CHAPTER 1

# Spin systems

## 1.1. Spin matrices

The building blocks of quantum spin systems are the *spin matrices*. We consider the algebra of $n \times n$ matrices with complex entries, and recall the notation for the *commutator*

$$[A, B] = AB - BA. \tag{1.1}$$

We will often use the convenient **Dirac notation** for the elements of a Hilbert space $\mathcal{H}$, the inner products, and projection operators. If $(e_i)_{i \in I}$ is a fixed orthonormal basis of $\mathcal{H}$, we introduce the "ket" $|i\rangle$ and the "bra" $\langle i|$:

$$\begin{aligned} &|i\rangle \equiv e_i, \\ &\langle i| \equiv e_i \in \mathcal{H}^*, \\ &|i\rangle\langle i| \equiv P_{e_i}, \text{ the orthogonal projection on } e_i \\ &\langle i|j\rangle \equiv \langle e_i, e_j\rangle. \end{aligned} \tag{1.2}$$

As we see above, the inner product is given by a bra and a ket, forming a "bracket". We also write $|i\rangle\langle j|$ for the operator such that

$$\big\langle e_k, \big(|i\rangle\langle j|\big) e_\ell \big\rangle = \langle k|i\rangle \, \langle j|\ell\rangle = \delta_{k,i}\delta_{j,\ell}. \tag{1.3}$$

Finally, the notation also involve operators, writing

$$\langle i|A|j\rangle \equiv \langle e_i, A\, e_j\rangle. \tag{1.4}$$

Notice that $A$ acts on the vector in the right by definition, even if the notation suggests symmetry (it does not matter when $A$ is hermitian).

DEFINITION 1.1. *Let $n \in \{2, 3, 4, \dots\}$.* Spin-matrices *are $n \times n$ hermitian matrices $S^{(1)}, S^{(2)}, S^{(3)}$ that satisfy the following:*

$$[S^{(1)}, S^{(2)}] = \mathrm{i}S^{(3)}, \quad [S^{(2)}, S^{(3)}] = \mathrm{i}S^{(1)}, \quad [S^{(3)}, S^{(1)}] = \mathrm{i}S^{(2)}, \tag{1.5}$$

$$(S^{(1)})^2 + (S^{(2)})^2 + (S^{(3)})^2 = \tfrac{n^2-1}{4}\mathbb{1}. \tag{1.6}$$

It is common to introduce the parameter $J = (n-1)/2 \in \frac{1}{2}\mathbb{N}$ (i.e. $n = 2J+1$); then (1.6) reads

$$(S^{(1)})^2 + (S^{(2)})^2 + (S^{(3)})^2 = J(J+1)\mathbb{1}. \tag{1.7}$$

The spin-matrices originate in the representation theory of the Lie algebra $\mathfrak{sl}_2(\mathbb{C})$, but we will not need much of the general theory. See Hall [**19**, Section 4.6] for more details about representation theory. Their relevance for quantum spins comes from the fact that they are the "infinitesimal generators" for rotations in three dimensions, and they describe the angular momentum of elementary particles. See Griffiths [**18**, Section 4.3] for the physics background.

The existence of such matrices follows by construction. Let $|a\rangle$, $a \in \{-J, -J+1, \dots, J\}$ denote an orthonormal basis of $\mathbb{C}^n$. Let $S^{(+)}, S^{(-)}$ be defined by

$$S^{(+)}|a\rangle = \sqrt{J(J+1) - a(a+1)}\ |a{+}1\rangle, \quad S^{(-)}|a\rangle = \sqrt{J(J+1) - (a-1)a}\ |a{-}1\rangle. \tag{1.8}$$

Note that $S^{(+)}|J\rangle = S^{(-)}|-J\rangle = 0$. Then define

$$S^{(1)} = \tfrac{1}{2}(S^{(+)} + S^{(-)}), \qquad S^{(2)} = \tfrac{1}{2\mathrm{i}}(S^{(+)} - S^{(-)}), \qquad S^{(3)}|a\rangle = a|a\rangle. \tag{1.9}$$

Lemma 1.2. *The matrices $S^{(1)}, S^{(2)}, S^{(3)}$ constructed above satisfy the relations (1.5) and (1.6).*

Proof. One can check the following commutation relations:

$$[S^{(3)}, S^{(+)}] = S^{(+)}, \qquad [S^{(3)}, S^{(-)}] = -S^{(-)}, \qquad [S^{(+)}, S^{(-)}] = 2S^{(3)}. \tag{1.10}$$

The relations (1.5) follow. Finally,

$$(S^{(1)})^2 + (S^{(2)})^2 + (S^{(3)})^2 = S^{(+)}S^{(-)} + [S^{(3)}]^2 - S^{(3)} = J(J+1)\mathbb{1}. \tag{1.11}$$

□

For $J = \frac{1}{2}$ ($n = 2$) we have

$$S^{(+)} = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \qquad S^{(-)} = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \tag{1.12}$$

and the choice above gives the Pauli matrices (multiplied by $\frac{1}{2}$):

$$S^{(1)} = \tfrac{1}{2}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \qquad S^{(2)} = \tfrac{1}{2}\begin{pmatrix} 0 & -\mathrm{i} \\ \mathrm{i} & 0 \end{pmatrix}, \qquad S^{(3)} = \tfrac{1}{2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \tag{1.13}$$

For $J = 1$ ($n = 3$), we get

$$S^{(+)} = \begin{pmatrix} 0 & \sqrt{2} & 0 \\ 0 & 0 & \sqrt{2} \\ 0 & 0 & 0 \end{pmatrix}, \qquad S^{(-)} = \begin{pmatrix} 0 & 0 & 0 \\ \sqrt{2} & 0 & 0 \\ 0 & \sqrt{2} & 0 \end{pmatrix}, \tag{1.14}$$

and thus

$$S^{(1)} = \frac{1}{\sqrt{2}}\begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}, \quad S^{(2)} = \frac{1}{\sqrt{2}}\begin{pmatrix} 0 & -\mathrm{i} & 0 \\ \mathrm{i} & 0 & -\mathrm{i} \\ 0 & \mathrm{i} & 0 \end{pmatrix}, \quad S^{(3)} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix}. \tag{1.15}$$

Notice that, for $J > 1$, the matrix $S^{(1)}$ is not proportional to $\delta_{|i-j|,1}$.

Spin matrices are not unique, but their spectrum is uniquely determined by the commutation relations.

LEMMA 1.3. *Assume that $S^{(1)}, S^{(2)}, S^{(3)}$ are hermitian $n \times n$ matrices that satisfy the relations (1.5) and (1.6). Then each $S^{(i)}$ has eigenvalues $\{-J, -J+1, \dots, J\}$.*

PROOF. Since the numbering $S^{(1)}, S^{(2)}, S^{(3)}$ of the matrices is arbitrary, it is enough to prove the claim for $S^{(3)}$. Define $S^{(+)} = S^{(1)} + \mathrm{i}S^{(2)}$ and $S^{(-)} = S^{(1)} - \mathrm{i}S^{(2)}$. One can check that

$$\begin{aligned} S^{(+)}S^{(-)} &= J(J+1)\mathbb{1} - [S^{(3)}]^2 + S^{(3)}, \\ S^{(-)}S^{(+)} &= J(J+1)\mathbb{1} - [S^{(3)}]^2 - S^{(3)}. \end{aligned} \tag{1.16}$$

Let $|a\rangle$ be an eigenvector of $S^{(3)}$ with eigenvalue $a$. It follows from Eq. (1.16) that

$$\begin{aligned} \big\|S^{(+)}|a\rangle\big\|^2 &= \langle a|S^{(-)}S^{(+)}|a\rangle = J(J+1) - a^2 - a \geq 0, \\ \big\|S^{(-)}|a\rangle\big\|^2 &= \langle a|S^{(+)}S^{(-)}|a\rangle = J(J+1) - a^2 + a \geq 0. \end{aligned} \tag{1.17}$$

Then $|a| \leq J$, and $S^{(+)}|a\rangle \neq 0$ if $a \neq J$. Next, observe that $[S^{(3)}, S^{(+)}] = S^{(+)}$. Then

$$S^{(3)}S^{(+)}|a\rangle = (a+1)S^{(+)}|a\rangle. \tag{1.18}$$

Then if $a \neq J$ is an eigenvalue, $a+1$ is also an eigenvalue. There are similar relations with $S^{(-)}$, so that if $a \neq -J$ is an eigenvalue, $a-1$ is also an eigenvalue. It follows that $\{-J, -J+1, \dots J\}$ is the set of eigenvalues. □

Notice that the relations (1.8) always hold for spin-matrices; this follows from (1.18) and (1.17). It follows from the parallelogram identity that $\|S^{\pm}\| = \sqrt{2}J$:

$$\begin{aligned} \|S^{(+)}\|^2 &= \tfrac14 (2\|S^{(+)}\|^2 + 2\|S^{(-)}\|^2) = \tfrac14 (\|S^{(+)} + S^{(-)}\|^2 + \|S^{(+)} - S^{(-)}\|^2) \\ &= \tfrac14 (4\|S^{(1)}\|^2 + 4\|S^{(2)}\|^2) = 2J^2. \end{aligned} \tag{1.19}$$

## 1.2. Rotation of spins and symmetries

Spin operators are related to rotations in $\mathbb{R}^3$, and the matrices $S^{(1)}, S^{(2)}, S^{(3)}$ in many ways behave like orthonormal vectors. Let $\vec{S} = (S^{(1)}, S^{(2)}, S^{(3)})$. Given $\vec{a} \in \mathbb{R}^3$, let

$$S^{\vec{a}} = \vec{a} \cdot \vec{S} = a_1 S^{(1)} + a_2 S^{(2)} + a_3 S^{(3)}. \tag{1.20}$$

By linearity, the commutation relations (1.5) generalize as

$$[S^{\vec{a}}, S^{\vec{b}}] = \mathrm{i} S^{\vec{a} \times \vec{b}}. \tag{1.21}$$

Finally, let $R_{\vec{a}}\vec{b}$ denote the vector $\vec{b}$ rotated around $\vec{a}$ by the angle $\|\vec{a}\|$. Rotations of spins are represented by conjugating with appropriate unitary matrices:

LEMMA 1.4.
$$e^{-iS^{\vec{a}}}\, S^{\vec{b}}\, e^{iS^{\vec{a}}} = S^{R_{\vec{a}}\vec{b}}.$$

PROOF. We replace $\vec{a}$ by $s\vec{a}$, and we check that both sides of the identity satisfy the same differential equation. We find
$$\frac{d}{ds}\, e^{-iS^{s\vec{a}}}\, S^{\vec{b}}\, e^{iS^{s\vec{a}}} = -i[S^{\vec{a}},\, e^{-iS^{s\vec{a}}}\, S^{\vec{b}}\, e^{iS^{s\vec{a}}}\,], \tag{1.22}$$
and
$$\frac{d}{ds} S^{R_{s\vec{a}}\vec{b}} = \Big(\frac{d}{ds} R_{s\vec{a}}\vec{b}\Big) \cdot \vec{S} = \Big(\vec{a} \times R_{s\vec{a}}\vec{b}\Big) \cdot \vec{S} = -i[S^{\vec{a}}, S^{R_{s\vec{a}}\vec{b}}]. \tag{1.23}$$
We used (1.21) for the last identity. □

It follows from Lemmas 1.3 and 1.4 that any matrix $S^{\vec{a}}$, $\vec{a} \in \mathbb{R}^3$ with $\|\vec{a}\| = 1$, has eigenvalues $\{-J, -J+1, \dots, J\}$.

COROLLARY 1.5. *Let $\psi_{\vec{b},c}$ be the eigenvector of $S^{\vec{b}}$ with eigenvalue $c$. Then $e^{-iS^{\vec{a}}}\, \psi_{\vec{b},c}$ is the eigenvector of $S^{R_{\vec{a}}\vec{b}}$ with eigenvalue $c$.*

PROOF. Using Lemma 1.4,
$$S^{R_{\vec{a}}\vec{b}}\, e^{-iS^{\vec{a}}}\, \psi_{\vec{b},c} = e^{-iS^{\vec{a}}}\, S^{\vec{b}} \psi_{\vec{b},c} = c\, e^{-iS^{\vec{a}}}\, \psi_{\vec{b},c}. \tag{1.24}$$
□

In contrast to rotations, reflections do not preserve the spin matrices since the commutation relations are destroyed (consider the example $S^{(1)} \mapsto -S^{(1)}$ with $S^{(2)}$ and $S^{(3)}$ fixed).

## 1.3. Spin systems

We now consider systems with an arbitrary finite number of spins. We describe two equivalent settings: 1. The one based on tensor products, which is popular with physicists and algebraists. 2. The one based on classical spin configurations, which is popular with probabilists.

### 1.3.1. Systems defined with tensor products.

Let $\mathcal{H}^{(1)}$ and $\mathcal{H}^{(2)}$ be two Hilbert spaces. The tensor product $\mathcal{H}^{(1)} \otimes \mathcal{H}^{(2)}$ consists of vectors $v \otimes w$ where $v \in \mathcal{H}^{(1)}$ and $w \in \mathcal{H}^{(2)}$, and of their linear combinations. The scalar multiplication rule is that
$$\lambda(v \otimes w) = \lambda v \otimes w = v \otimes \lambda w, \quad \lambda \in \mathbb{C}. \tag{1.25}$$
The inner product on $\mathcal{H}^{(1)} \otimes \mathcal{H}^{(2)}$ is constructed using the inner products on $\mathcal{H}^{(1)}$ and $\mathcal{H}^{(2)}$. If $v, v' \in \mathcal{H}^{(1)}$ and $w, w' \in \mathcal{H}^{(2)}$, we set
$$\langle v \otimes w, v' \otimes w' \rangle_{\mathcal{H}^{(1)} \otimes \mathcal{H}^{(2)}} = \langle v, v' \rangle_{\mathcal{H}^{(1)}}\, \langle w, w' \rangle_{\mathcal{H}^{(2)}}. \tag{1.26}$$

This definition extends by linearity to arbitrary vectors of $\mathcal{H}^{(1)} \otimes \mathcal{H}^{(2)}$. Finally, if $\{e_i\}$ and $\{f_j\}$ are orthonormal bases of $\mathcal{H}^{(1)}$ and $\mathcal{H}^{(2)}$ respectively, then $\{e_i \otimes f_j\}$ is an orthonormal basis of $\mathcal{H}^{(1)} \otimes \mathcal{H}^{(2)}$. This implies that the dimension of the tensor product space satisfies

$$\dim \mathcal{H}^{(1)} \otimes \mathcal{H}^{(2)} = \dim \mathcal{H}^{(1)} \cdot \dim \mathcal{H}^{(2)}. \tag{1.27}$$

The tensor product should not be confused with the direct sum, where $\dim \mathcal{H}^{(1)} \oplus \mathcal{H}^{(2)} = \dim \mathcal{H}^{(1)} + \dim \mathcal{H}^{(2)}$.

Tensor products of more than two spaces are defined similarly.

Given $\Lambda \Subset \mathbb{Z}^d$, we consider the tensor space

$$\mathcal{H}_\Lambda = \otimes_{x \in \Lambda} \mathbb{C}^n. \tag{1.28}$$

We extend the action of the spin operators to $\mathcal{H}_\Lambda$ by setting

$$S_x^{(i)} = S^{(i)} \otimes \mathbb{1}_{\Lambda \setminus \{x\}}. \tag{1.29}$$

This means that $S_x^{(i)}$ acts as $S^{(i)}$ on the vector at $x$, and as the identity on the other sites. To be more precise, let us denote the sites of $\Lambda$ by $x_1, x_2, \dots, x_{|\Lambda|}$. Let $v \in \mathcal{H}_\Lambda$ a vector of the form $v = \otimes_{j=1}^{|\Lambda|} v_{x_j}$; then

$$S_{x_j}^{(i)} v = v_{x_1} \otimes \dots \otimes v_{x_{j-1}} \otimes S^{(i)} v_{x_j} \otimes v_{x_{j+1}} \otimes \dots \otimes v_{x_{|\Lambda|}}. \tag{1.30}$$

The action of $S_x^{(i)}$ extends by linearity to general vectors of the tensor product of spaces.

A **local observable** is an operator on $\mathcal{H}_\Lambda$ for some $\Lambda \Subset \mathbb{Z}^d$. We let $\mathcal{B}_\Lambda$ denote the space of local observables in $\Lambda$. If $A$ is such an observable, it has a straightforward counterpart $A'$ in $\mathcal{H}_{\Lambda'}$ for $\Lambda \supset \Lambda$ by writing

$$A' = A \otimes \mathbb{1}_{\Lambda' \setminus \Lambda}. \tag{1.31}$$

The **support** of a local observable $A$ is the smallest set $\Lambda \Subset \mathbb{Z}^d$ such that $A \in \mathcal{B}_\Lambda$ (more precisely, one can look at all $\Lambda'$ such that $A$ has a counterpart in $\mathcal{B}_{\Lambda'}$, and take the intersection of all domains $\Lambda'$).

**1.3.2. Systems defined with classical spin configurations.** An alternative definition of $\mathcal{H}_\Lambda$ makes use the space of classical configurations. It is equivalent to the previous construction, and it has the advantage of clarifying that quantum systems are generalisations of classical spin systems (with finite spin state). Namely, given $n = 2, 3, \dots$ and $J \in \frac{1}{2}\mathbb{N}$ such that $n = 2J + 1$, let $\Omega_\Lambda$ denote the set of **classical spin configurations**, namely

$$\Omega_\Lambda = \{-J, -J+1, \dots, J\}^\Lambda. \tag{1.32}$$

We then define $\mathcal{H}_\Lambda$ to be the linear span of $\Omega_\Lambda$, that is, $v \in \mathcal{H}_\Lambda$ is a vector $v = (v_\omega)_{\omega \in \Omega_\Lambda}$, with $v_\omega \in \mathbb{C}$ for each $\omega$. The inner product is

$$\langle v, w, \rangle = \sum_{\omega \in \Omega_\Lambda} \overline{v}_\omega w_\omega. \tag{1.33}$$

A natural basis is formed by the vectors

$$|\omega\rangle = (0, \dots, 0, \underbrace{1}_{\text{at } \omega}, 0, \dots, 0). \tag{1.34}$$

The dimension of $\mathcal{H}_\Lambda$ is equal to the number of basis vectors, i.e. $n^{|\Lambda|}$.

The spin operators can be defined through their actions on spin configurations. In the case $n = 2$, we set

$$\begin{aligned} S_x^{(1)}|\omega\rangle &= \tfrac{1}{2}|\omega^{(x)}\rangle; \\ S_x^{(2)}|\omega\rangle &= \mathrm{i}\,\omega_x|\omega^{(x)}\rangle; \\ S_x^{(3)}|\omega\rangle &= \omega_x|\omega^{(x)}\rangle. \end{aligned} \tag{1.35}$$

Here, $\omega^{(x)}$ is equal to the configuration $\omega$, but with $\omega_x$ flipped. For general $n$, we can define the operators $S_x^{(\pm)}$ and $S_x^{(3)}$:

$$\begin{aligned} S_x^{(+)}|\omega\rangle &= \begin{cases} \sqrt{(J(J+1) - \omega_x(\omega_x+1)}\,|\omega + \delta_x\rangle & \text{if } \omega_x < J; \\ 0 & \text{if } \omega_x = J; \end{cases} \\ S_x^{(-)}|\omega\rangle &= \begin{cases} \sqrt{(J(J+1) - (\omega_x-1)\omega_x}\,|\omega - \delta_x\rangle & \text{if } \omega_x > -J; \\ 0 & \text{if } \omega_x = -J; \end{cases} \\ S_x^{(3)}|\omega\rangle &= \omega_x|\omega^{(x)}\rangle. \end{aligned} \tag{1.36}$$

Then we define $S_x^{(1)} = \frac{1}{2}(S_x^{(+)} + S_x^{(-)})$ and $S_x^{(2)} = \frac{1}{2\mathrm{i}}(S_x^{(+)} - S_x^{(-)})$.

## 1.4. Hamiltonians and their symmetries

The quantum spin systems of statistical mechanics, as their classical counterparts, are defined on a finite domain $\Lambda \Subset \mathbb{Z}^d$. We write $\mathcal{E}(\Lambda)$ for the set of nearest-neighbour edges in $\Lambda$. The Hilbert space is $\mathcal{H}_\Lambda$ defined in (1.28).

We consider the following family of hamiltonians, that depends on parameters $J^{(1)}, J^{(2)}, J^{(3)}, h \in \mathbb{R}$.

**XYZ hamiltonian:**

$$H_{\Lambda,h} = -\sum_{xy \in \mathcal{E}(\Lambda)} \big( J^{(1)} S_x^{(1)} S_y^{(1)} + J^{(2)} S_x^{(2)} S_y^{(2)} + J^{(3)} S_x^{(3)} S_y^{(3)} \big) - h \sum_{x \in \Lambda} S_x^{(3)}. \tag{1.37}$$

The symmetries in the system are important, since the phase transitions are often associated to symmetry breaking. Given a unitary matrix $U$ in $\mathbb{C}^n$, we consider the following tensored matrix on $\mathcal{H}_\Lambda$:

$$U_\Lambda = \otimes_{x \in \Lambda} U. \tag{1.38}$$

The hamiltonian is invariant under the symmetry $U$ if $[U_\Lambda, H_{\Lambda,h}] = 0$, or equivalently $H_{\Lambda,h} = U_\Lambda^{-1} H_{\Lambda,h} U_\Lambda$.

Let us discuss special cases of the model (1.37) and the corresponding symmetries.

- The case $J^{(1)} = J^{(2)} \neq J^{(3)}$ is the XXZ-model. For $h = 0$ it is invariant under rotations of the circle, i.e. the group SO(2) represented by the unitaries $U = \mathrm{e}^{\mathrm{i}\alpha S^{(3)}}$ with $\alpha \in [0, 2\pi]$. It is also invariant under the 'spin-flip' $S^{(3)} \mapsto -S^{(3)}$ (represented by $U = \mathrm{e}^{\mathrm{i}\pi S_x^{(1)}}$), thus it has SO(2)$\times\mathbb{Z}_2$-symmetry.
- The case $J^{(1)} = J^{(2)} = J^{(3)}$ is the Heisenberg (or XXX-) model. For $h = 0$ it is invariant under rotations of the sphere, i.e. SO(3). Indeed, it is invariant under $U = \mathrm{e}^{\mathrm{i}S^{(\vec{a})}}$ for any $\vec{a} \in \mathbb{R}^3$.
- The case $J^{(1)} = J^{(2)} = 0$ is the classical Ising model. For $h = 0$ it is invariant under the discrete group $\mathbb{Z}_2$. The XXZ-model also has this symmetry.
- The case $J^{(2)} = J^{(3)} = 0$ is known as the quantum Ising model. It is equivalent to a classical Ising model in $d + 1$ dimensions (one dimension being continuous), which allows to prove many properties. It is invariant under the discrete group $\mathbb{Z}_2$.

This family of interactions also has lattice symmetries (lattice translations and rotations). Lattice translations can be broken in 'antiferromagnetic' models with negative coupling constants.

## 1.5. Gibbs states and correlation functions

Given the hamiltonian $H_{\Lambda,h}$, the corresponding finite-volume Gibbs states in domain $\Lambda \Subset \mathbb{Z}^d$ at inverse temperature $\beta$ is the state $\langle\cdot\rangle_{\Lambda,\beta,h} : \mathcal{B}(\mathcal{H}_\Lambda) \to \mathbb{C}$ defined as

$$\langle A\rangle_{\Lambda,\beta,h} = \frac{1}{Z_{\Lambda,\beta,h}} \mathrm{Tr}\, A\, \mathrm{e}^{-\beta H_{\Lambda,h}}\,. \tag{1.39}$$

The trace is in the Hilbert space $\mathcal{H}_\Lambda$ and the normalisation $Z_{\Lambda,\beta,h}$ is the partition function, $Z_{\Lambda,\beta,h} = \mathrm{Tr}\, \mathrm{e}^{-\beta H_{\Lambda,h}}$. This definition is similar to the classical case, with free boundary conditions. (Boundary conditions are notoriously tricky in quantum systems.) Let $M_\Lambda$ be the magnetisation operator in 3rd direction of spins:

$$M_\Lambda = \sum_{x\in\Lambda} S_x^{(3)}. \tag{1.40}$$

The average magnetisation is given by

$$m_{\Lambda,\beta,h} = \Big\langle \frac{1}{|\Lambda|} M_\Lambda \Big\rangle_{\Lambda,\beta,h}. \tag{1.41}$$

The **spontaneous magnetisation** is then given by

$$m^*(\beta) = \lim_{h\downarrow 0} \lim_{\Lambda\Uparrow\mathbb{Z}^d} m_{\Lambda,\beta,h}. \tag{1.42}$$

Notice that the existence of these limits is not mathematically obvious (one could use lim inf instead). The order of limits is important, since we usually have by symmetries that $m_{\Lambda,\beta,h=0} = 0$. We see below that $m^*(\beta)$ is zero when $\beta$ is small but that it is positive in some cases when $\beta$ is large.

The two-point correlation functions between sites 0 and $x$ are given by

$$\langle S_0^{(i)} S_x^{(i)} \rangle_{\Lambda,\beta,h} = \frac{1}{Z_{\Lambda,\beta,h}} \operatorname{Tr}\left( S_0^{(i)} S_x^{(i)} \, \mathrm{e}^{-\beta H_{\Lambda,h}} \right). \tag{1.43}$$

We also consider the states $\langle \cdot \rangle^{\text{per}}_{\Lambda_\ell,\beta,h}$ with periodic boundary conditions, where we use $H^{\text{per}}_{\Lambda_\ell,h}$ instead of $H_{\Lambda_\ell,h}$. Often the system has *short range correlations* in the sense that

$$\langle S_0^{(i)} S_x^{(i)} \rangle_{\Lambda,\beta,h} \approx \langle S_0^{(i)} \rangle_{\Lambda,\beta,h} \, \langle S_0^{(i)} \rangle_{\Lambda,\beta,h}, \tag{1.44}$$

as $x$ is far from the origin. This happens e.g. at high temperatures, when $\beta$ is small. At low temperatures the system may exhibit long-range correlations, or have the following property of long-range order.

DEFINITION 1.6. *The system exhibits* **long-range order** *if there exists a sequence of domains $\Lambda_n$, where either $\Lambda_n \Uparrow \mathbb{Z}^d$, or $\Lambda_n = \{1, \dots, m_n\}^d_{\text{per}}$ with $m_n \to \infty$, such that*

$$\frac{1}{|\Lambda_n|^2} \sum_{x,y \in \Lambda_n} \langle S_x^{(3)} S_y^{(3)} \rangle_{\Lambda_n,\beta,0} \geq c > 0,$$

*for all $n$.*

## 1.6. Phase diagrams of ferromagnetic models

We review the phase diagrams of the XXZ hamiltonians with nonnegative coupling constants and for dimensions $d \geq 2$. Parts of the phase diagrams are proved, parts still lack a mathematically rigorous proof.

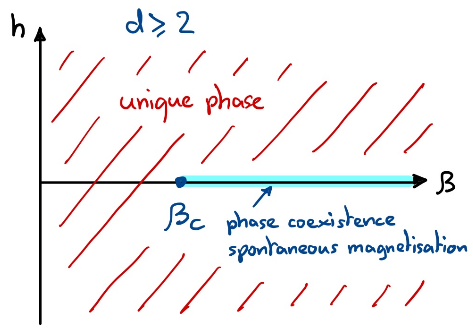


FIGURE 1.1. Phase diagram of the **Ising model**, and of the **XXZ model** for $J^{(3)} > J^{(1)} = J^{(2)} \geq 0$, for all dimensions $d \geq 2$.

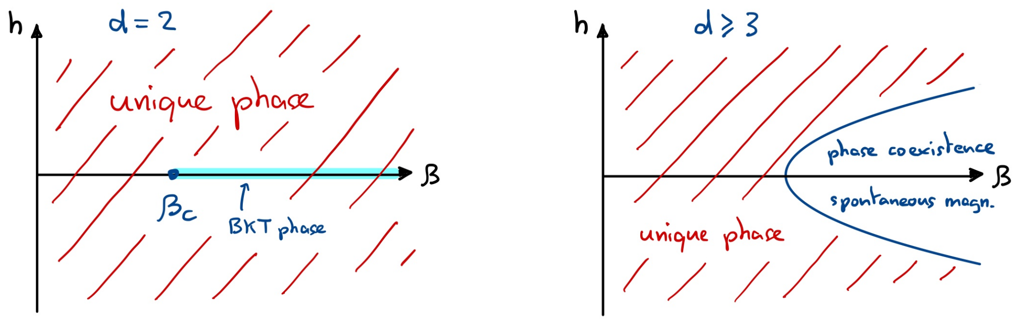


FIGURE 1.2. Phase diagrams of the **XXZ model** for $J^{(1)} = J^{(2)} > |J^{(3)}|$. The BKT phase is the Berezinsky-Kosterlitz-Thouless phase where the two point correlation function has power-law decay (in the unique phase, decay is exponential).

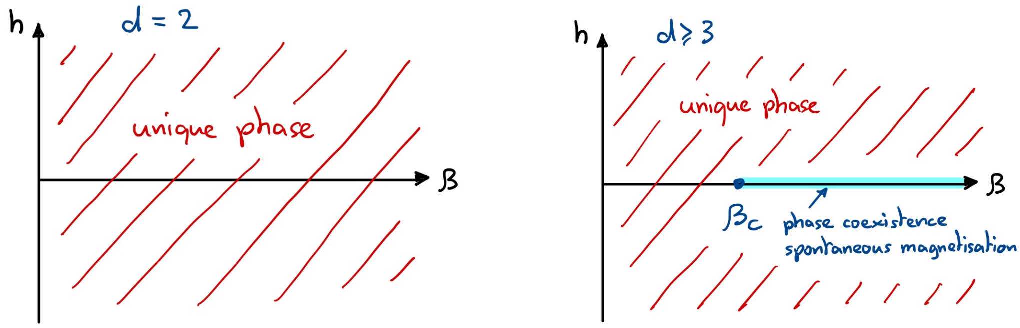


FIGURE 1.3. Phase diagrams of the **Heisenberg** or **XXX model** for which $J^{(1)} = J^{(2)} = J^{(3)} \geq 0$.

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## 1.7. Exercises

EXERCISE 1.1. *Show the following properties, using the results of this chapter or direct calculations.*

(a) *Let* $\vec{a} = \frac{\pi}{2}(1,0,0)$*; the unitary* $U = \mathrm{e}^{\mathrm{i}\frac{\pi}{2}S^{(1)}}$ *then maps* $S^{(1)}$ *to itself,* $S^{(2)}$ *to* $S^{(3)}$*, and* $S^{(3)}$ *to* $-S^{(2)}$*.*

(b) *Describe the cases* $\vec{a} = \frac{\pi}{2}(0,1,0)$ *and* $\vec{a} = \frac{\pi}{2}(0,0,1)$*.*

(c) *Let* $\vec{a} = \frac{2\pi}{3\sqrt{3}}(1,1,1)$*; the unitary* $U = \mathrm{e}^{\mathrm{i}S^{(\vec{a})}}$ *then maps* $S^{(1)} \mapsto S^{(2)} \mapsto S^{(3)} \mapsto S^{(1)}$*.*

(d) *Check that*

$$\mathrm{e}^{-\mathrm{i}aS^{(3)}}\,S^{(+)}\,\mathrm{e}^{\mathrm{i}aS^{(3)}} = \mathrm{e}^{-\mathrm{i}a}\,S^{(+)},$$
$$\mathrm{e}^{-\mathrm{i}aS^{(3)}}\,S^{(-)}\,\mathrm{e}^{\mathrm{i}aS^{(3)}} = \mathrm{e}^{\mathrm{i}a}\,S^{(-)}.$$

EXERCISE 1.2. *Let $\mathcal{H}^{(1)}$ and $\mathcal{H}^{(2)}$ be two Hilbert spaces of dimensions $n_1$ and $n_2$, respectively. Find an element of the tensor space $\mathcal{H}^{(1)} \otimes \mathcal{H}^{(2)}$ that cannot be written $v \otimes w$ with $v \in \mathcal{H}^{(1)}$, $w \in \mathcal{H}^{(2)}$. Can you find such a vector for any choice of $n_1, n_2$?*

EXERCISE 1.3. *Check that the map $\mathcal{H}^{(1)} \otimes \mathcal{H}^{(2)} \to \mathbb{C}$ defined in Eq. (1.26) satisfies all the properties of an inner product.*

EXERCISE 1.4. *Check that the spins operators defined in (1.35) and (1.36) satisfy the relations of Definition 1.1.*

EXERCISE 1.5. *For $\ell \in \{1, 2, \dots\}$, let $\mathcal{H}_\ell$ denote the space of homogeneous degree $\ell$ polynomials $p(x_1, x_2, x_3)$ in 3 variables (with complex coefficients) satisfying $\Delta p = 0$. Check that the differential operators*

$$L^{(1)} := \tfrac{1}{\mathrm{i}}\big(x_2\tfrac{\partial}{\partial x_3} - x_3\tfrac{\partial}{\partial x_2}\big), \quad L^{(2)} := \tfrac{1}{\mathrm{i}}\big(x_3\tfrac{\partial}{\partial x_1} - x_1\tfrac{\partial}{\partial x_3}\big), \quad L^{(3)} := \tfrac{1}{\mathrm{i}}\big(x_1\tfrac{\partial}{\partial x_2} - x_2\tfrac{\partial}{\partial x_1}\big)$$

*on $\mathcal{H}_\ell$ are spin operators, i.e. satisfy Definition 1.1, with spin $J = \ell$. Also check that $(x_1 + \mathrm{i}x_2)^\ell$ is an eigenvector for $L^{(3)}$ with eigenvalue $\ell$ (i.e. a highest weight vector) and for the case $\ell = 1$ compute eigenvectors for the remaining eigenvalues.*

EXERCISE 1.6. *Calculate $[H_{\Lambda,h}, M_\Lambda]$ where $H_{\Lambda,h}$ is the* XYZ*-Hamiltonian defined in (1.37) and $M_\Lambda$ is the magnetisation operator defined in (1.40). When does the commutator vanish?*

EXERCISE 1.7. *Consider the* XXX*-Heisenberg interaction on $(\mathbb{C}^n)^{\otimes\{x,y\}}$:*

$$\vec{S}_x \cdot \vec{S}_y = S_x^{(1)} S_y^{(1)} + S_x^{(2)} S_y^{(2)} + S_x^{(3)} S_y^{(3)}.$$

(a) *In the case $n = 2$ (spin $\frac{1}{2}$) show that*

$$\vec{S}_x \cdot \vec{S}_y = \tfrac{1}{2}T - \tfrac{1}{4},$$

*where $T$ is the* transposition operator *defined by $T|a, b\rangle = |b, a\rangle$.*

(b) *Still in the case $n = 2$, show that $\vec{S}_x \cdot \vec{S}_y \le \frac{1}{4}$, i.e. that $\frac{1}{4}\mathbb{1} - \vec{S}_x \cdot \vec{S}_y$ is non-negative definite.*

(c) *Now consider the case $n = 3$ (spin 1). Write the transposition operator $T$ as a polynomial in $\vec{S}_x \cdot \vec{S}_y$.*

EXERCISE 1.8. *Fix a basis $\mathbf{e}_1, \dots, \mathbf{e}_n$ for $\mathbb{C}^n$ and consider the tensor product $\mathbb{C}^n \otimes \mathbb{C}^n$. Define a vector*

$$\psi = \frac{1}{\sqrt{n}} \sum_{j=1}^{n} \mathbf{e}_j \otimes \mathbf{e}_j.$$

*(a) Let $A$ be any orthogonal matrix, $A^{-1} = A^{\intercal}$. Show that $(A \otimes A)\psi = \psi$. Deduce that $(A \otimes A)$ commutes with the projector $Q = |\psi\rangle\langle\psi|$.*

*(b) Now let $n = 2$ and consider the following subspace of $(\mathbb{C}^2)^{\otimes 2}$:*

$$\mathcal{H}_1 = \mathrm{span}\{|+,-\rangle - |-,+\rangle\}.$$

*Find a unitary matrix $U$ acting on $(\mathbb{C}^2)^{\otimes 2}$ such that $U^*QU$ is the projection onto the subspace $\mathcal{H}_1$.*