APPENDIX B

Solutions to some exercises

EXERCISE 2.5: An example using spin operators is $\sum_{x\in\mathbb{Z}^d} \mathrm{e}^{-|x|} S_x^{(3)}$. It does not belong to $\mathcal{A}_X$ for any $X \Subset \mathbb{Z}^d$, but it can be defined as the injective limit of $\sum_{x\in X} \mathrm{e}^{-|x|} S_x^{(3)}$.

EXERCISE 2.7: Recall that $\|\langle\cdot\rangle\| = \sup_{\|a\|=1} |\langle a\rangle|$. It is clear that $\|\langle\cdot\rangle\| \geq 1$ by choosing $a = \mathbb{1}$. Conversely, since local observables are dense in $\mathcal{A}$, it is enough to take the supremum over local observables. Let $a \in \mathcal{A}_\Lambda$, and consider the restriction of $\langle\cdot\rangle$ to $\mathcal{A}_\Lambda$. It is a state and it is represented by the density matrix ρ_Λ by Proposition 2.8. Then

$$|\langle a\rangle| = \bigl|\operatorname{Tr}\rho_\Lambda a\bigr| \leq \bigl|\operatorname{Tr}\rho_\Lambda\bigr|\,\|a\| = \|a\|.$$

The inequality is Hölder, Proposition 2.3.

EXERCISE 2.9: For (i), write $a = b + \mathrm{i}c$ where b, c are hermitian. For hermitian b we have $\langle b\rangle \in \mathbb{R}$ since $b + \|b\|\mathbb{1} \geq 0$ so that $\langle b\rangle + \|b\| \geq 0$, in particular $\in \mathbb{R}$. For (ii), let $x \in \mathbb{R}$ and expand the left-hand-side of the inequality $\langle (xa+b)^*(xa+b)\rangle \geq 0$ to deduce from non-positivity of the discriminant and part (i) that $\operatorname{Re}(\langle a^*b\rangle)^2 \leq \langle a^*a\rangle\langle b^*b\rangle$. Replacing a with $\mathrm{e}^{\mathrm{i}\theta} a$, with θ chosen to make $\mathrm{e}^{-\mathrm{i}\theta}\langle a^*b\rangle$ real, gives the result.

EXERCISE 3.1: Both bounds follow from Peierls inequality (Proposition 2.7). Use a basis of eigenvectors of a for the lower bound, and a basis of eigenvectors of $a+b$ for the upper bound.

EXERCISE 4.1: For the first equation, let $\mathcal{K} \in \mathcal{H}$ denote the kernel of ρ. Since $\rho + \rho'$ is block diagonal with respect to $\mathcal{K}$, and $\operatorname{Tr}\rho = \operatorname{Tr}\rho' = 1$, we have

$$\begin{aligned} S\bigl((1-\varepsilon)\rho + \varepsilon\rho'\bigr) &= \operatorname{Tr}_{\mathcal{K}^\perp}[(1-\varepsilon)\rho]\log[(1-\varepsilon)\rho] + \operatorname{Tr}_{\mathcal{K}}(\varepsilon\rho')\log(\varepsilon\rho') \\ &= (1-\varepsilon)\log(1-\varepsilon) + (1-\varepsilon)S(\rho) + \varepsilon\log\varepsilon + \varepsilon S(\rho'). \end{aligned}$$

The equation follows. The other two equations are straightforward; observe that ρ_0 is indeed invertible.

EXERCISE 8.3: The answer is yes. Notice that $F = F^* = F^{-1}$. In the representation given by Eq. (8.3) we have $FS_x^{(1)}F = S_x^{(1)}$, $FS_x^{(2)}F = -S_x^{(2)}$, and $FS_x^{(3)}F = -S_x^{(3)}$. Then $F = \mathrm{e}^{\mathrm{i}\pi\sum_x S_x^{(1)}}$. In the case of spin $S = \frac{1}{2}$ we also have $F = \prod_{x\in\Lambda} 2S_x^{(1)}$.

EXERCISE A.1: This amounts to showing that for $\alpha, \beta \in [0,1]$ we have

$$\alpha\log\tfrac{\alpha}{\beta} + (1-\alpha)\log\tfrac{1-\alpha}{1-\beta} \geq 2(\alpha-\beta)^2.$$

For fixed α let $f(\beta) = \alpha \log \frac{\alpha}{\beta} + (1-\alpha)\log\frac{1-\alpha}{1-\beta} - 2(\alpha-\beta)^2$. We have $f(\alpha) = 0$ and $f'(\beta) = (\beta-\alpha)(1-2\beta)^2$, which is negative for $\beta < \alpha$ and positive for $\beta > \alpha$. It is then clear that $f(\beta)$ is always above zero.