CHAPTER 4

Uniqueness and non-uniqueness of Gibbs states

4.1. Uniqueness of KMS states at high temperature

We give here a sufficient condition (high temperature) that guarantees the uniqueness of the infinite-volume Gibbs state. The method of proof is inspired by [**16**]. Recall that existence of Gibbs states at any temperature can be established by a compactness argument.

Recall that $\dim \mathcal{H}_0 = N$.

THEOREM 4.1. *Assume that there exists $r > 0$ such that $\|\Phi\|_r < \frac{r}{2}$ and that*

$$\frac{\|\Phi\|_r}{\frac{r}{2} - \|\Phi\|_r} < \mathrm{e}^{-r} \, \frac{2^{3/2}}{N^{5/2}}.$$

Then the KMS state for the interaction Φ is unique.

In this section we denote the state $\rho(\cdot) = \langle \cdot \rangle$. Notice that the interaction Φ is small enough so that the evolution operator α_{i} is bounded on $\mathcal{A}_\Lambda$ for all $\Lambda \Subset \mathbb{Z}^2$, see (3.45).

The starting point is the following rearrangement of the KMS condition:

$$\rho([A, B]) = \rho\big(B(\alpha_{\mathrm{i}} - \mathbb{1})A\big). \tag{4.1}$$

In order to use this equation we need to turn an operator into a sum of commutators. This is the content of the next lemma.

LEMMA 4.2. *Let A be a hermitian $N \times N$ matrix with the property that $\operatorname{Tr} A = 0$. Then there exist hermitian $N \times N$ matrices $b_1, \dots, b_{N-1}$ and $c_1, \dots, c_{N-1}$ (that depend on A) such that*

$$A = \mathrm{i} \sum_{k=1}^{N-1} [b_k, c_k]$$

and

$$\sum_{k=1}^{N-1} \|b_k\| \, \|c_k\| \leq \tfrac{1}{2} N \, \|A\|.$$

Proof. Let $\alpha_1, \dots, \alpha_N$ be the eigenvalues of A (repeated according to their multiplicity). We have that $\sum_{i=1}^N \alpha_i = 0$. Let us order the eigenvalues so that

$$\Big| \sum_{i=1}^{k} \alpha_i \Big| \leq \|A\| \tag{4.2}$$

for all $1 \leq k \leq N-1$. This is indeed possible, as can be seen by induction using $\sum \alpha_i = 0$: If $0 \leq \sum^k \alpha_i \leq \|A\|$, we can find $\alpha_{k+1} \leq 0$ among the remaining eigenvalues such that $|\sum^{k+1} \alpha_i| \leq \|A\|$. And if the partial sum is negative, we can find $\alpha_{k+1} \geq 0$ among the remaining eigenvalues, with the same conclusion.

We work in a basis such that A is diagonal and its eigenvalues are ordered so they satisfy the properties above. Let $\tilde{\alpha}_k = \sum_{i=1}^k \alpha_i$, and let $\sigma^1_{j,j+1}, \sigma^2_{j,j+1}, \sigma^3_{j,j+1}$ be $N \times N$ matrices that are equal to Pauli matrices on the 2×2 block that contains (j,j) and $(j+1,j+1)$, and that are equal to zero everywhere else. It is not hard to check that

$$A = \sum_{j=1}^{N-1} \tilde{\alpha}_j \, \sigma^3_{j,j+1}. \tag{4.3}$$

We therefore have that

$$A = -\tfrac{\mathrm{i}}{2} \sum_{j=1}^{N-1} \tilde{\alpha}_j \, [\sigma^1_{j,j+1}, \sigma^2_{j,j+1}], \tag{4.4}$$

which proves the first claim. The bound follows from $|\tilde{\alpha}_j| \leq \|A\|$ and $\|\sigma^i_{j,j+1}\| = 1$. □

Recall that $\|A\|_2 = \sqrt{\operatorname{tr} A^* A}$ is the normalised Hilbert-Schmidt norm. For $A \in \mathcal{A}_\Lambda$ we have $\frac{1}{\sqrt{\dim \mathcal{H}_\Lambda}} \|A\| \leq \|A\|_2 \leq \|A\|$.

Proof of Theorem 4.1. Let $\rho^{(0)}$ a fixed KMS state for Φ. We want to show that any KMS state satisfies $\rho(A) = \rho^{(0)}(A)$ for all $A \in \mathcal{A}$. It is enough to show it for all $A \in \mathcal{A}^{\rm sa}_\Lambda$ and all $\Lambda \Subset \mathbb{Z}^d$, where $\mathcal{A}^{\rm sa}_\Lambda$ is the set of self-adjoint operators with support inside Λ.

We proceed by induction on Λ: We assume the result to hold for Λ and we prove it for $\Lambda \cup \{x\}$ where $x \in \mathbb{Z}^d \setminus \Lambda$. (The base case is $A = \mathbb{1}$ for which certainly $\rho(\mathbb{1}) = \rho^{(0)}(\mathbb{1}) = 1$.)

Let $(e_j)_{j=0}^{N^2-1}$ be an orthogonal hermitian basis of $\mathcal{M}_N(\mathbb{C})$ such that $e_0 = \mathbb{1}$, $\operatorname{Tr} e_j = 0$ if $j \neq 0$, $\operatorname{Tr} e_i e_j = 0$ if $i \neq j$, e_j is hermitian, and $\|e_j\| = 1$, for all j. Any operator $A \in \mathcal{A}^{\rm sa}_{\{x\} \cup \Lambda}$ has a unique decomposition as

$$A = \sum_{j=0}^{N^2-1} e_j \otimes C_j, \tag{4.5}$$

where $C_j \in \mathcal{A}_\Lambda^{\rm sa}$. We use Lemma 4.2 to write $e_j = {\rm i} \sum_{k=1}^{N-1} [b_j^{(k)}, c_j^{(k)}]$ for $j > 0$. The KMS condition then implies that

$$\begin{aligned} \rho(A) &= \sum_{j=0}^{N^2-1} \rho(e_j \otimes C_j) \\ &= \rho(\mathbb{1}_{\{x\}} \otimes C_0) + \sum_{j=1}^{N^2-1} \sum_{k=1}^{N-1} \rho\big({\rm i}[b_j^{(k)} \otimes \mathbb{1}_\Lambda, c_j^{(k)} \otimes C_j]\big) \\ &= \rho(\mathbb{1}_{\{x\}} \otimes C_0) + \sum_{j=1}^{N^2-1} \sum_{k=1}^{N-1} \rho\big({\rm i}(c_j^{(k)} \otimes C_j)(\alpha_{\rm i} - \mathbb{1})(b_j^{(k)} \otimes \mathbb{1}_\Lambda)\big). \end{aligned} \tag{4.6}$$

Let ρ_0 be the linear functional on $\mathcal{A}_\Lambda^{\rm sa}$ such that $\rho_0(A) = \rho^{(0)}(e_0 \otimes C_0)$. We have $\rho(\mathbb{1}_{\{x\}} \otimes C_0) = \rho_0(A)$ by induction. Let us introduce the operator $K : \mathcal{L}(\Lambda_{\{x\}\cup\Lambda}) \to \mathcal{L}(\Lambda_{\{x\}\cup\Lambda})$ such that, with respect to the decomposition (4.5),

$$(K\phi)(A) = \sum_{j=1}^{N^2-1} \sum_{k=1}^{N-1} \phi\big({\rm i}(c_j^{(k)} \otimes C_j)(\alpha_{\rm i} - \mathbb{1})(b_j^{(k)} \otimes \mathbb{1}_\Lambda)\big). \tag{4.7}$$

The KMS condition can then be cast as an identity about linear functionals, namely

$$\rho = \rho_0 + K\rho. \tag{4.8}$$

This is equivalent to $(1-K)\rho = \rho_0$. We now show that $1-K$ is invertible so there is a unique solution for ρ.

We work in the real linear vector space $(\mathcal{A}_{\{x\}\cup\Lambda}^{\rm sa}, \|\cdot\|_2)$. Then $\|\phi\| = \sup_{\|A\|_2=1} |\phi(A)|$. We have

$$\begin{aligned} |(K\phi)(A)| &\le \sum_{j=1}^{N^2-1} \sum_{k=1}^{N-1} \big|\phi\big((c_j^{(k)} \otimes C_j)(\alpha_{\rm i} - \mathbb{1})(b_j^{(k)} \otimes \mathbb{1}_\Lambda)\big)\big| \\ &\le \|\phi\| \sum_{j=1}^{N^2-1} \sum_{k=1}^{N-1} \big\|(c_j^{(k)} \otimes C_j)(\alpha_{\rm i} - \mathbb{1})(b_j^{(k)} \otimes \mathbb{1}_\Lambda)\big)\big\|_2 \\ &\le \|\phi\| \, {\rm e}^r \, \frac{\|\Phi\|_r}{\frac{r}{2} - \|\Phi\|_r} \sum_{j=1}^{N^2-1} \|C_j\|_2 \sum_{k=1}^{N-1} \|c_j^{(k)}\| \, \|b_j^{(k)}\|. \end{aligned} \tag{4.9}$$

We used Eq. (3.44) to get

$$\|(\alpha_{\rm i} - \mathbb{1})(b_j^{(k)} \otimes \mathbb{1}_\Lambda)\| \le \, {\rm e}^r \, \frac{\|\Phi\|_r}{\frac{r}{2} - \|\Phi\|_r} \|b_j^{(k)}\|. \tag{4.10}$$

The sum over k is less than $N/2$ by Lemma 4.2. We check in Exercise 4.1 that $\sum_j \|C_j\|_2 \leq \frac{N^{3/2}}{\sqrt{2}} \|A\|_2$. We have proved that

$$\|K\| \leq \frac{N^{5/2}}{2^{3/2}} \, \mathrm{e}^r \, \frac{\|\Phi\|_r}{\frac{r}{2} - \|\Phi\|_r}. \tag{4.11}$$

Then $\|K\| < 1$ and $1 - K$ is invertible when Φ satisfies the condition of the theorem. □

4.2. Long-range order in the XXZ-model

4.2.1. The classical Ising model. The classical Ising model is obtained by taking $J_1 = J_2 = 0$ and $n = 2$ (spin $\frac{1}{2}$) in the XYZ-hamiltonian (1.37). It is convenient to take $J_3 = 2$ and to add a constant.

Ising hamiltonian:

$$H_\Lambda^{\mathrm{ISING}} = -2 \sum_{xy \in \mathcal{E}(\Lambda)} \big(S_x^{(3)} S_y^{(3)} - \tfrac{1}{4} \big) = -\tfrac{1}{2} \sum_{xy \in \mathcal{E}(\Lambda)} \big(\sigma_x^{(3)} \sigma_y^{(3)} - 1 \big). \tag{4.12}$$

Here $\sigma^{(3)} = \left(\begin{smallmatrix} 1 & 0 \\ 0 & -1 \end{smallmatrix}\right)$ is the Pauli matrix. The hamiltonian is diagonal in the usual product-basis:

$$H_\Lambda^{\mathrm{ISING}} |\omega\rangle = \big(-\tfrac{1}{2} \textstyle\sum_{xy \in \mathcal{E}\Lambda} (\omega_x \omega_y - 1) \big) |\omega\rangle, \tag{4.13}$$

where $\omega = \{\omega_x\}_{x \in \Lambda} \in \{-1, +1\}^\Lambda$ denotes a configuration of classical spins. Thus, the Gibbs factor $\mathrm{e}^{-\beta H_\Lambda} / Z_\Lambda$ is also diagonal:

$$\frac{1}{Z_\Lambda} \mathrm{e}^{-\beta H_\Lambda^{\mathrm{ISING}}} |\omega\rangle = \frac{1}{Z_\Lambda} \exp\big(\tfrac{1}{2} \beta \textstyle\sum_{xy \in \mathcal{E}(\Lambda)} (\omega_x \omega_y - 1) \big) |\omega\rangle. \tag{4.14}$$

Because of this, it is customary to consider the probability measure on the set $\{-1, +1\}^\Lambda$ of spin configurations given by

$$\mathbb{P}_\Lambda(\omega) = \frac{1}{Z_\Lambda} \exp\big(\tfrac{1}{2} \beta \textstyle\sum_{xy \in \mathcal{E}(\Lambda)} (\omega_x \omega_y - 1) \big), \qquad \omega \in \{-1, +1\}^\Lambda. \tag{4.15}$$

The spin-spin correlation becomes an expected value:

$$\langle \sigma_x^{(3)} \sigma_y^{(3)} \rangle_\Lambda = \mathbb{E}_\Lambda[\omega_x \omega_y]. \tag{4.16}$$

The fact that it is trivial to diagonalize the hamiltonian does not mean that the model is trivial — far from it. An excellent introduction to the subject is given in [**10**]. A central feature of the theory is that the $\mathbb{Z}_2$-symmetry of the model, obtained by simultaneously mapping all $\omega_x \mapsto -\omega_x$, is broken at low temperature (large β). For this we take $\Lambda = \Lambda_N = \{-N, -N+1, \dots, N\}^d$ to be a box in $\mathbb{Z}^d$ for $d \geq 2$:

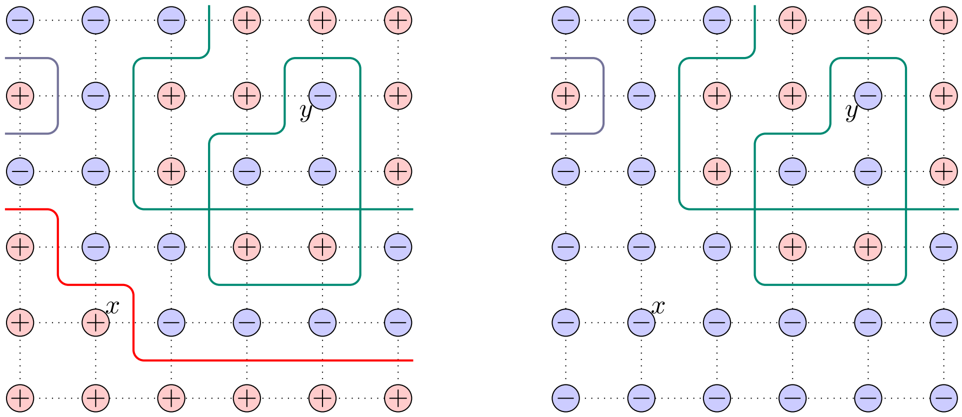

FIGURE 4.1. Ising configurations and their sets of contours. If x and y carry different values, there must be a contour separating them (possibly by surrounding one of them). Reversing all signs on the left side of a given contour γ in the set g (the red one in the picture) produces a configuration with set $g \setminus \{\gamma\}$.

THEOREM 4.3. *Consider the model (4.12) with $d \geq 2$. There exist $\beta_0 < \infty$ and $c(\beta) > 0$ (that depend on d but not on N) such that for $\beta > \beta_0$, we have*

$$\langle \sigma_x^{(3)} \sigma_y^{(3)} \rangle_\Lambda \geq c \tag{4.17}$$

for all $x, y \in \Lambda_N$.

We state below the same theorem for the asymmetric quantum Heisenberg model. The theorem above is then a special case. We nonetheless explain its proof in details (the "Peierls argument") because it is a useful warm-up.

PROOF. Although the proof applies to all $d \geq 2$, it is best to think of the case $d = 2$. The key idea is to represent configurations with the help of *contours*. These are made of the dual edges that separate neighbouring the $+$ and $-$ spins. A contour is a connected component. (In $d = 3$ one considers dual plaquettes; more generally, in d dimensions one considers dual $(d-1)$-dimensional cubes.)

We let Γ_Λ denote the set of contours in Λ. We let G_Λ denote the set of sets of disjoint contours. Notice that to a given spin configuration ω, there corresponds a unique set of contours $g \in G$. To a given $g \in G$, there correspond two spin configurations (they are related by spin flips). See Fig. 4.1 for an illustration in $d = 2$.

Let us introduce the weight of contours by

$$w(\gamma) = \mathrm{e}^{-\beta|\gamma|}\,, \tag{4.18}$$

where γ denote the number of dual edges (or plaquettes or hypercubes depending on d). We can express the partition function as a sum of contour configurations:

$$Z_{\Lambda,\beta} = \operatorname{Tr} \mathrm{e}^{-\beta H_\Lambda^{\text{Ising}}} = 2 \sum_{g \in G_\Lambda} \prod_{\gamma \in g} w(\gamma). \tag{4.19}$$

As for the correlation function, it satisfies

$$\langle \sigma_x^{(3)} \sigma_y^{(3)} \rangle_\Lambda = \mathbb{E}_\Lambda[\omega_x \omega_y] = 1 - 2\mathbb{P}_\Lambda(\omega_x \neq \omega_y). \tag{4.20}$$

For $\omega_x \neq \omega_y$ to hold, the must be an odd number of contours in g that separate x from y. We get an upper bound by summing over one contour that surrounds either x or y.

$$\begin{aligned} \mathbb{P}_\Lambda(\omega_x \neq \omega_y) &\leq \frac{2}{Z_{\Lambda,\beta}} \sum_{\substack{g \in G_\Lambda \\ x,y \text{ separated}}} \prod_{\gamma \in g} w(\gamma) \\ &\leq \frac{2}{Z_{\Lambda,\beta}} \Bigl(\sum_{\gamma_0 : \mathrm{Int}\gamma_0 \ni x} w(\gamma_0) \sum_{g : g \cup \{\gamma_0\} \in G_\Lambda} \prod_{\gamma \in g} w(\gamma) + [\text{same with } y] \Bigr). \end{aligned} \tag{4.21}$$

Here, $\mathrm{Int}\gamma_0$ denotes the interior of γ_0. The definition is cumbersome but intuitive when γ_0 does not touch the boundary of Λ. When it touches the boundary, we take the interior to be the smallest set of sites.

The following bound is clear and vital. For any γ_0,

$$\frac{2}{Z_{\Lambda,\beta}} \sum_{g : g \cup \{\gamma_0\} \in G_\Lambda} \prod_{\gamma \in g} w(\gamma) \leq 1. \tag{4.22}$$

We then obtain

$$\mathbb{P}_\Lambda(\omega_x \neq \omega_y) \leq \sum_{\gamma_0 : \mathrm{Int}\gamma_0 \ni x} w(\gamma_0) + \sum_{\gamma_0 : \mathrm{Int}\gamma_0 \ni y} w(\gamma_0). \tag{4.23}$$

We show that both sums above as as small as we want by taking β large, uniformly in N and in $x, y \in \Lambda_N$. We do it in a slightly more complicated way than necessary, but it will be useful when dealing with quantum spins. Let $D_x(\gamma_0)$ denote the length of a shortest path along neighbouring sites, that connects x with a site close (at distance $\frac{1}{2}$) of γ_0. Whether the contour touches the boundary of Λ_N or not, and since $x \in \mathrm{Int}\gamma_0$, we always have

$$D_x(\gamma_0) \leq |\gamma_0|. \tag{4.24}$$

Let $\delta > 1$. We then have

$$\sum_{\gamma_0 : \mathrm{Int}\gamma_0 \ni x} w(\gamma_0) \leq \sum_{\gamma_0 \in \Gamma_\Lambda} \mathrm{e}^{-(\beta - \log \delta)|\gamma_0|} \, \delta^{-D_x(\gamma_0)}. \tag{4.25}$$

This can be estimated by summing over the paths leading from x to the site that is close to the contour, and then over contours that start at this site. The sum over paths can be written as geometric series in d directions and can be bounded by $(\frac{2}{\delta-1})^d$. The number of contours starting close to a give site of length ℓ is less than c_d^ℓ where c_d is a constant that depends on the dimension; for $d = 2$ we can take $c_2 = 3$. We then get

$$\sum_{\gamma_0 \in \Gamma_\Lambda} e^{-(\beta - \log \delta)|\gamma_0|} \, \delta^{-D_x(\gamma_0)} \leq \big(\tfrac{2}{\delta-1}\big)^d \big(c_d^{-1} \, e^{\beta - \log \delta} - 1\big)^{-1}. \tag{4.26}$$

We have found that

$$\mathbb{P}_\Lambda(\omega_x \neq \omega_y) \leq 2\big(\tfrac{2}{\delta-1}\big)^d \big(c_d^{-1} \, e^{\beta - \log \delta} - 1\big)^{-1}. \tag{4.27}$$

This is indeed as small as we want by first choosing $\delta > 1$ and then taking β large enough. □

4.2.2. The Ising regime of the xxz-model. We consider the $S = \frac{1}{2}$ XXZ-model viewed as a perturbation of the Ising model H_Λ^{Ising} defined in (4.12). Namely, with $t \in [0, 1)$, we let

$$H_\Lambda^{\text{XXZ}} = -\tfrac{1}{2} \sum_{xy \in \mathcal{E}(\Lambda)} \big(t\sigma_x^{(1)}\sigma_y^{(1)} + t\sigma_x^{(2)}\sigma_y^{(2)} + \sigma_x^{(3)}\sigma_y^{(3)} - 1\big) = H_\Lambda^{\text{Ising}} + tV_\Lambda. \tag{4.28}$$

It is natural to expect the model to behave like the Ising model when t is small. Ginibre proved that the existence of long-range order when $d \geq 2$, t is small, and β is large [**17**]. Remarkably, Tom Kennedy [**21**] could extend this result to all $t \in [0, 1)$, provided β is large enough (depending on how close t is from 1).

THEOREM 4.4. *Let $d \geq 2$. Consider the model (4.28) with $t \in [0, 1)$. There exists $c > 0$ and $\beta_0(t) < \infty$ such that for all $\beta > \beta_0(t)$, all finite boxes Λ, and all $x, y \in \Lambda$, we have*

$$\langle \sigma_x^{(3)} \sigma_y^{(3)} \rangle_\Lambda \geq c. \tag{4.29}$$

It follows that the set of infinite-volume Gibbs states $\mathcal{G}_\beta$ is not a singleton: Taking $\Lambda \to \mathbb{Z}^d$ (along a subsequence if necessary so as to guarantee convergence of the state), we get a Gibbs state that does *not* have short-range correlations. It is therefore not extremal, and $\mathcal{G}_\beta$ has more than one element. An important open problem is to characterise the set of translation-invariant, extremal Gibbs states.

When $t = 1$ and $d = 2$ there is no long-range order at any positive temperature, as proved in Chapter 6. When $t = 1$ and $d \geq 3$, long-range order is expected at low temperatures but proving this is a famously open problem.

The proof of Theorem 4.4 uses an extension of the Peierls argument explained above for the classical Ising model. The method is again to express deviations from the ground-state vectors $|+\rangle$ or $|-\rangle$ in terms of contours separating + and − entries, and to show that these contours are 'costly' when β is large. The starting-point for the quantum model (4.28) is to write the Gibbs factor $e^{-\beta H_\Lambda^{\text{XXZ}}}$

as a sequence of classical Ising configurations evolving over time. We are then faced with the task of controlling an *evolving* sequence of Peierls contours.

We now describe how to express $\mathrm{e}^{-\beta H_\Lambda^{\mathrm{XXZ}}}$ in terms of evolving Ising-models. The technique, which relies on the Lie–Trotter expansion, is frequently used in studying quantum systems. We treat all dimensions $d \geq 2$ but people should have the $d = 2$ case in mind.

4.2.3. Lie–Trotter expansion. We use an expansion for the exponential of the sum of two non-commuting matrices, which will allow us to write the Gibbs factor $\mathrm{e}^{-\beta(H_\Lambda^{\mathrm{Ising}}+tV_\Lambda)}$ as a sequence of classical Ising models: For any $n \times n$ matrices a, b we have

$$\mathrm{e}^{a+b} = \lim_{N\to\infty} \Big[\mathrm{e}^{\frac{1}{N}a} \big(1 + \tfrac{1}{N} b\big) \Big]^N. \tag{4.30}$$

This is proved in Proposition A.5. We use it with $a = -\beta H_\Lambda^{\mathrm{Ising}}$ and $b = -\beta t V_\Lambda$. We obtain

$$\mathrm{e}^{-\beta H_\Lambda^{\mathrm{XXZ}}} = \lim_{N\to\infty} \Big[\mathrm{e}^{-\frac{\beta}{N} H_\Lambda^{\mathrm{Ising}}} \big(1 - \tfrac{\beta t}{N} V_\Lambda\big) \Big]^N. \tag{4.31}$$

Inserting the resolution of the identity $\mathbb{1} = \sum_{\omega_\Lambda \in \{-1,+1\}^\Lambda} |\omega_\Lambda\rangle\langle\omega_\Lambda|$ between the different factors in the product above, we get

$$\begin{aligned}
\operatorname{Tr} \mathrm{e}^{-\beta H_\Lambda^{\mathrm{XXZ}}} &= \lim_{N\to\infty} \sum_{\omega_\Lambda^{(1)},\dots,\omega_\Lambda^{(N)} \in \{-1,+1\}^\Lambda} \Big\langle \omega_\Lambda^{(1)} \Big| \mathrm{e}^{-\frac{\beta}{N} H_\Lambda^{\mathrm{Ising}}} \big(1 - \tfrac{\beta t}{N} V_\Lambda\big) \Big| \omega_\Lambda^{(2)} \Big\rangle \\
&\qquad \dots \Big\langle \omega_\Lambda^{(N)} \Big| \mathrm{e}^{-\frac{\beta}{N} H_\Lambda^{\mathrm{Ising}}} \big(1 - \tfrac{\beta t}{N} V_\Lambda\big) \Big| \omega_\Lambda^{(1)} \Big\rangle, \\
\operatorname{Tr} \sigma_x^{(3)} \sigma_y^{(3)} \mathrm{e}^{-\beta H_\Lambda^{\mathrm{XXZ}}} &= \lim_{N\to\infty} \sum_{\omega_\Lambda^{(1)},\dots,\omega_\Lambda^{(N)} \in \{-1,+1\}^\Lambda} \Big\langle \omega_\Lambda^{(1)} \Big| \omega_x^{(1)} \omega_y^{(1)} \mathrm{e}^{-\frac{\beta}{N} H_\Lambda^{\mathrm{Ising}}} \big(1 - \tfrac{\beta t}{N} V_\Lambda\big) \Big| \omega_\Lambda^{(2)} \Big\rangle \\
&\qquad \dots \Big\langle \omega_\Lambda^{(N)} \Big| \mathrm{e}^{-\frac{\beta}{N} H_\Lambda^{\mathrm{Ising}}} \big(1 - \tfrac{\beta t}{N} V_\Lambda\big) \Big| \omega_\Lambda^{(1)} \Big\rangle.
\end{aligned} \tag{4.32}$$

Let us focus on the expression for the partition function. The Ising hamiltonian is diagonal in the basis of spin configurations. Further, introducing the set of contours $g \in G_\Lambda$ exactly as in the classical model, we have

$$\Big\langle \omega_\Lambda \Big| \mathrm{e}^{-\frac{\beta}{N} H_\Lambda^{\mathrm{Ising}}} \Big| \omega_\Lambda \Big\rangle = \mathrm{e}^{\frac{\beta}{2N} \sum_{xy} (1 - \omega_x \omega_y)} = \prod_{\gamma \in g} \mathrm{e}^{-\frac{\beta}{N} |\gamma|}. \tag{4.33}$$

As for the terms involving $V_\Lambda = t \sum_{(x,y)} \sigma_x^{(+)} \sigma_y^{(-)}$, we have

$$\Big\langle \omega_\Lambda^{(i)} \Big| \big(1 - \tfrac{\beta t}{N} V_\Lambda\big) \Big| \omega_\Lambda^{(i+1)} \Big\rangle = \begin{cases} 1 & \text{if } \omega_\Lambda^{(i)} = \omega_\Lambda^{(i+1)}, \\ \frac{\beta t}{N} & \text{if } |\omega_\Lambda^{(i)}\rangle = \sigma_x^{(+)} \sigma_y^{(-)} |\omega_\Lambda^{(i+1)}\rangle \text{ for some neighbours } x, y \in \Lambda, \\ 0 & \text{otherwise.} \end{cases} \tag{4.34}$$

We obtain a "space-time" contour model that involves Peierls contours $g^{(1)}, \dots, g^{(N)} \in G_\Lambda$, $i = 1, \dots, N$, such that $g^{(i)}$ and $g^{(i+1)}$ are identical, except for one possible change corresponding to interchanging neighbouring $+$ and $-$.

4.2.4. Space-time contours and Peierls argument. We now define the connected components. We say that two Peierls contours are connected in the space-time if they share at least one edge. A space-time contour is then a collection $\gamma = (\gamma^{(1)}, \dots, \gamma^{(N)})$ of Peierls contours at times $i = 1, \dots, N$. Each $\gamma^{(i)}$ is a collection of mutually disjoint Peierls contours, but γ must be connected. Let $\mathbf{\Gamma}_\Lambda$ denote the set of space-time contours. The weight of a contour $\gamma \in \mathbf{\Gamma}_\Lambda$ is

$$w(\gamma) = \exp\Big(-\frac{\beta}{N} \sum_{i=1}^{N} |\gamma_i| \Big) \Big(\frac{\beta t}{N} \Big)^{n(\gamma)}, \tag{4.35}$$

where $n(\gamma)$ is the number of changes (corresponding to interchanging neighbouring $+$ and $-$). We recover (4.18) when $n(\gamma) = 0$ and all contours $\gamma^{(i)}$ are identical. We then get the generalisation of (4.19):

$$\operatorname{Tr} \mathrm{e}^{-\beta H_\Lambda^{\mathrm{XXZ}}} = 2 \sum_{\{\gamma_1, \dots, \gamma_k\}} \prod_{j=1}^{k} w(\gamma_j). \tag{4.36}$$

The sum is over sets of mutually disjoint space-time contours of $\mathbf{\Gamma}_\Lambda$. Notice that, for $t = 0$, the contours are constant in time and we recover the classical setting.

As in the classical case, we need to prove that the sum over contours that surround a given site is less than $\frac{1}{2}$ if β is large enough, uniformly in N and in the size of the box Λ. Let $x \in \Lambda$, and let $\gamma = (\gamma^{(1)}, \dots, \gamma^{(N)}) \in \mathbf{\Gamma}_\Lambda$ a space-time contour such that $\gamma^{(1)}$ surrounds x. Let $D_x(\gamma^{(i)})$ the minimal distance between all the connected components of $\gamma^{(i)}$ and with x. We necessarily have for each $i = 1, \dots, N$ (compare with (4.24))

$$D_x(\gamma^{(i)}) \leq |\gamma^{(i)}| + n(\gamma). \tag{4.37}$$

Let $\eta > 0$ and $\delta > 1$. We can write

$$\begin{aligned}\sum_{\gamma\in\boldsymbol{\Gamma}_\Lambda:\gamma^{(1)}\text{ surrounds }x} w(\gamma) &\leq \sum_{\gamma:\gamma^{(1)}\text{ surrounds }x} \mathrm{e}^{-\frac{\beta\eta}{N}\sum_{j=1}^N|\gamma^{(j)}|}\,\mathrm{e}^{-\frac{\beta}{N}(1-\eta)\sum_{i=1}^N|\gamma^{(i)}|}\Big(\frac{\beta t}{N}\Big)^{n(\gamma)}\\ &\leq \sum_{\gamma\in\boldsymbol{\Gamma}_\Lambda:\gamma^{(1)}\text{ surrounds }x}\frac{1}{N}\sum_{j=1}^N \mathrm{e}^{-\beta\eta|\gamma^{(j)}|}\,\mathrm{e}^{-\frac{\beta}{N}(1-\eta)\sum_{i=1}^N|\gamma^{(i)}|}\Big(\frac{\beta t}{N}\Big)^{n(\gamma)}\\ &\leq \sum_{\gamma\in\boldsymbol{\Gamma}_\Lambda}\frac{1}{N}\sum_{j=1}^N \mathrm{e}^{-(\beta\eta-\log\delta)|\gamma^{(j)}|}\,\delta^{-D_x(\gamma^{(j)})}\,\mathrm{e}^{-\frac{\beta}{N}(1-\eta)\sum_{i=1}^N|\gamma^{(i)}|}\Big(\frac{\beta t\delta}{N}\Big)^{n(\gamma)}\\ &= \sum_{\gamma^{(0)}\in G_\Lambda}\mathrm{e}^{-(\beta\eta-\log\delta)|\gamma^{(0)}|}\,\delta^{-D_x(\gamma^{(0)})}\sum_{\gamma\in\boldsymbol{\Gamma}_\Lambda:\gamma^{(1)}=\gamma^{(0)}}\mathrm{e}^{-\frac{\beta}{N}(1-\eta)\sum_{i=1}^N|\gamma^{(i)}|}\Big(\frac{\beta t\delta}{N}\Big)^{n(\gamma)}.\end{aligned} \tag{4.38}$$

The second inequality was Jensen. We now choose $\eta > 0, \delta > 1$ such that $1-\eta \geq t\delta$. Reverse expanding, we get a surprisingly easy bound for the sum over quantum contours:

$$\sum_{\gamma\in\boldsymbol{\Gamma}_\Lambda:\gamma^{(1)}=\gamma^{(0)}}\mathrm{e}^{-\frac{\beta}{N}(1-\eta)\sum_{i=1}^N|\gamma^{(i)}|}\Big(\frac{\beta t\delta}{N}\Big)^{n(\gamma)} \leq \operatorname{Tr} P_{\gamma^{(0)}}\,\mathrm{e}^{-\beta t\delta H_\Lambda^{\mathrm{XXX}}} \leq 2. \tag{4.39}$$

Here $P_{\gamma^{(0)}}$ is the projector onto the two configurations with contour $\gamma^{(0)}$. H_Λ^{XXX} is the hamiltonian (4.28) with $t = 1$. The last inequality follows from $H_\Lambda^{\mathrm{XXX}} \geq 0$.

The last step is the following estimate, which does not involve N. It generalises (4.26).

Proposition 4.5. *For any $\delta > 1$ and $\varepsilon > 0$ there is a constant β_0 (that depends on d, ε, δ but not on Λ) such that for any $\beta \geq \beta_0$, we have*

$$\sum_{\gamma^{(0)}\in G_\Lambda}\mathrm{e}^{-\beta|\gamma^{(0)}|}\,\delta^{-D_x(\gamma^{(0)})} \leq \varepsilon.$$

Sketch proof. We proceed by induction on K. Let $G_\Lambda^{(K)} \subset G_\Lambda$ denote the set of sets of contours where the number of contours is less than or equal to K. We show that for any K we have for $\beta > \beta_0$ that

$$\sum_{\gamma^{(0)}\in G_\Lambda^{(K)}}\mathrm{e}^{-\beta|\gamma^{(0)}|}\,\delta^{-D_x(\gamma^{(0)})} \leq \varepsilon. \tag{4.40}$$

When $K = 1$ we have $G_\Lambda^{(1)} = \Gamma_\Lambda$ and we get the Ising case, see Eq. (4.26). This establishes the beginning of the induction. We now prove the estimate for $K + 1$. From the site x, we first sum over the starting site y of the closest connected component, and then over $\gamma \in \Gamma_\Lambda^{(y)}$ that "starts" at y. Then we sum over departing sites $x_1, \dots, x_k$ (that are located on the path to the contour, or on the contour itself) and we sum over composites with less than K connected components. We get the bound

$$\begin{aligned}
\sum_{\gamma^{(0)} \in G_\Lambda^{(K+1)}} \mathrm{e}^{-\beta|\gamma^{(0)}|}\, \delta^{-D_x(\gamma^{(0)})} &\leq 2^d \sum_{\ell_1,\dots,\ell_d \geq 1} \delta^{-\ell_1-\dots-\ell_d} \sup_{y\in\Lambda} \sum_{\gamma\in\Gamma_\Lambda^{(y)}} \mathrm{e}^{-\beta|\gamma|} \\
&\qquad \sum_{k=0}^{K} \frac{(\ell_1+\dots+\ell_d+|\gamma|)^k}{k!} \sum_{\gamma_1,\dots\gamma_k \in G_\Lambda^{(K)}} \prod_{i=1}^{k} \mathrm{e}^{-\beta|\gamma_i|}\, \delta^{-D_{x_i}(\gamma_i)} \\
&\leq 2^d \sum_{\ell_1,\dots,\ell_d\geq 1} (\delta\,\mathrm{e}^{-\varepsilon})^{-\ell_1-\dots-\ell_d} \sup_{y\in\Lambda} \sum_{\gamma\in\Gamma_\Lambda^{(y)}} \mathrm{e}^{-(\beta-\varepsilon)|\gamma|} \\
&\leq 2^d (\delta\,\mathrm{e}^{-\varepsilon}-1)^{-d} \sup_{y\in\Lambda} \sum_{\gamma\in\Gamma_\Lambda^{(y)}} \mathrm{e}^{-(\beta-\varepsilon)|\gamma|}.
\end{aligned} \tag{4.41}$$

We can suppose that ε is small enough so that $\delta\,\mathrm{e}^{-\varepsilon} > 1$. The last term is smaller than ε provided β is large enough. □

4.3. Long-range order using infrared bounds

Using the method of reflection positivity, it is possible to obtain a bound on the Fourier transform of the correlation function, that is called "Infrared bound" because it captures the physics of large scales (infrared light has small frequency / large wavelength). We do not explain how to get the bound, this can be found in [**9, 12, 13, 2, 3**]. Here we explain how to get long-range order from this bound.

It is necessary here to work in a box with periodic boundary conditions. That is, we take $\Lambda = (\mathbb{Z}/\ell\mathbb{Z})^d$. We can think of Λ_ℓ as being the box $\{0,1,\dots,\ell-1\}^d$ where the set of edges $\mathcal{E}(\Lambda)$ contains the usual nearest-neighbours, but also edges between $(0,x_2,\dots,x_d)$ and $(\ell-1,x_2,\dots,x_d)$, and similarly in other spatial directions. The hamiltonian is H_Λ^{XYZ} defined in Eq. (1.37) with $h=0$.

Let us recall the basic formulæ about Fourier transforms of functions on Λ_ℓ. The dual of Λ_ℓ in Fourier theory is

$$\Lambda_\ell^* = \tfrac{2\pi}{\ell}\big\{-\tfrac{\ell}{2}+1,\dots,\tfrac{\ell}{2}\big\}^d \subset [-\pi,\pi]^d. \tag{4.42}$$

The Fourier transform of a function $f : \Lambda_\ell \to \mathbb{C}$ is

$$\widehat{f}(k) = \sum_{x\in\Lambda_\ell} \mathrm{e}^{-\mathrm{i}kx} f(x), \qquad k \in \Lambda_\ell^*, \tag{4.43}$$

where we write kx for the usual inner product $\sum_{i=1}^d k_i x_i$. One can check that the inverse relation is then

$$f(x) = \frac{1}{\ell^d} \sum_{k \in \Lambda_\ell^*} \mathrm{e}^{\mathrm{i}kx} \widehat{f}(k). \tag{4.44}$$

The main object here is the correlation function $\langle S_0^{(3)} S_x^{(3)} \rangle_{\Lambda_\ell,\beta}$, which we view as a function of $x \in \Lambda_\ell$. In order to state the infrared bound, let us introduce the function

$$\varepsilon(k) = 2 \sum_{i=1}^d (1 - \cos k_i), \qquad k \in \Lambda_\ell^*. \tag{4.45}$$

Notice that $\varepsilon(k) \geq 0$ and $\varepsilon(k) \approx k^2$ around $k = 0$.

THEOREM 4.6 (Infrared bound). *Assume that $\ell \in 2\mathbb{N}$ and that*

$$J^{(1)}, J^{(3)} \geq 0 \geq J^{(2)}.$$

Then we have for all $k \in \Lambda_\ell^* \setminus \{0\}$ that

$$\widehat{\langle S_0^{(3)} S_x^{(3)} \rangle}_{\Lambda_\ell,\beta}(k) \leq S \sqrt{\frac{d(J^{(1)} - J^{(2)})}{J^{(3)}}} \frac{1}{\sqrt{\varepsilon(k)}} + \frac{1}{2\beta J^{(3)} \varepsilon(k)}.$$

We refer to [**9, 12, 13**] for a proof of this important theorem, and for detailed information about the method of reflection positivity. It allows to prove the existence of long-range oder in some cases, that include models where the broken symmetry is continuous.

THEOREM 4.7. *Assume that $\ell \in 2\mathbb{N}$ and that*

$$J^{(3)} \geq J^{(1)} \geq -J^{(2)} \geq 0.$$

Then

$$\frac{1}{\ell^d} \sum_{x \in \Lambda_\ell} \langle S_0^{(3)} S_x^{(3)} \rangle_{\Lambda_\ell,\beta} \geq \tfrac{1}{3} S(S+1) - \frac{1}{\ell^d} \sum_{k \in \Lambda_\ell^* \setminus \{0\}} \left(S \sqrt{\frac{d(J^{(1)} - J^{(2)})}{J^{(3)}}} \frac{1}{\sqrt{\varepsilon(k)}} + \frac{1}{2\beta J^{(3)} \varepsilon(k)} \right). \tag{4.46}$$

As $\ell \to \infty$ the last line converges to an integral over $k \in [-\pi, \pi]^d$. The integral of $\frac{1}{\sqrt{\varepsilon(k)}}$ converges when $d \geq 2$. The integral of $\frac{1}{\varepsilon(k)}$ converges when $d \geq 3$. The lower bound is then strictly positive when $d \geq 3$ and S, β are large enough. This theorem also establishes long-range order in the ground state (i.e. $\beta \to \infty$) when $d \geq 2$.

PROOF OF THEOREM 4.7. It is not too hard to establish the following correlation inequality:

$$\langle S_0^{(3)} S_0^{(3)} \rangle_{\Lambda_\ell,\beta} \geq \tfrac{1}{3} \textstyle\sum_{i=1}^{3} \langle S_0^{(i)} S_0^{(i)} \rangle_{\Lambda_\ell,\beta} = \tfrac{1}{3} S(S+1). \tag{4.47}$$

For this we use that $J_x^{(3)} \geq J_x^{(1)} \geq -J_x^{(2)} \geq 0$.

Using the inverse Fourier transform on the two-point correlation function, we get

$$\tfrac{1}{3} S(S+1) \leq \langle S_0^{(3)} S_0^{(3)} \rangle_{\Lambda_\ell,\beta} = \frac{1}{\ell^d} \langle \widehat{S_0^{(3)} S_x^{(3)}} \rangle_{\Lambda_\ell,\beta}(0) + \frac{1}{\ell^d} \sum_{k \in \Lambda_\ell^* \setminus \{0\}} \langle \widehat{S_0^{(3)} S_x^{(3)}} \rangle_{\Lambda_\ell,\beta}(k). \tag{4.48}$$

Notice that the first term of the right side is equal to the long-range order parameter. Then

$$\frac{1}{\ell^d} \sum_{x \in \Lambda_\ell} \langle S_0^{(3)} S_x^{(3)} \rangle_{\Lambda_\ell,\beta} = \frac{1}{\ell^d} \langle \widehat{S_0^{(3)} S_x^{(3)}} \rangle_{\Lambda_\ell,\beta}(0) \geq \tfrac{1}{3} S(S+1) - \frac{1}{\ell^d} \sum_{k \in \Lambda_\ell^* \setminus \{0\}} \langle \widehat{S_0^{(3)} S_x^{(3)}} \rangle_{\Lambda_\ell,\beta}(k). \tag{4.49}$$

Invoking the infrared bound, Theorem 4.6, for the last term, we obtain Theorem 4.7. □

EXERCISE 4.1. *Recall the decomposition (4.5). Show that*

$$\sum_{j=0}^{N^2-1} \|C_j\|_2 \leq \frac{N^{3/2}}{\sqrt{2}} \|A\|_2.$$

EXERCISE 4.2.

(1) *Show that the ground states of the model (4.28) with* $t \in [0,1]$ *are the constant vectors* $|+\rangle$ *and* $|-\rangle$.

(2) *Now consider the antiferromagnetic models with hamiltonians* $-H_\Lambda^{\text{Ising}} + tV_\Lambda$ *and* $-H_\Lambda^{\text{Ising}} - tV_\Lambda$, $t \in [0,1]$. *Are the ground states given by the antiferromagnetic configurations* $|\omega\rangle$ *where* $\omega_x = (-1)^{\|x\|_1}$ *or* $\omega_x = -(-1)^{\|x\|_1}$*?*

EXERCISE 4.3. *Show that* $H_\Lambda^{\text{xxx}} \geq 0$.