalities - spin $\frac{1}{2}$

We consider the hamiltonian

$$H_\Lambda = -\sum_{A\subset\Lambda}\Big(J_A^{(1)}\prod_{x\in A}S_x^{(1)} + J_A^{(2)}\prod_{x\in A}S_x^{(2)}\Big). \tag{8.27}$$

Here, $J_A^{(i)}$ is a nonnegative coupling constant for each subset of $A \subset \Lambda$ and each spin direction $i \in \{1, 2\}$. The expected value of an observable a (that is, an operator on $\mathcal{H}_\Lambda$) in the Gibbs state with hamiltonian H_Λ and at inverse temperature $\beta > 0$ is

$$\langle a\rangle = \frac{1}{Z(\Lambda)}\operatorname{Tr} a\,\mathrm{e}^{-\beta H_\Lambda}\,, \tag{8.28}$$

where the normalisation $Z(\Lambda)$ is the partition function

$$Z(\Lambda) = \operatorname{Tr}\,\mathrm{e}^{-\beta H_\Lambda}\,. \tag{8.29}$$

Traces are taken in $\mathcal{H}_\Lambda$. We also consider Schwinger functions that are defined for $s \in [0,1]$ by

$$\langle a; b\rangle_s = \frac{1}{Z(\Lambda)} \operatorname{Tr} a\, \mathrm{e}^{-s\beta H_\Lambda}\, b\, \mathrm{e}^{-(1-s)\beta H_\Lambda}\,. \tag{8.30}$$

Our first result holds for $S = \frac{1}{2}$ and all temperatures.

THEOREM 8.7. *Assume that $J_A^i \geq 0$ for all $A \subset \Lambda$ and all $i \in \{1,2\}$. Assume also that $S = \frac{1}{2}$. Then for all $A, B \subset \Lambda$, and all $s \in [0,1]$, we have*

$$\Big\langle \prod_{x\in A} S_x^{(1)}; \prod_{x\in B} S_x^{(1)} \Big\rangle_s - \Big\langle \prod_{x\in A} S_x^{(1)} \Big\rangle \Big\langle \prod_{x\in B} S_x^{(1)} \Big\rangle \geq 0;$$

$$\Big\langle \prod_{x\in A} S_x^{(1)}; \prod_{x\in B} S_x^{(2)} \Big\rangle_s - \Big\langle \prod_{x\in A} S_x^{(1)} \Big\rangle \Big\langle \prod_{x\in B} S_x^{(2)} \Big\rangle \leq 0.$$

Clearly, other inequalities can be generated using spin symmetries.

It is convenient to work with the hamiltonian with interactions in the 1-3 spin directions, namely

$$H_\Lambda = -\sum_{A\subset\Lambda} \Big(J_A^{(1)} \prod_{x\in A} S_x^{(1)} + J_A^{(3)} \prod_{x\in A} S_x^{(3)} \Big). \tag{8.31}$$

We introduce the product space $\mathcal{H}_\Lambda \otimes \mathcal{H}_\Lambda$. Given an operator a on $\mathcal{H}_\Lambda$, we consider the operators a_+ and a_- on the product space, defined by

$$a_\pm = a \otimes \mathbb{1} \pm \mathbb{1} \otimes a. \tag{8.32}$$

The Gibbs state in the product space is

$$\langle\cdot\rangle = \frac{1}{Z(\Lambda)^2} \operatorname{Tr} \cdot\, \mathrm{e}^{-H_{\Lambda,+}}\,, \tag{8.33}$$

where $H_{\Lambda,+} = H_\Lambda \otimes \mathbb{1} + \mathbb{1} \otimes H_\Lambda$. Without loss of generality, we set $\beta = 1$ in this section. We also need the Schwinger functions in the product space, namely

$$\langle \cdot;\cdot\rangle_s = \frac{1}{Z(\Lambda)^2} \operatorname{Tr} \cdot\, \mathrm{e}^{-sH_{\Lambda,+}} \cdot\, \mathrm{e}^{-(1-s)H_{\Lambda,+}}\,. \tag{8.34}$$

LEMMA 8.8. *For all observables a, b on $\mathcal{H}_\Lambda$, we have*

$$\langle ab\rangle - \langle a\rangle\langle b\rangle = \tfrac{1}{2}\langle a_- b_-\rangle,$$

$$\langle a; b\rangle_s - \langle a\rangle\langle b\rangle = \tfrac{1}{2}\langle a_-; b_-\rangle_s.$$

PROOF. It is enough to prove the second line. The right side is equal to

$$\begin{aligned}\langle a_-; b_-\rangle_s = \frac{1}{Z(\Lambda)^{(2)}} \Big[&\mathrm{Tr}\,(a\otimes 1\!\!1)\,\mathrm{e}^{-sH_{\Lambda,+}}\,(b\otimes 1\!\!1)\,\mathrm{e}^{-(1-s)H_{\Lambda,+}} \\ &+ \mathrm{Tr}\,(1\!\!1\otimes a)\,\mathrm{e}^{-sH_{\Lambda,+}}\,(1\!\!1\otimes b)\,\mathrm{e}^{-(1-s)H_{\Lambda,+}} \\ &- \mathrm{Tr}\,(1\!\!1\otimes a)\,\mathrm{e}^{-sH_{\Lambda,+}}\,(b\otimes 1\!\!1)\,\mathrm{e}^{-(1-s)H_{\Lambda,+}} \\ &- \mathrm{Tr}\,(a\otimes 1\!\!1)\,\mathrm{e}^{-sH_{\Lambda,+}}\,(1\!\!1\otimes b)\,\mathrm{e}^{-(1-s)H_{\Lambda,+}} \Big].\end{aligned} \tag{8.35}$$

The first two lines of the right side give $2\langle a;b\rangle_s$ and the last two lines give $2\langle a\rangle\langle b\rangle$. □

Next, a simple lemma with a useful formula.

LEMMA 8.9. *For all operators a, b on $\mathcal{H}_\Lambda$, we have*

$$(ab)_\pm = \tfrac{1}{2}a_+b_\pm + \tfrac{1}{2}a_-b_\mp.$$

The proof is straightforward algebra. Notice that both terms of the right side have *positive* factors. Now comes the key observation that leads to positive (and negative) correlations.

LEMMA 8.10. *There exists an orthonormal basis on $\mathbb{C}^2\otimes\mathbb{C}^2$ such that $S_+^{(1)}, S_-^{(1)}, S_+^{(3)}, -S_-^{(3)}$ have nonnegative matrix elements.*

As a consequence, there exists an orthonormal basis of $\mathcal{H}_\Lambda\otimes\mathcal{H}_\Lambda$ such that $S_{x,+}^{(1)}, S_{x,-}^{(1)}, S_{x,+}^{(3)}$, and $-S_{x,-}^{(3)}$ have nonnegative matrix elements.

PROOF OF LEMMA 8.10. For $\varepsilon_1,\varepsilon_2=\pm$, let $|\varepsilon_1,\varepsilon_2\rangle$ denote the eigenvectors of $S^{(3)}\otimes 1\!\!1$ and $1\!\!1\otimes S^{(3)}$ with respective eigenvalues $\frac{1}{2}\varepsilon_1$ and $\frac{1}{2}\varepsilon_2$. It is well-known that $S^{(1)}\otimes 1\!\!1\,|\varepsilon_1,\varepsilon_2\rangle = \frac{1}{2}|-\varepsilon_1,\varepsilon_2\rangle$ and similarly for $1\!\!1\otimes S^{(1)}$. The convenient basis in $\mathbb{C}^2\otimes\mathbb{C}^2$ consists of the following four elements:

$$\begin{aligned} p_+ &= \tfrac{1}{\sqrt{2}}\big(|++\rangle + |--\rangle\big), & q_+ &= \tfrac{1}{\sqrt{2}}\big(|-+\rangle + |+-\rangle\big), \\ p_- &= \tfrac{1}{\sqrt{2}}\big(|++\rangle - |--\rangle\big), & q_- &= \tfrac{1}{\sqrt{2}}\big(|-+\rangle - |+-\rangle\big). \end{aligned} \tag{8.36}$$

Direct calculations show that

$$\begin{aligned} \langle p_+, S_+^{(1)} q_+\rangle &= \langle q_+, S_+^{(1)} p_+\rangle = 1, \\ \langle p_-, S_-^{(1)} q_-\rangle &= \langle q_-, S_-^{(1)} p_-\rangle = 1, \\ \langle p_+, S_+^{(3)} p_-\rangle &= \langle p_-, S_+^{(3)} p_+\rangle = 1, \\ \langle q_+, S_-^{(3)} q_-\rangle &= \langle q_-, S_-^{(3)} q_+\rangle = -1. \end{aligned} \tag{8.37}$$

All other matrix elements are zero. □

PROOF OF THEOREM 8.7 FOR $S = \frac{1}{2}$. We use Lemma 8.8 in order to get

$$\Big\langle \prod_{x\in A} S_x^{(1)}; \prod_{x\in B} S_x^{(1)} \Big\rangle_s - \Big\langle \prod_{x\in A} S_x^{(1)} \Big\rangle \Big\langle \prod_{x\in B} S_x^{(1)} \Big\rangle = \tfrac{1}{2} \Big\langle\!\Big\langle \Big(\prod_{x\in A} S_x^{(1)} \Big)_-; \Big(\prod_{x\in B} S_x^{(1)} \Big)_- \Big\rangle\!\Big\rangle_s. \tag{8.38}$$

In order to make visible the sign of the right side, we expand the exponentials in Taylor series, so as to get a positive linear combination of terms of the form

$$\operatorname{Tr}_{\mathcal{H}_\Lambda \otimes \mathcal{H}_\Lambda} \Big(\prod_{x\in A} S_x^{(1)} \Big)_- (-H_{\Lambda,+})^k \Big(\prod_{x\in B} S_x^{(1)} \Big)_- (-H_{\Lambda,+})^\ell \tag{8.39}$$

with $k, \ell \in \mathbb{N}$. Expanding $(-H_{\Lambda,+})^k$ and $(-H_{\Lambda,+})^\ell$, we get a positive linear combination of

$$\operatorname{Tr}_{\mathcal{H}_\Lambda \otimes \mathcal{H}_\Lambda} \Big(\prod_{x\in A} S_x^{(1)} \Big)_- \prod_{i=1}^{k} \Big(\prod_{x\in A_i} S_x^{\varepsilon_i} \Big)_+ \Big(\prod_{x\in B} S_x^{(1)} \Big)_- \prod_{j=1}^{\ell} \Big(\prod_{x\in A'_j} S_x^{\varepsilon'_j} \Big)_+ \tag{8.40}$$

with $\varepsilon_i, \varepsilon'_j \in \{1,3\}$. Further, all products $(\prod S_x^{(i)})_\pm$ can be expanded using Lemma 8.9 in polynomials of $S_{x,\pm}^i$, still with positive coefficients. Finally, observe that the total number of operators $S_{x,-}^{(3)}$, $x \in \Lambda$, is always even; then each $S_{x,-}^{(3)}$ can be replaced by $-S_{x,-}^{(3)}$. We now have the trace of a polynomial, with positive coefficients, of matrices with nonnegative elements (by Lemma 8.10). This is positive.

The second inequality (with $S^{(3)}$ instead of $S^{(2)}$) is similar. The only difference is that $(\prod S_x^{(3)})_-$ gives a polynomial where the number of $S_{x,-}^{(3)}$ is odd. Hence the negative sign. □

EXERCISE 8.1. *For $S = 1$, check that the following matrices satisfy the spin relations.*

$$S^{(1)} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -\mathrm{i} \\ 0 & \mathrm{i} & 0 \end{pmatrix}, \qquad S^{(2)} = \begin{pmatrix} 0 & 0 & -\mathrm{i} \\ 0 & 0 & 0 \\ \mathrm{i} & 0 & 0 \end{pmatrix}, \qquad S^{(3)} = \begin{pmatrix} 0 & -\mathrm{i} & 0 \\ \mathrm{i} & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

EXERCISE 8.2. *For $S = 1$, check that the following matrices do not satisfy the spin relations.*

$$S^{(1)} = \frac{1}{2}\begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \qquad S^{(2)} = \frac{1}{2}\begin{pmatrix} 0 & -\mathrm{i} & 0 \\ \mathrm{i} & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \qquad S^{(3)} = \frac{1}{2}\begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

EXERCISE 8.3. *Let F be the linear operator for spin flips: If $|(\sigma_x)_{x\in\Lambda}\rangle$ denotes the ket associated with the classical configuration (σ_x), then $F|(\sigma_x)_{x\in\Lambda}\rangle = |(-\sigma_x)_{x\in\Lambda}\rangle$. Can F be written using spin operators?*