CHAPTER 2

General setting

2.1. Hilbert spaces, tensor products, operators

Let $\mathcal{H}$ be a separable Hilbert space and $\{e^{(i)}\}_{i\geq 1}$ be a finite or countable orthonormal basis. The **linear span** span$\{\varphi_1, \varphi_2, ...\}$ of the vectors $\varphi_1, \varphi_2, ...$ is the space of finite linear combinations of these vectors.

Recall that $\mathcal{H}$ is isomorphic to the completion of span$\{e^{(i)}\}_{i\geq 1}$.

We let $\mathcal{B}(\mathcal{H})$ denote the space of bounded linear operators on $\mathcal{H}$.

DEFINITION 2.1 (Tensor product). *Let $\mathcal{H}_1, \mathcal{H}_2$ be two Hilbert spaces with respective bases $\{e_1^{(i)}\}, \{e_2^{(j)}\}$. The* **tensor product** *of $\mathcal{H}_1$ and $\mathcal{H}_2$, denoted $\mathcal{H}_1 \otimes \mathcal{H}_2$, is the completion of the linear span of $\{(e_1^{(i)}, e_2^{(j)})\}_{i,j\geq 1}$.*

The dimension of the tensor product space satisfies

$$\dim \mathcal{H}_1 \otimes \mathcal{H}_2 = \dim \mathcal{H}_1 \cdot \dim \mathcal{H}_2. \tag{2.1}$$

Given two vectors $\varphi_1 \in \mathcal{H}_1$ and $\varphi_2 \in \mathcal{H}_2$, we can construct the element $\varphi_1 \otimes \varphi_2$ as follows. Let $\sum_i a_i e_1^{(i)}$ and $\sum_j b_j e_2^{(j)}$ be the decompositions of φ_1, φ_2 in the bases $\{e_1^{(i)}\}$ and $\{e_2^{(j)}\}$, respectively. Then

$$\varphi_1 \otimes \varphi_2 = \sum_{i,j\geq 1} a_i b_j e_1^{(i)} \otimes e_2^{(j)}. \tag{2.2}$$

Notice that $\varphi_1 \otimes 2\varphi_2 = 2\varphi_1 \otimes \varphi_2 = 2(\varphi_1 \otimes \varphi_2)$. Not all elements of $\mathcal{H}_1 \otimes \mathcal{H}_2$ can be written as tensor product vectors (see Exercise 2.1).

The inner product on $\mathcal{H}_1 \otimes \mathcal{H}_2$ is

$$\langle \varphi_1 \otimes \varphi_2, \psi_1 \otimes \psi_2 \rangle = \langle \varphi_1, \psi_1 \rangle \cdot \langle \varphi_2, \psi_2 \rangle, \tag{2.3}$$

where the inner products in the right side are in $\mathcal{H}_1$ and $\mathcal{H}_2$, respectively. This extends to general elements of $\mathcal{H}_1 \otimes \mathcal{H}_2$ by linearity.

Let $a_1 \in \mathcal{B}(\mathcal{H}_1)$ and $a_2 \in \mathcal{B}(\mathcal{H}_2)$ be two bounded operators. The tensor product operator $a_1 \otimes a_2$ is an operator acting on $\mathcal{H}_1 \otimes \mathcal{H}_2$ and its action on tensor product vectors is

$$(a_1 \otimes a_2)(\varphi_1 \otimes \varphi_2) = a_1\varphi_1 \otimes a_2\varphi_2. \tag{2.4}$$

Its action on general vectors is obtained by linearity. This construction is easily generalised to an arbitrary finite number of Hilbert spaces.

A lattice system involves a (finite) set of sites, denoted Λ, and a complex Hilbert space at every site; we assume that each site involves an identical copy of the same Hilbert space $\mathcal{H}_0$. The Hilbert space for the whole lattice system is

$$\mathcal{H}_\Lambda = \otimes_{x\in\Lambda}\mathcal{H}_0. \tag{2.5}$$

Then $\mathcal{H}_\Lambda$ is the completion of the linear span of the vectors $\otimes_{x\in\Lambda}e_x^{(i_x)}$, where $1 \leq i_x \leq \dim\mathcal{H}_0$.

Readers who are familiar with classical lattice systems may prefer an alternate description using classical configurations. This is completely equivalent. Let Ω_0 denote a finite or countable set (e.g. $\Omega_0 = \{-1, +1\}$ for the Ising model) and $\Omega_\Lambda = \Omega_0^\Lambda$. The linear span of Ω_0 is the set of vectors of the form $\psi = (\psi^{(i)})_{i\in\Omega_0}$, $\psi^{(i)} \in \mathbb{C}$, with finitely-many nonzero entries. The inner product between vectors $\psi_1, \psi_2 \in \operatorname{span}\Omega_0$ is

$$\langle\psi_1, \psi_2\rangle = \sum_{i\in\Omega_0} \overline{\psi_1^{(i)}}\psi_2^{(i)} \tag{2.6}$$

and the norm is $\|\psi\| = \sqrt{\langle\psi,\psi\rangle}$. We then define $\mathcal{H}_0 = \overline{\operatorname{span}\Omega_0}$, where the completion is with respect to the norm $\|\cdot\|$. In a similar fashion, the Hilbert space $\mathcal{H}_\Lambda$ is the completion of the linear span of Ω_Λ.

The **Dirac notation** is a convenient way of writing some elements of the Hilbert space, the inner products, and projection operators. If $(\psi_i)_{i\in I}$ is a fixed orthonormal basis of $\mathcal{H}$, we introduce the "ket" $|i\rangle$ and the "bra" $\langle i|$:

$$\begin{aligned} |i\rangle &\equiv \psi_i, \\ \langle i| &\equiv \psi_i \in \mathcal{H}^*, \\ |i\rangle\langle i| &\equiv P_{\psi_i}, \\ \langle i|j\rangle &\equiv \langle\psi_i, \psi_j\rangle. \end{aligned} \tag{2.7}$$

As we see above, the inner product is given by a bra and a ket, forming a "bracket". We can also write $|i\rangle\langle j|$ for the operator such that

$$\big\langle\psi_k, \big(|i\rangle\langle j|\big)\psi_\ell\big\rangle = \langle k|i\rangle\,\langle j|\ell\rangle = \delta_{k,i}\delta_{j,\ell}. \tag{2.8}$$

Finally, the notation can involve operators, writing

$$\langle i|a|j\rangle \equiv \langle\psi_i, a\,\psi_j\rangle. \tag{2.9}$$

Notice that a acts on the vector in the right by definition, even if the notation suggests symmetry (it does not matter when a is hermitian).

The tensor product is sometimes confused with the direct sum; let us clarify that it is different.

DEFINITION 2.2. *The* **direct sum** *of two Hilbert spaces* $\mathcal{H}_1$ *and* $\mathcal{H}_2$*, denoted* $\mathcal{H}_1 \oplus \mathcal{H}_2$*, is the space of pairs* (φ_1, φ_2) *with* $\varphi_1 \in \mathcal{H}_1$*,* $\varphi_2 \in \mathcal{H}_2$*, with operations*

$$\alpha(\varphi_1, \varphi_2) + \beta(\psi_1, \psi_2) = (\alpha\varphi_1 + \beta\psi_1, \alpha\varphi_2 + \beta\psi_2), \qquad \alpha, \beta \in \mathbb{C}.$$

If $\{e_i^{(1)}\}$ and $\{e_j^{(2)}\}$ are bases of $\mathcal{H}_1$ and $\mathcal{H}_2$, then $\{(e_i^{(1)}, 0)\} \cup \{(0, e_j^{(2)})\}$ is a basis of $\mathcal{H}_1 \oplus \mathcal{H}_2$. It follows that

$$\dim \mathcal{H}_1 \oplus \mathcal{H}_2 = \dim \mathcal{H}_1 + \dim \mathcal{H}_2. \tag{2.10}$$

Notice the difference between (2.1) and (2.10). In terms of matrix representation, the direct sum corresponds to block decomposition. The tensor product consists in replacing each entry of the first matrix by the whole of the second matrix, multiplied by the entry (this is called the “Kronecker product”).

2.2. Basic tools

We collect here a series of useful properties of square matrices. Recall that the “absolute value” of a matrix is $|a| = (a^* a)^{\frac{1}{2}}$, where the square root of a nonnegative hermitian matrix can be defined by diagonalising and taking the non-negative square root of the eigenvalues. Then $|a|$ is hermitian. If a is an hermitian matrix and f is a real-valued function whose domain contains the spectrum of a, we similarly define $f(a)$ by diagonalising a and applying f to each of the eigenvalues. The p-norm of a matrix is then defined as

$$\|a\|_p = (\operatorname{Tr} |a|^p)^{1/p}. \tag{2.11}$$

Note that $\|a\|_\infty = \lim_{p\to\infty} \|a\|_p = \|a\|$, the operator norm of a (see Exercise 2.4).

PROPOSITION 2.3 (**Hölder inequality for matrices**). *If* $1 \le p, q, r \le \infty$ *with* $\frac{1}{p} + \frac{1}{q} = \frac{1}{r}$*, we have*

$$\|ab\|_r \le \|a\|_p \|b\|_q.$$

It follows from a simple induction that

$$\Big\| \prod_{j=1}^n a_j \Big\|_r \le \prod_{j=1}^n \|a_j\|_{p_j} \tag{2.12}$$

whenever $1 \le r, p_1, \dots, p_n$ with $\sum_{j=1}^n \frac{1}{p_j} = \frac{1}{r}$. The proof of Proposition 2.3 can be found in the appendix.

PROPOSITION 2.4 (**Golden–Thompson inequality**). *Let* a, b *be hermitian matrices. Then*

$$\operatorname{Tr}\big(\mathrm{e}^{a+b}\big) \le \operatorname{Tr}\big(\mathrm{e}^{a}\,\mathrm{e}^{b}\big).$$

PROOF. Hölder's inequality, in the form (2.12) with $r = 1$, $p_j = \frac{1}{n}$ and $a_j = ab$, implies that $|\mathrm{Tr}\,(ab)^n| \leq \|ab\|_n^n$. The latter is equal to $\mathrm{Tr}\,(a^2b^2)^{n/2}$ since a, b are hermitian. Letting n be a power of 2, we can iterate and we get

$$\mathrm{Tr}\,(ab)^n \leq \mathrm{Tr}\, a^n b^n. \tag{2.13}$$

We use this inequality with $a \mapsto \mathrm{e}^{\frac{1}{n}a}$ and $b \mapsto \mathrm{e}^{\frac{1}{n}b}$, which gives

$$\mathrm{Tr}\,\Big(\mathrm{e}^{\frac{1}{n}a}\,\mathrm{e}^{\frac{1}{n}b}\Big)^n \leq \mathrm{Tr}\,\mathrm{e}^{a}\,\mathrm{e}^{b}. \tag{2.14}$$

The left side converges to $\mathrm{Tr}\,\mathrm{e}^{a+b}$ as $n \to \infty$ by the Trotter formula (Proposition A.4). □

PROPOSITION 2.5 (**Klein inequality**). *Let f be a convex differentiable function, and a, b be hermitian matrices with eigenvalues in the domain of f. Then*

$$\mathrm{Tr}\,\big[f(a) - f(b) - (a-b)f'(b)\big] \geq 0.$$

With $f(s) = \mathrm{e}^{s}$, exchanging a and b, we get

$$\mathrm{Tr}\,\big(\mathrm{e}^{a} - \mathrm{e}^{b}\big) \leq \mathrm{Tr}\,(a-b)\,\mathrm{e}^{a}. \tag{2.15}$$

PROOF. Let (ϕ_i) and (ψ_i) be orthonormal bases of eigenvectors of a and b, and let (α_i) and (β_i) the eigenvalues. Let $c_{ij} = \langle \phi_i, \psi_j \rangle$. Then

$$\begin{aligned}\big\langle \phi_i, \big[f(a) - f(b) - &(a-b)f'(b)\big]\phi_i \big\rangle \\ &= f(\alpha_i) - \sum_j |c_{ij}|^2 f(\beta_j) - \sum_j |c_{ij}|^2 (\alpha_i - \beta_j) f'(\beta_j) \\ &= \sum_j |c_{ij}|^2 \big[f(\alpha_i) - f(\beta_j) - (\alpha_i - \beta_j) f'(\beta_j)\big] \\ &\geq 0.\end{aligned} \tag{2.16}$$

□

PROPOSITION 2.6 (**Peierls–Bogolubov inequality**). *Let f be convex on $\mathbb{R}$ and a, h be hermitian matrices such that $\mathrm{Tr}\,\mathrm{e}^{-h} = 1$. Then*

$$f\big(\mathrm{Tr}\, a\,\mathrm{e}^{-h}\big) \leq \mathrm{Tr}\, f(a)\,\mathrm{e}^{-h}.$$

PROOF. Let (ϕ_i) and (η_i) be the eigenvectors and eigenvalues of h. Using Jensen's inequality twice,

$$\begin{aligned} f\big(\mathrm{Tr}\, a\,\mathrm{e}^{-h}\big) &= f\Big(\sum_i \langle \phi_i, a\phi_i \rangle\, \mathrm{e}^{-\eta_i}\Big) \leq \sum_i f\big(\langle \phi_i, a\phi_i \rangle\big)\, \mathrm{e}^{-\eta_i} \\ &\leq \sum_i \langle \phi_i, f(a)\phi_i \rangle\, \mathrm{e}^{-\eta_i} = \mathrm{Tr}\, f(a)\,\mathrm{e}^{-h}. \end{aligned} \tag{2.17}$$

□

PROPOSITION 2.7 (**Peierls inequality**). *Let a be a hermitian matrix and (ϕ_i) an orthonormal set of vectors. Then*

$$\sum_i \mathrm{e}^{\langle \phi_i, a\phi_i \rangle} \leq \mathrm{Tr}\, \mathrm{e}^a .$$

PROOF. Let α_j, ψ_j be the eigenvalues and eigenvectors of a. Then

$$\mathrm{e}^{\langle \phi_i, a\phi_i \rangle} = \exp\Bigl\{ \sum_j \alpha_j |\langle \phi_i, \psi_j \rangle|^2 \Bigr\} \leq \sum_j |\langle \phi_i, \psi_j \rangle|^2 \, \mathrm{e}^{\alpha_j} . \tag{2.18}$$

We used Jensen's inequality. The claim then follows by summing over i, using $\sum_i |\langle \phi_i, \psi_j \rangle|^2 \leq 1$ (Bessel inequality). □

2.3. States on finite domains

We now describe states for finite systems, i.e. systems with finitely many sites. Later we will consider states for infinite systems with no reference to a Hilbert space. But for now we have a Hilbert space $\mathcal{H}$ (whose dimension is possibly infinite) and its space of bounded operators $\mathcal{B}(\mathcal{H}) = \{\text{linear } a : \mathcal{H} \to \mathcal{H} : \|a\| = \sup_{h \in \mathcal{H}} \|ah\|/\|h\| < \infty\}$, and this allows to identify states with density operators.

A **state** is a positive, normalised linear functional on $\mathcal{B}(\mathcal{H})$; that is, it is a map $\langle \cdot \rangle : \mathcal{B}(\mathcal{H}) \to \mathbb{C}$ that is

(i) linear: $\langle sa + tb \rangle = s\langle a \rangle + t\langle b \rangle$ for all $s, t \in \mathbb{C}$ and $a, b \in \mathcal{B}(\mathcal{H})$;
(ii) positive: $\langle a^* a \rangle \geq 0$ for all $a \in \mathcal{B}(\mathcal{H})$;
(iii) normalised: $\langle \mathbb{1} \rangle = 1$.

One can check that the operator norm $\|\langle \cdot \rangle\|$ of any state $\langle \cdot \rangle$ is 1. It turns out that states can be represented by density operators, a very useful property. A **density operator** ρ is a trace-class (meaning that $\mathrm{Tr}\,|\rho| < \infty$) positive-definite hermitian operator such that $\mathrm{Tr}\,\rho = 1$. Given a density operator, there corresponds the state

$$\langle a \rangle = \mathrm{Tr}\, \rho\, a. \tag{2.19}$$

The converse is also true, each state is represented by a density operator.

PROPOSITION 2.8 (**Riesz representation of states**). *Let $\langle \cdot \rangle$ be a state. Then there exists a unique density operator ρ such that $\langle a \rangle = \mathrm{Tr}\, \rho a$ for all $a \in \mathcal{B}(\mathcal{H})$.*

In the case when $\mathcal{H}$ is finite-dimensional, the density matrix ρ can be defined using the matrix elements $\rho_{i,j} = \langle |j\rangle\langle i| \rangle$ (with respect to some orthonormal basis) where we used the Dirac notation (2.8).

In the infinite-dimensional case, we consider the subspace $\mathcal{B}_2(\mathcal{H}) \subset \mathcal{B}(\mathcal{H})$ that consists of bounded operators with finite **Hilbert–Schmidt norm**

$$\|a\|_{\mathrm{HS}} = \sqrt{\operatorname{Tr} a^* a}. \tag{2.20}$$

This norm derives from the inner product

$$\langle a, b\rangle_{\mathrm{HS}} = \operatorname{Tr} a^* b, \qquad a, b \in \mathcal{B}_2(\mathcal{H}). \tag{2.21}$$

One can check that $\mathcal{B}_2(\mathcal{H})$ is complete with respect to the Hilbert–Schmidt norm, and is therefore a Hilbert space. It is dense in $\mathcal{B}(\mathcal{H})$. Since $\|a\| \leq \|a\|_{\mathrm{HS}}$ we have $\mathcal{B}_2(\mathcal{H}) \subset \mathcal{B}(\mathcal{H})$ and a bounded linear functional on $\mathcal{B}(\mathcal{H})$ is also a bounded linear functional on $\mathcal{B}_2(\mathcal{H})$. We prove Proposition 2.8 in $\mathcal{B}_2(\mathcal{H})$.

Proof of Proposition 2.8. The standard Riesz representation theorem implies that $\mathcal{B}_2(\mathcal{H})$ is self-dual, that is, every linear functional $\langle\cdot\rangle : \mathcal{B}_2(\mathcal{H}) \to \mathbb{C}$ is represented by a unique operator ρ such that

$$\langle a\rangle = \langle \rho, a\rangle_{\mathrm{HS}} = \operatorname{Tr} \rho^* a. \tag{2.22}$$

for all $a \in \mathcal{B}_2(\mathcal{H})$. We now check that ρ is a density operator. We have for all $\varphi \in \mathcal{H}$ with $\|\varphi\| = 1$ that

$$\langle \varphi, \rho^* \varphi\rangle = \operatorname{Tr} P_\varphi \rho^* = \langle P_\varphi\rangle = \langle P_\varphi^2\rangle \geq 0. \tag{2.23}$$

(Here, P_φ denotes the projector onto φ.) Then ρ^* is positive-definite; it is therefore hermitian (Exercise 2.6) so $\rho \geq 0$ as well. Next, let $(\varphi_i)_{i=1}^\infty$ denote an orthonormal basis of $\mathcal{H}$ and P_n the orthogonal projector onto the span of $(\varphi)_{i=1}^n$. Since $P_n \in \mathcal{B}_2(\mathcal{H})$, we have

$$\operatorname{Tr} \rho = \lim_{n\to\infty} \operatorname{Tr} \rho P_n = \lim_{n\to\infty} \langle P_n\rangle \leq 1. \tag{2.24}$$

The latter inequality follows from $\langle \mathbb{1} - P_n\rangle \geq 0$ since $\mathbb{1} - P_n \geq 0$. Then ρ is a trace-class operator with $\|\rho\|_1 \leq 1$.

If (a_n) is a sequence of operators in $\mathcal{B}_2(\mathcal{H})$ that converges to $a \in \mathcal{B}(\mathcal{H})$, we have from Hölder's inequality (Proposition 2.3) that

$$\big|\operatorname{Tr} \rho\, a_n - \operatorname{Tr} \rho\, a\big| \leq \|\rho(a_n - a)\|_1 \leq \|\rho\|_1 \|a_n - a\|_\infty \leq \|a_n - a\|. \tag{2.25}$$

Then $\langle a\rangle = \operatorname{Tr} \rho a$ for all $a \in \mathcal{B}(\mathcal{H})$. Finally, the relation $\langle \mathbb{1}\rangle = 1$ implies that ρ is a density operator. □

The set of states is *convex*: If $\langle\cdot\rangle_1, \langle\cdot\rangle_2$ are two states, the convex combination $\alpha\langle\cdot\rangle_1 + (1-\alpha)\langle\cdot\rangle_2$ is also a state for all $\alpha \in [0, 1]$. A state is **mixed** if it can be written as a convex combination of distinct states. A state is **pure** if it is not mixed; in other words, pure states are the extremal points of the convex set of states.

Given $\varphi \in \mathcal{H}$, the corresponding projector P_φ is a special case of a density operator, hence it gives a state. It is perhaps expected that this state is pure, and that all pure states are represented by projectors.

PROPOSITION 2.9. *A state is pure if and only if its density operator is equal to P_φ for some $\varphi \in \mathcal{H}$.*

PROOF. The density operator ρ of the state is hermitian and it can be written as

$$\rho = \sum_{i=1}^{n} \lambda_i P_{\varphi_i}, \tag{2.26}$$

where the φ_is form a basis of eigenvectors. This can be viewed as a convex combination of density operators. This shows that if ρ is not equal to a projector, then the corresponding state is mixed.

There remains to show that if $\rho = P_\varphi$ is a projector, then the state is pure. Assume that $\langle\cdot\rangle = t\langle\cdot\rangle_1 + (1-t)\langle\cdot\rangle_2$ with $t \in (0,1)$. By Proposition 2.8, there exist density matrices ρ_1, ρ_2 such that $\langle\cdot\rangle_i = \operatorname{Tr} \rho_i \cdot$ for $i = 1, 2$. Then

$$P_\varphi = t\rho_1 + (1-t)\rho_2. \tag{2.27}$$

We have

$$\begin{aligned} 1 = \operatorname{Tr} \rho = \operatorname{Tr} P_\varphi = \operatorname{Tr} P_\varphi^2 &= \operatorname{Tr} P_\varphi\big(t\rho_1 + (1-t)\rho_2\big) \\ &= t\langle\varphi, \rho_1\varphi\rangle + (1-t)\langle\varphi, \rho_2\varphi\rangle \leq t\|\rho_1\| + (1-t)\|\rho_2\|. \end{aligned} \tag{2.28}$$

The last inequality follows from Cauchy–Schwarz and the definition of the operator norm. Since $\|\rho_i\| \leq 1$, we find that $\|\rho_i\| = 1$ (so ρ_i is a projector) and that the inequality is actually an identity. Both ρ_1 and ρ_2 must then project onto φ, so we necessarily have $\rho_1 = \rho_2 = P_\varphi$. □

2.4. Local observables and interactions

The goal of statistical mechanics is to describe the "bulk properties" of the system, far away from its boundaries. The large system is approximated by an infinite regular graph, the "lattice". For simplicity we consider $\mathbb{Z}^d$, although all of the setting and many of the properties hold more generally. The notation $\Lambda \Subset \mathbb{Z}^d$ means that Λ is a *finite* subset of $\mathbb{Z}^d$.

At each site $x \in \mathbb{Z}^d$ is associated a Hilbert space $\mathcal{H}_x$. We write $N = \dim \mathcal{H}_x$, which we assume to be finite and independent of x. For $X \Subset \mathbb{Z}^d$ we let $\mathcal{H}_X = \otimes_{x\in X} \mathcal{H}_x$ and we let $\mathcal{A}_X = \mathcal{B}(\mathcal{H}_X)$ denote the algebra of bounded linear operators on $\mathcal{H}_X$. We consider two norms on $\mathcal{A}_X$. First, the usual operator norm $\|a\| = \sup_{\varphi\in\mathcal{H}_X} \|a\varphi\|$. Second, the Hilbert-Schmidt norm $\|a\|_2 = \sqrt{\operatorname{tr} a^* a}$, where tr denotes the **normalised trace**

$$\operatorname{tr} a = \frac{1}{N^{|X|}} \operatorname{Tr} a. \tag{2.29}$$

If $X \subset Y \Subset \mathbb{Z}^d$, there is a natural injection $\iota : \mathcal{A}_X \to \mathcal{A}_Y$ defined by

$$\iota a = a \otimes \mathbb{1}_{Y\setminus X}. \tag{2.30}$$

Notice that $\|\iota a\| = \|a\|$ and $\|\iota a\|_2 = \|a\|_2$ (because of the normalised trace). Then we can view $\mathcal{A}_X$ as a subalgebra of $\mathcal{A}_Y$. We define the algebra of **local observables** as the inductive union

$$\mathcal{A}_{\rm loc} = \bigvee_{X \Subset \mathbb{Z}^d} \mathcal{A}_X. \tag{2.31}$$

If $a \in \mathcal{A}_{\rm loc}$, then there exists $X \Subset \mathbb{Z}^d$ such that $a \in \mathcal{A}_X$. Finally, we let $\mathcal{A}$ denote the completion of $\mathcal{A}_{\rm loc}$ with respect to the operator norm; this is the algebra of **quasi-local observables**. We also let $\mathcal{A}_{\rm h}$ denote the algebra of hermitian quasi-local operators.

From the definition of $\mathcal{A}$ above we see that a local operator can be represented by several distinct elements, since $a \in \mathcal{A}_X$ has a counterpart $\iota(a) \in \mathcal{A}_Y$ for all $Y \supset X$. If we define a map α on operators $a \in \mathcal{A}_X$, simultaneously for all $X \Subset \mathbb{Z}^d$, we need to check that it is **consistent**, namely that $\alpha(a) = \alpha(\iota(a))$.

We denote by τ_x the lattice translation by $x \in \mathbb{Z}^d$. The action of τ_x is intuitive but let us define it formally. First we view τ_x as a linear map between $\mathcal{H}_X$ and $\mathcal{H}_{X+x}$ for any $X \Subset \mathbb{Z}^d$ such that

$$\tau_x \otimes_{y \in X} \varphi_y = \otimes_{y \in X} \varphi_{y+x} \tag{2.32}$$

for any set of vectors (φ_y) in $\mathcal{H}_0$. This extends by linearity to all vectors of $\mathcal{H}_X$. Notice that $\tau_x^{-1} = \tau_{-x}$. Then we define $\tilde{\tau}_x$ as the linear map $\mathcal{A}_X \to \mathcal{A}_{X+x}$ whose action on the local observable $a \in \mathcal{A}_X$ is

$$(\tilde{\tau}_x a)\varphi = \tau_x a(\tau_x^{-1}\varphi), \qquad \forall \varphi \in \mathcal{H}_{X+x}. \tag{2.33}$$

We dismiss the tilde from now on and we write τ_x for translations of vectors and of operators.

An **interaction** is a collection of local self-adjoint observables indexed by finite subsets, $\Phi = (\Phi_X)_{X \Subset \mathbb{Z}^d}$. We only consider translation-invariant interactions, i.e. we assume that $\Phi_{X+x} = \tau_x \Phi_X$ for all $X \Subset \mathbb{Z}^d$ and all $x \in \mathbb{Z}^d$. Interactions form a (real) linear space and we consider the following norms:

$$\begin{aligned} |\!|\!|\Phi|\!|\!| &= \sum_{X \ni 0} \frac{\|\Phi_X\|}{|X|}, \\ \|\Phi\|_r &= \sum_{X \ni 0} \mathrm{e}^{r|X|} \, \|\Phi_X\| \qquad \text{for } r \geq 0. \end{aligned} \tag{2.34}$$

We denote $\mathcal{I}, \mathcal{I}_r$ the corresponding Banach spaces of interactions.

The **hamiltonian** in a finite domain $\Lambda \Subset \mathbb{Z}^d$ is

$$H_\Lambda^\Phi = \sum_{X \subset \Lambda} \Phi_X. \tag{2.35}$$

In the finite domain $\Lambda \Subset \mathbb{Z}^d$ the **Gibbs state** for the interaction Φ is the linear functional $\langle\cdot\rangle_\Lambda : \mathcal{A}_\Lambda \to \mathbb{C}$ given by

$$\langle a\rangle_\Lambda = \frac{1}{Z_\Lambda(\Phi)} \operatorname{Tr} a\, \mathrm{e}^{-H_\Lambda^\Phi}, \tag{2.36}$$

where $Z_\Lambda(\Phi) = \operatorname{Tr} \mathrm{e}^{-H_\Lambda^\Phi}$. We want to extend this notion to the infinite lattice $\mathbb{Z}^d$. There is no hamiltonian on the infinite lattice. In fact, we also avoid the Hilbert space for $\mathbb{Z}^d$ since it would be an infinite tensor product and would not be separable; this would cause many pathologies. The way out is to extend the linear functionals from $\mathcal{A}_\Lambda$ to $\mathcal{A}$, the space of quasi-local observables.

We define a **state** $\langle\cdot\rangle$ as a normalised, positive linear functional on $\mathcal{A}$. That is, $\langle\cdot\rangle$ satisfies

(i) $\langle sa + tb\rangle = s\langle a\rangle + t\langle b\rangle$ for all $a, b \in \mathcal{A}$ and $s, t \in \mathbb{C}$.
(ii) $\langle \mathbb{1}\rangle = 1$.
(iii) $\langle a^* a\rangle \geq 0$ for all $a \in \mathcal{A}$.

All states have norm 1 (Exercise 2.7); by the Banach–Alaoglu theorem the set of states is compact in the weak-* topology (that is, the topology of pointwise convergence of linear functionals). In plain words, this means that from any sequence of states (ρ_n), we can extract a subsequence (n_k) such that $\rho_{n_k}(a)$ converges for any $a \in \mathcal{A}$.

Here is a first definition for infinite-volume equilibrium states.

DEFINITION 2.10 (State as cluster point). *Let $\Phi \in \mathcal{I}$, and let Ψ_n be a sequence of interactions in $\mathcal{I}$ such that $\|\Psi_n\| \to 0$ as $n \to \infty$. Let $\Lambda_n = \{-n, \dots, n\}^d$ and for $a \in \mathcal{A}_{\mathrm{loc}}$, let*

$$\langle a\rangle_{\Lambda_n} = \frac{1}{Z_{\Lambda_n}(\Phi + \Psi_n)} \operatorname{Tr} a\, \mathrm{e}^{-H_{\Lambda_n}^{\Phi+\Psi_n}}.$$

The cluster points of the sequence $(\langle\cdot\rangle_{\Lambda_n})$ are infinite-volume Gibbs states for the interaction Φ.

In the next chapters we will see three other definitions of infinite-volume equilibrium states: via tangent functionals, the Gibbs variational principle, and the KMS condition. The states of Definition 2.10 satisfy all three other definitions. To motivate the terminology *equilibrium* state, we also verify that KMS states are invariant under the relevant time evolution.

EXERCISE 2.1. *Show that there exist no $\varphi_1 \in \mathcal{H}_1$, $\varphi_2 \in \mathcal{H}_2$ such that*

$$e_1^{(1)} \otimes e_1^{(2)} + e_2^{(1)} \otimes e_2^{(2)} = \varphi_1 \otimes \varphi_2.$$

EXERCISE 2.2. *Show that*

(a) $\|\varphi_1 \otimes \varphi_2\| = \|\varphi_1\| \cdot \|\varphi_2\|$ *for all $\varphi_1 \in \mathcal{H}_1$, $\varphi_2 \in \mathcal{H}_2$.*

(b) $\|A_1 \otimes A_2\| = \|A_1\| \cdot \|A_2\|$ *for all* $A_1 \in \mathcal{B}(\mathcal{H}_1)$, $A_2 \in \mathcal{B}(\mathcal{H}_2)$.

Exercise 2.3. *Show that*

$$\underbrace{\mathcal{H} \oplus \cdots \oplus \mathcal{H}}_{n \text{ times}} \simeq \mathcal{H} \otimes \mathbb{C}^n.$$

Exercise 2.4.

(i) *Show that* $\|a\|_p$ *is decreasing in* p, *and that* $\lim_{p\to\infty} \|a\|_p = \|a\|$.
(ii) *Prove that* $\|a\|_p$ *is a norm for all* $1 \le p \le \infty$.
(iii) *Use Hölder inequality to show that* $\|a\|_p$ *is submultiplicative, that is,* $\|ab\|_p \le \|a\|_p \|b\|_p$.

Exercise 2.5. *Give an example of a quasi-local observable that is not a local observable.*

Exercise 2.6. *Show that any* $a \in \mathcal{B}(\mathcal{H})$ *can be written as* $a = b + \mathrm{i}c$ *where* $b, c \in \mathcal{B}(\mathcal{H})$ *are hermitian. Deduce that if* a *is positive-definite then it is hermitian.*

Exercise 2.7. *Prove that the norm of a state is 1.*

Exercise 2.8. *If* $\langle \cdot \rangle$ *is a state for finite-dimensional* $\mathcal{H}$, *show that* ρ *defined with respect to some orthonormal basis by* $\rho_{i,j} = \langle |j\rangle\langle i| \rangle$ *is a density-matrix such that* $\langle a \rangle = \operatorname{Tr} \rho a$.

Exercise 2.9. *Show that a state* $\langle \cdot \rangle$ *on* $\mathcal{B}(\mathcal{H})$ *satisfies (i)* $\langle a^* \rangle = \overline{\langle a \rangle}$, *and (ii) the Cauchy–Schwarz inequality,* $|\langle a^* b \rangle|^2 \le \langle a^* a \rangle \langle b^* b \rangle$.