

APPENDIX B

Solutions to some exercises

B.1. Spin systems

EXERCISE 1.1 Unitaries act on operators by conjugation ($S \mapsto U^* S U$), so we appeal to Lemma 1.4.

- (a) $S^{(1)}, S^{(2)}, S^{(3)}$ are the same as $S^{e_1}, S^{e_2}, S^{e_3}$ where the e_i are the standard basis of \mathbb{R}^3 . Rotation around e_1 by $\frac{\pi}{2} = 90^\circ$ indeed has the effect described.
- (b) Rotation around $\vec{a} = \frac{\pi}{2}(0, 1, 0)$ maps $S^{(1)} \mapsto -S^{(3)}, S^{(3)} \mapsto S^{(1)}, S^{(2)} \mapsto S^{(2)}$. Rotation around $\vec{a} = \frac{\pi}{2}(0, 0, 1)$ maps $S^{(1)} \mapsto S^{(2)}, S^{(2)} \mapsto -S^{(1)}, S^{(3)} \mapsto S^{(3)}$.
- (c) Since $\|\vec{a}\| = 2\pi/3$ this is indeed the case.
- (d) $U = e^{iaS^{(3)}}$ acts by rotation of the 1, 2-plane by angle a , so we can use the rotation matrix:

$$\begin{pmatrix} \cos a & -\sin a \\ \sin a & \cos a \end{pmatrix}.$$

We have

$$\begin{aligned} e^{-iaS^{(3)}} S^{(+)} e^{iaS^{(3)}} &= e^{-iaS^{(3)}} (S^{(1)} + iS^{(2)}) e^{iaS^{(3)}} \\ &= (\cos a - i \sin a) S^{(1)} + (\sin a + i \cos a) S^{(2)} \\ &= (\cos(-a) + i \sin(-a)) S^{(1)} + i(\cos(-a) + i \sin(-a)) S^{(2)} = e^{-ia} S^{(+)}. \end{aligned}$$

The calculation for $S^{(-)}$ is similar.

EXERCISE 1.2 One can find such vectors provided $n_1, n_2 \geq 2$ (if one of them, say $\mathcal{H}^{(1)}$, is one-dimensional then it is $= \mathbb{C}$ so $\mathcal{H}^{(1)} \otimes \mathcal{H}^{(2)} = \mathcal{H}^{(2)}$ and tensoring with a scalar is just multiplication with that scalar). Assuming $n_1, n_2 \geq 2$, take a basis e_1, e_2, \dots for $\mathcal{H}^{(1)}$ and a basis f_1, f_2, \dots for $\mathcal{H}^{(2)}$, and consider the vector

$$x = e_1 \otimes f_2 + e_2 \otimes f_1.$$

Assuming we could write $x = v \otimes w$, expand $v = \sum_{i \geq 1} a_i e_i$ and $w = \sum_{i \geq 1} b_i f_i$, then by multilinearity $x = \sum_{i,j \geq 1} a_i b_j e_i \otimes f_j$. The definition of x requires that $a_1 b_2 = a_2 b_1 = 1 \neq 0$ (so $a_1, b_1, a_2, b_2 \neq 0$) but also $a_1 b_1 = a_2 b_2 = 0$, which is a contradiction.

EXERCISE 1.3 Perhaps this is mainly an exercise in remembering the definition of an inner product: we need $\langle x, y \rangle = \overline{\langle y, x \rangle}$, $\langle \alpha x + \beta y, z \rangle = \overline{\alpha} \langle x, z \rangle + \overline{\beta} \langle y, z \rangle$, and $\langle x, x \rangle \geq 0$ with $= 0$ if and only if $x = 0$. All these properties are inherited for the tensor product.

EXERCISE 1.4 Since (1.35) is the special case $J = \frac{1}{2}$ of (1.36), it suffices to do the latter. Expanding the commutator we get

$$[S^{(1)}, S^{(2)}] = \frac{1}{2i} [S^{(-)}, S^{(+)}].$$

We also get

$$(S^{(1)})^2 + (S^{(2)})^2 + (S^{(3)})^2 = \frac{1}{2} (S^{(+)} S^{(-)} + S^{(-)} S^{(+)}) + (S^{(3)})^2$$

so we should consider the products $S^{(+)}S^{(-)}$ and $S^{(-)}S^{(+)}$. We get:

$$S^{(-)}S^{(+)}|\omega\rangle = \begin{cases} 0 & \text{if } \omega_x = J, \\ (J(J+1) - \omega_x(\omega_x + 1))|\omega\rangle, & \text{if } \omega_x < J \end{cases}$$

$$S^{(+)}S^{(-)}|\omega\rangle = \begin{cases} 0 & \text{if } \omega_x = -J, \\ (J(J+1) - \omega_x(\omega_x - 1))|\omega\rangle, & \text{if } \omega_x > -J \end{cases}$$

Checking all the cases we get

$$[S^{(1)}, S^{(2)}]|\omega\rangle = \frac{1}{2i}[S^{(-)}, S^{(+)}]|\omega\rangle = \frac{1}{2i}(-2\omega_x)|\omega\rangle = iS^{(3)}|\omega\rangle,$$

as required. One should similarly check the other commutation relations. As to the Casimir operator, we have

$$(S^{(-)}S^{(+)} + S^{(+)}S^{(-)})|\omega\rangle = \begin{cases} 2J|\omega\rangle & \text{if } \omega_x = \pm J, \\ 2(J(J+1) - \omega_x^2)|\omega\rangle, & \text{otherwise} \end{cases}$$

which in all cases gives

$$((S^{(1)})^2 + (S^{(2)})^2 + (S^{(3)})^2)|\omega\rangle = (\frac{1}{2}(S^{(+)}S^{(-)} + S^{(-)}S^{(+)}) + (S^{(3)})^2)|\omega\rangle = J(J+1)|\omega\rangle,$$

as required.

EXERCISE 1.5 The basic fact we need for computation is that

$$x_i \frac{\partial}{\partial x_j} x_k \frac{\partial}{\partial x_l} = \begin{cases} x_i x_k \frac{\partial^2}{\partial x_j \partial x_l} & \text{if } j \neq k, \\ x_i \left(\frac{\partial}{\partial x_l} + x_j \frac{\partial^2}{\partial x_j \partial x_l} \right) & \text{if } j = k. \end{cases}$$

With this one may check that

$$L^{(1)}L^{(2)} = (-1) \left[x_2 \left(\frac{\partial}{\partial x_1} + x_3 \frac{\partial^2}{\partial x_1 x_3} \right) - x_1 x_2 \frac{\partial^2}{\partial x_3^2} - x_3^2 \frac{\partial^2}{\partial x_1 x_2} + x_1 x_3 \frac{\partial^2}{\partial x_2 x_3} \right]$$

$$L^{(2)}L^{(1)} = (-1) \left[x_1 \left(\frac{\partial}{\partial x_2} + x_3 \frac{\partial^2}{\partial x_2 x_3} \right) - x_1 x_2 \frac{\partial^2}{\partial x_3^2} - x_3^2 \frac{\partial^2}{\partial x_1 x_2} + x_2 x_3 \frac{\partial^2}{\partial x_1 x_3} \right]$$

so the commutator

$$[L^{(1)}, L^{(2)}] = (-1) \left[x_2 \frac{\partial}{\partial x_1} - x_1 \frac{\partial}{\partial x_2} \right] = iL^{(3)}.$$

Similarly, the other commutation relations hold. We also need to check the Casimir operator: somewhat lengthy calculations give

$$\sum_{i=1}^3 (L^{(i)})^2 = (-1) \left[x_1^2 \left(\frac{\partial^2}{\partial x_2^2} + \frac{\partial^2}{\partial x_3^2} \right) + x_2^2 \left(\frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_3^2} \right) + x_3^2 \left(\frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2} \right) \right. \\ \left. - 2 \left(x_1 x_2 \frac{\partial^2}{\partial x_1 x_2} + x_1 \frac{\partial}{\partial x_1} + x_1 x_3 \frac{\partial^2}{\partial x_1 x_3} + x_3 \frac{\partial}{\partial x_3} + x_2 x_3 \frac{\partial^2}{\partial x_2 x_3} + x_2 \frac{\partial}{\partial x_2} \right) \right].$$

Using that the Laplacian is 0 on \mathcal{H}_ℓ we can write this as

$$x_1^2 \frac{\partial^2}{\partial x_1^2} + x_2^2 \frac{\partial^2}{\partial x_2^2} + x_3^2 \frac{\partial^2}{\partial x_3^2} + 2 \left(x_1 x_2 \frac{\partial^2}{\partial x_1 x_2} + x_1 \frac{\partial}{\partial x_1} + x_1 x_3 \frac{\partial^2}{\partial x_1 x_3} + x_3 \frac{\partial}{\partial x_3} + x_2 x_3 \frac{\partial^2}{\partial x_2 x_3} + x_2 \frac{\partial}{\partial x_2} \right),$$

which in turn we can recognize as

$$\left(1 + \sum_{i=1}^3 x_i \frac{\partial}{\partial x_i} \right) \left(\sum_{i=1}^3 x_i \frac{\partial}{\partial x_i} \right).$$

Since the operator $\sum_{i=1}^3 x_i \frac{\partial}{\partial x_i}$ acts as multiplication by ℓ on degree ℓ monomials, the last display is indeed $\ell(\ell+1)\mathbb{1}$ on \mathcal{H}_ℓ .

Checking that $(x_1 + ix_2)^\ell$ is a highest weight vector is straightforward, just apply $L^{(3)}$. To construct the other eigenvectors for $L^{(3)}$, we can use the lowering operator:

$$L^{(-)} = L^{(1)} - iL^{(2)} = x_1 \frac{\partial}{\partial x_3} - x_3 \frac{\partial}{\partial x_1} - ix_2 \frac{\partial}{\partial x_3} + ix_3 \frac{\partial}{\partial x_2}.$$

Then $(L^{(-)})^k (x_1 + ix_2)^\ell$ will give eigenvectors for all eigenvalues. (Since we checked that the L 's are spin operators this is guaranteed to work, see Lemma 1.3.) For $\ell = 1$ it is easy to compute that

$$L^{(-)}(x_1 + ix_2) = -2x_3, \quad L^{(-)}x_3 = x_1 - ix_2$$

are such eigenvectors (not normalized, indeed we did not define a norm on \mathcal{H}_ℓ).

EXERCISE 1.6 First expand M_Λ :

$$[H_{\Lambda,h}, M_\Lambda] = \sum_{x \in \Lambda} [H_{\Lambda,h}, S_x^{(3)}].$$

Since $S_x^{(3)}$ commutes with the external-field term, and with any term not involving the site x , we get

$$[H_{\Lambda,h}, S_x^{(3)}] = - \sum_{y \sim x} [J^{(1)} S_x^{(1)} S_y^{(1)} + J^{(2)} S_x^{(2)} S_y^{(2)} + J^{(3)} S_x^{(3)} S_y^{(3)}, S_x^{(3)}]$$

where the sum is over all sites y neighbouring x . Since $S_x^{(3)}$ commutes with itself and using the commutation relations,

$$\begin{aligned} [H_{\Lambda,h}, S_x^{(3)}] &= - \sum_{y \sim x} J^{(1)} [S_x^{(1)}, S_x^{(3)}] S_y^{(1)} + J^{(2)} [S_x^{(2)}, S_x^{(3)}] S_y^{(2)} \\ &= -i \sum_{y \sim x} (J^{(2)} S_x^{(1)} S_y^{(2)} - J^{(1)} S_x^{(2)} S_y^{(1)}). \end{aligned}$$

Summing over x we get

$$[H_{\Lambda,h}, M_\Lambda] = i \sum_{xy \in \mathcal{E}(\Lambda)} (J^{(1)} - J^{(2)}) (S_x^{(1)} S_y^{(2)} + S_x^{(2)} S_y^{(1)})$$

where the sum is over the edges. The commutator vanishes if and only if $J^{(1)} = J^{(2)}$.

EXERCISE 1.7

- (a) To verify that $\vec{S}_x \cdot \vec{S}_y = \frac{1}{2}T - \frac{1}{4}$, check it on the product basis $|\pm, \mp\rangle$. It is convenient to work with $4\vec{S}_x \cdot \vec{S}_y = \vec{\sigma}_x \cdot \vec{\sigma}_y$. Expanding $\vec{\sigma}_x \cdot \vec{\sigma}_y + \mathbb{1} = \sigma^{(1)} \otimes \sigma^{(1)} + \sigma^{(2)} \otimes \sigma^{(2)} + \sigma^{(3)} \otimes \sigma^{(3)} + \mathbb{1}$ we can compute:

$$(\vec{\sigma}_x \cdot \vec{\sigma}_y + \mathbb{1}) \begin{Bmatrix} |+, +\rangle \\ |+, -\rangle \\ |-, +\rangle \\ |-, -\rangle \end{Bmatrix} = \begin{Bmatrix} |-, -\rangle + i^2 |-, -\rangle + |+, +\rangle + |+, +\rangle \\ |-, +\rangle + i(-i) |-, +\rangle - |+, -\rangle + |+, -\rangle \\ |+, -\rangle + i(-i) |+, -\rangle - |-, +\rangle + |-, +\rangle \\ |+, +\rangle + (-i)^2 |-, -\rangle + |-, -\rangle + |-, -\rangle \end{Bmatrix} = \begin{Bmatrix} 2|+, +\rangle \\ 2|-, +\rangle \\ 2|+, -\rangle \\ 2|-, -\rangle \end{Bmatrix}$$

Thus $\vec{\sigma}_x \cdot \vec{\sigma}_y + \mathbb{1} = 2T$ as claimed.

Another way to see it is to start from $S^{(+)}$ and $S^{(-)}$: we have

$$(S_x^{(+)} S_y^{(-)} + S_x^{(-)} S_y^{(+)}) \begin{Bmatrix} |+, +\rangle \\ |+, -\rangle \\ |-, +\rangle \\ |-, -\rangle \end{Bmatrix} = \begin{Bmatrix} 0 \\ |-, +\rangle \\ |+, -\rangle \\ 0 \end{Bmatrix}$$

so $S_x^{(+)}S_y^{(-)} + S_x^{(-)}S_y^{(+)}$ is ‘almost’ the transposition-operator. At the same time

$$(4S_x^{(3)}S_y^{(3)} + \mathbb{1}) \begin{cases} |+, +\rangle \\ |+, -\rangle \\ |-, +\rangle \\ |-, -\rangle \end{cases} = \begin{cases} 2|+, +\rangle \\ 0 \\ 0 \\ 2|-, -\rangle \end{cases}$$

so $4S_x^{(3)}S_y^{(3)} + \mathbb{1}$ acts like a scalar perpendicular to $S_x^{(+)}S_y^{(-)} + S_x^{(-)}S_y^{(+)}$. Combining these, we see that

$$2(S_x^{(+)}S_y^{(-)} + S_x^{(-)}S_y^{(+)}) + (4S_x^{(3)}S_y^{(3)} + \mathbb{1}) = 2T,$$

which after expanding $S^{(\pm)}$ in $S^{(1)}$ and $S^{(2)}$ gives the claim.

- (b) Use $\frac{1}{4} - \vec{S}_x \cdot \vec{S}_y = \frac{1}{2}(\mathbb{1} - T)$. Since $T^2 = \mathbb{1}$, the eigenvalues of T are ± 1 . Then $\frac{1}{2}(\mathbb{1} - T)$ is non-negative since its eigenvalues are 0 and 1.
- (c) We have $T = (\vec{S}_x \cdot \vec{S}_y)^2 + \vec{S}_x \cdot \vec{S}_y - \mathbb{1}$. This can be verified through brute force.

EXERCISE 1.8

- (a) We compute

$$\begin{aligned} (A \otimes A)\psi &= \sum_{j=1}^n (A\mathbf{e}_j) \otimes (A\mathbf{e}_j) = \sum_{j=1}^n \left(\sum_{k=1}^n A_{k,j}\mathbf{e}_k \right) \otimes \left(\sum_{\ell=1}^n A_{\ell,j}\mathbf{e}_\ell \right) \\ &= \sum_{k,\ell=1}^n \left(\sum_{j=1}^n A_{k,j}A_{\ell,j} \right) \mathbf{e}_k \otimes \mathbf{e}_\ell = \sum_{k,\ell=1}^n \left(\sum_{j=1}^n A_{k,j}A_{j,\ell}^T \right) \mathbf{e}_k \otimes \mathbf{e}_\ell \\ &= \sum_{k,\ell=1}^n \delta_{k,\ell} \mathbf{e}_k \otimes \mathbf{e}_\ell = \psi. \end{aligned}$$

Any vector φ can be decomposed as $\varphi = a\psi + \psi'$ with $a \in \mathbb{C}$ and $\langle \psi, \psi' \rangle = 0$. Then (using the shorthand A for $A \otimes A$) we have $\langle \psi, A\psi' \rangle = \langle A^*\psi, \psi' \rangle$ and $A^*\psi = A^{-1}\psi = \overline{A^{-1}\psi} = \psi$ since $A^{-1}\psi = \psi$ and $\overline{\psi} = \psi$. Then $AQ\varphi = A(a\psi) = a\psi$ while $QA\varphi = Q(a\psi) + Q(A\psi') = a\psi$ also so Q and A commute.

- (b) Write $\psi' = \frac{1}{\sqrt{2}}(|+, -\rangle - |-, +\rangle)$. Then $\psi' = (\sigma^{(3)} \otimes \mathbb{1})\psi$. So $U = \sigma^{(3)} \otimes \mathbb{1} = U^*$ works.

B.2. Fermionic and bosonic systems

EXERCISE 2.2: Recall that

$$P_- e_{x_1, \dots, x_n} = \frac{1}{n!} \sum_{\sigma \in \mathfrak{S}_n} (-1)^\sigma e_{x_{\sigma(1)}, \dots, x_{\sigma(n)}}.$$

We use the change of variables $\sigma \mapsto \sigma'$ where $\sigma' = \sigma \circ \tau_{i,j}$ with $\tau_{i,j}$ the transposition of i and j . The sum over σ can be replaced by a sum over σ' . We have $(-1)^\sigma = -(-1)^{\sigma'}$ and $e_{x_{\sigma(1)}, \dots, x_{\sigma(n)}} = e_{x_{\sigma'(1)}, \dots, x_{\sigma'(n)}}$. We obtain

$$P_- e_{x_1, \dots, x_n} = \frac{1}{n!} \sum_{\sigma' \in \mathfrak{S}_n} -(-1)^{\sigma'} e_{x_{\sigma'(1)}, \dots, x_{\sigma'(n)}} = -P_- e_{x_1, \dots, x_n},$$

so that $P_- e_{x_1, \dots, x_n} = 0$.

EXERCISE 2.3: We have

$$\begin{aligned} \|P_+ e_{x_1, \dots, x_n}\|^2 &= \langle P_+ e_{x_1, \dots, x_n} | P_+ e_{x_1, \dots, x_n} \rangle = \frac{1}{n!} \sum_{\sigma \in \mathfrak{S}_n} \langle e_{x_{\sigma(1)}, \dots, x_{\sigma(n)}} | e_{x_1, \dots, x_n} \rangle \\ &= \frac{1}{n!} \#\{\sigma \in \mathfrak{S}_n : (x_{\sigma(1)}, \dots, x_{\sigma(n)}) = (x_1, \dots, x_n)\} = \frac{1}{n!} \prod_{x \in \Lambda} n_x!. \end{aligned}$$

The case of $P_- e_{x_1, \dots, x_n}$ is similar and actually simpler. Since all the sites x_i are distinct (otherwise the vector is 0 by the previous exercise), the permutation must be the identity.

EXERCISE 2.4: The cardinality of $\mathcal{N}_{\Lambda, n}^{(+)}$ is equal to the number of integer partitions with $|\Lambda|$ elements and total length n . Such partitions can be obtained by selecting $|\Lambda| - 1$ separations in the interval $\{1, 2, \dots, n + |\Lambda|\}$, so we get $\binom{n + |\Lambda| - 1}{|\Lambda| - 1}$.

The case of $\mathcal{N}_{\Lambda, n}$ is immediate.

EXERCISE 2.8: The first two parts are straightforward. For the third part, use that

$$\mathbb{E}((X_\ell - \rho_c)^2) = \mathbb{E}((X_\ell - \mathbb{E}(X_\ell))^2) + (\mathbb{E}(X_\ell^2) - \rho_c^2), \quad (\text{B.1})$$

where the second term goes to zero and the first term is the variance of X_ℓ . By properties of the variance of geometric random variables we get

$$\mathbb{E}((X_\ell - \mathbb{E}(X_\ell))^2) = \frac{1}{\ell^{2d}} \sum_{k \in \Lambda^* \setminus \{0\}} \frac{e^{-\beta \varepsilon(k)}}{(1 - e^{-\beta \varepsilon(k)})}. \quad (\text{B.2})$$

Here

$$\frac{1}{\ell^d} \sum_{k \in \Lambda^* \setminus \{0\}} \frac{e^{-\beta \varepsilon(k)}}{(1 - e^{-\beta \varepsilon(k)})} \quad (\text{B.3})$$

converges to a Riemann integral, so due to the additional factor $1/\ell^d$ in front, we see that $\mathbb{E}((X_\ell - \mathbb{E}(X_\ell))^2) \rightarrow 0$.

For the next part, Markov's inequality gives, for any $\delta > 0$,

$$\mathbb{P}(X_\ell \leq \rho_c + \delta) \geq 1 - \frac{\mathbb{E}[(X_\ell - \rho_c)^2]}{\delta^2} \rightarrow 1. \quad (\text{B.4})$$

We have

$$\mathbb{E}[X_\ell | X_\ell \leq \rho_c + \delta] - \rho_c = \frac{\mathbb{E}[X_\ell - \rho_c]}{\mathbb{P}(X_\ell \leq \rho_c + \delta)} - \frac{\mathbb{E}[(X_\ell - \rho_c) \mathbb{1}\{X_\ell > \rho_c + \delta\}]}{\mathbb{P}(X_\ell \leq \rho_c + \delta)}. \quad (\text{B.5})$$

The first term goes to zero. The second term satisfies

$$\begin{aligned} 0 &\leq \frac{\mathbb{E}[(X_\ell - \rho_c) \mathbb{1}\{X_\ell > \rho_c + \delta\}]}{\mathbb{P}(X_\ell \leq \rho_c + \delta)} \leq \frac{\mathbb{E}[(X_\ell - \rho_c)^2]^{1/2} \mathbb{P}(X_\ell > \rho_c + \delta)^{1/2}}{\mathbb{P}(X_\ell \leq \rho_c + \delta)} \\ &\leq \frac{\mathbb{E}[(X_\ell - \rho_c)^2]^{1/2}}{\mathbb{P}(X_\ell \leq \rho_c + \delta)} \rightarrow 0, \end{aligned} \quad (\text{B.6})$$

by the Cauchy-Schwarz inequality. Thus $\mathbb{E}[X_\ell | X_\ell \leq \rho] \rightarrow \rho_c$ whenever $\rho > \rho_c$.

The computation of $\Lambda(t)$ is similar to the convergence of the pressure (??): we get

$$\Lambda(t) = \begin{cases} \int_{[-\pi, \pi]^d} dk \log \frac{1 - e^{t - \beta \varepsilon(k)}}{1 - e^{-\beta \varepsilon(k)}}, & \text{if } t \leq 0, \\ -\infty & \text{otherwise.} \end{cases} \quad (\text{B.7})$$

The resulting formula for $\lim_{\ell \rightarrow \infty} -\frac{1}{\ell^d} \log \mathbb{P}(X_\ell \leq x)$ is a version of Cramér's Theorem, see e.g. [?]. We then get for any $\rho \leq \rho_c$ and any $\delta > 0$ that

$$\begin{aligned} \mathbb{E}[X_\ell \mid X_\ell \leq \rho] &\geq \mathbb{E}[X_\ell \mathbb{1}\{X_\ell \geq \rho - \delta\} \mid X_\ell \leq \rho] \geq (\rho - \delta) \mathbb{P}(X_\ell \geq \rho - \delta \mid X_\ell \leq \rho) \\ &= (\rho - \delta)(1 - \mathbb{P}(X_\ell < \rho - \delta \mid X_\ell \leq \rho)) \\ &= (\rho - \delta) \left(1 - \frac{\mathbb{P}(X_\ell < \rho - \delta)}{\mathbb{P}(X_\ell \leq \rho)}\right) \\ &\geq (\rho - \delta) \left(1 - \frac{\mathbb{P}(X_\ell < \rho - \delta)}{\mathbb{P}(X_\ell < \rho - \delta/2)}\right). \end{aligned} \tag{B.8}$$

Now

$$-\lim_{\ell \rightarrow \infty} \frac{1}{\ell^d} \log \frac{\mathbb{P}(X_\ell < \rho - \delta)}{\mathbb{P}(X_\ell < \rho - \delta/2)} = \Lambda^*(\rho - \delta) - \Lambda^*(\rho - \delta/2) > 0, \tag{B.9}$$

the last inequality holding since $\Lambda^*(x)$ is strictly monotone for $x < \rho_c$. We conclude that

$$\frac{\mathbb{P}(X_\ell < \rho - \delta)}{\mathbb{P}(X_\ell < \rho - \delta/2)} \rightarrow 0. \tag{B.10}$$

Thus $\lim_{\ell \rightarrow \infty} \mathbb{E}[X_\ell \mid X_\ell \leq \rho] \geq \rho - \delta$ whenever $\rho \leq \rho_c$. Since $\delta > 0$ was arbitrary, $\lim_{\ell \rightarrow \infty} \mathbb{E}[X_\ell \mid X_\ell \leq \rho] \geq \rho$ whenever $\rho \leq \rho_c$.

EXERCISE 2.14: In addition to the operator $U_{x,\sigma}^{\text{ph}}$, let us introduce $\tilde{U}_{x,\sigma}^{\text{ph}} = c_{x,\sigma} - c_{x,\sigma}^*$. We then have

$$(\tilde{U}_{x,\sigma}^{\text{ph}})^{-1} c_{x,\sigma} \tilde{U}_{x,\sigma}^{\text{ph}} = -c_{x,\sigma}^*, \quad (\tilde{U}_{x,\sigma}^{\text{ph}})^{-1} c_{x,\sigma}^* \tilde{U}_{x,\sigma}^{\text{ph}} = -c_{x,\sigma}.$$

Let $U_\Lambda = \prod_{x \in \Lambda_A} (U_{x,\uparrow}^{\text{ph}} U_{x,\downarrow}^{\text{ph}}) \prod_{x \in \Lambda_B} (\tilde{U}_{x,\uparrow}^{\text{ph}} \tilde{U}_{x,\downarrow}^{\text{ph}})$. One can check that $U_\Lambda^{-1} T_\Lambda U_\Lambda = T_\Lambda$ and that $U_\Lambda^{-1} n_{x,\sigma} U_\Lambda = 1 - n_{x,\sigma}$. Then $U_\Lambda^{-1} H_\Lambda^{(0)} U_\Lambda = H_\Lambda^{(0)}$. The hamiltonian is then invariant, so that

$$\langle n_x \rangle_{\Lambda,\beta} = \langle U_\Lambda^{-1} n_x U_\Lambda \rangle_{\Lambda,\beta} = 2 - \langle n_x \rangle_{\Lambda,\beta}.$$

Then $\langle n_x \rangle_{\Lambda,\beta} = 1$.

B.3. Equilibrium states

EXERCISE 3.1 (a) We define $B = A + A^*$ and $C = i(A - A^*)$. (b) We have

$$\|AB\| = \sup_{v \in \mathcal{H}} \frac{\|A(Bv)\|}{\|Bv\|} \frac{\|Bv\|}{\|v\|} \leq \sup_{v \in \mathcal{H}} \frac{\|A(Bv)\|}{\|Bv\|} \sup_{v \in \mathcal{H}} \frac{\|Bv\|}{\|v\|} \leq \|A\| \|B\|.$$

EXERCISE 3.2

- (a) Consider $\langle (A + \mathbb{1})^* (A + \mathbb{1}) \rangle$ and $\langle (A + i\mathbb{1})^* (A + i\mathbb{1}) \rangle$. Both are non-negative, in particular real, and expanding gives identities for the real and imaginary parts of $\langle A \rangle$ and $\langle A^* \rangle$.
(b) Let $t, \theta \in \mathbb{R}$. Then, using the previous part,

$$0 \leq \langle (A + t e^{i\theta} B)^* (A + t e^{i\theta} B) \rangle = \langle A^* A \rangle + t(e^{i\theta} \langle A^* B \rangle + \overline{e^{i\theta} \langle A^* B \rangle}) + t^2 \langle B^* B \rangle.$$

Choose θ so that $e^{i\theta} \langle A^* B \rangle \in \mathbb{R}$, then this gives

$$\langle A^* A \rangle + 2t e^{i\theta} \langle A^* B \rangle + t^2 \langle B^* B \rangle \geq 0, \quad \text{for all } t \in \mathbb{R}.$$

Then the discriminant is ≤ 0 , i.e.

$$0 \geq (2e^{i\theta} \langle A^* B \rangle)^2 - 4 \langle A^* A \rangle \langle B^* B \rangle = 4 |\langle A^* B \rangle|^2 - 4 \langle A^* A \rangle \langle B^* B \rangle$$

as claimed.

(c) We have

$$|\langle A \rangle|^2 = |\langle \mathbb{1}^* A \rangle|^2 \leq \langle \mathbb{1}^* \mathbb{1} \rangle \langle A^* A \rangle = \langle A^* A \rangle.$$

Now assume $\|A\| = 1$. Note that $\mathbb{1} - A^* A \geq 0$, because for any $v \in \mathcal{H}$:

$$\langle v, (\mathbb{1} - A^* A)v \rangle = \|v\|^2 - \|Av\|^2 \geq \|v\|^2 - \|v\|^2 = 0.$$

Then

$$0 \leq \langle \mathbb{1} - A^* A \rangle = 1 - \langle A^* A \rangle,$$

so for all A with $\|A\| = 1$ we have

$$|\langle A \rangle|^2 \leq \langle A^* A \rangle \leq 1.$$

I.e. $\|\langle \cdot \rangle\| \leq 1$. For the opposite inequality, take $A = \mathbb{1}$: then $\|\langle \cdot \rangle\| \geq |\langle \mathbb{1} \rangle| = 1$.

EXERCISE 3.3

- (a) This is immediate from $\langle v \otimes w, v \otimes w \rangle = \langle v, v \rangle \langle w, w \rangle$.
 (b) One inequality is quick:

$$\|A \otimes B\| \geq \sup_{\substack{v \in \mathcal{H}_1, w \in \mathcal{H}_2 \\ \|v\| = \|w\| = 1}} \|(A \otimes B)(v \otimes w)\| = \|A\| \|B\|.$$

For the opposite inequality, one option is to first use

$$\|A \otimes B\| = \|(A \otimes \mathbb{1})(\mathbb{1} \otimes B)\| \leq \|A \otimes \mathbb{1}\| \|\mathbb{1} \otimes B\|,$$

which means it suffices to prove $\|A \otimes \mathbb{1}\| = \|A\|$. Expanding using orthonormal bases we get

$$\|A \otimes \mathbb{1}\|^2 = \sup \sum_{i,j} |\lambda_{i,j}|^2 \langle v_i \otimes w_j | A^* A \otimes \mathbb{1} | v_i \otimes w_j \rangle$$

where the supremum is over $\sum_{i,j} |\lambda_{i,j}|^2 = 1$. Then write

$$\sum_{i,j} |\lambda_{i,j}|^2 \langle v_i \otimes w_j | A^* A \otimes \mathbb{1} | v_i \otimes w_j \rangle = \sum_i |\mu_i|^2 \langle v_i | A^* A | v_i \rangle$$

where $|\mu_i|^2 = \sum_j |\lambda_{i,j}|^2$. We can reach the supremum over $\sum_i |\mu_i|^2 = 1$ using μ of the above form, which gives $\|A \otimes \mathbb{1}\|^2 = \|A\|^2$ as claimed.

EXERCISE 3.4

- (a) We can write out X explicitly in terms of a_1, a_2, a_3 :

$$X = \frac{1}{2} \begin{pmatrix} 1 + a_3 & a_1 - ia_2 \\ a_1 + ia_2 & 1 - a_3 \end{pmatrix}.$$

Then we see that

$$\det(X - \lambda \mathbb{1}) = \lambda^2 - \lambda - \frac{1}{4}(1 - \|\vec{a}\|^2).$$

Thus the eigenvalues are

$$\lambda = \frac{1 \pm \|\vec{a}\|}{2}$$

which means that $X \geq 0$ if and only if $\|\vec{a}\| \leq 1$.

- (b) It is clear from the explicit form of X that any Hermitian matrix can be written in this form.

(c) A short computation shows that

$$XY = \frac{1}{4}((1 + \vec{a} \cdot \vec{b})\mathbb{1} + (\vec{a} + \vec{b} + \vec{a} \times \vec{b}) \cdot \vec{\sigma}).$$

Thus

$$\text{Tr } XY = \frac{1}{2}(1 + \vec{a} \cdot \vec{b}).$$

Note that $x_1 = \frac{1+\|\vec{a}\|}{2}$, $x_2 = \frac{1-\|\vec{a}\|}{2}$, $y_1 = \frac{1+\|\vec{b}\|}{2}$, $y_2 = \frac{1-\|\vec{b}\|}{2}$. Thus

$$x_1 y_1 + x_2 y_2 = \frac{1}{2}(1 + \|\vec{a}\|\|\vec{b}\|), \quad x_1 y_2 + x_2 y_1 = \frac{1}{2}(1 - \|\vec{a}\|\|\vec{b}\|).$$

The claimed inequality is

$$\frac{1}{2}(1 - \|\vec{a}\|\|\vec{b}\|) \leq \frac{1}{2}(1 + \vec{a} \cdot \vec{b}) \leq \frac{1}{2}(1 + \|\vec{a}\|\|\vec{b}\|),$$

which holds since $\vec{a} \cdot \vec{b} = \|\vec{a}\|\|\vec{b}\| \cos \theta$ and $-1 \leq \cos \theta \leq 1$.

EXERCISE 3.5 We check the positivity by computing:

$$\text{Tr } A^* A e^{-\beta H} = \text{Tr } e^{-\frac{1}{2}\beta H} A^* A e^{-\frac{1}{2}\beta H} = \text{Tr } (A e^{-\frac{1}{2}\beta H})^* A e^{-\frac{1}{2}\beta H} \geq 0,$$

since the trace of a positive semidefinite operator is non-negative.

EXERCISE 3.6 Let H and A be fixed self-adjoint operators and consider the function $f : s \mapsto F(H + sA)$. It is concave and the derivative at $s = 0$ is equal to $\langle A \rangle_{H,\beta}$, which shows that $F_\beta(H + A) \leq F_\beta(A) + \langle A \rangle_{H,\beta}$. Uniqueness follows from the fact that the function is differentiable.

EXERCISE 3.7 We first check that the minimiser of $\mathcal{F}_\beta(\cdot)$ is in the interior of the set of density operators. Indeed, let ρ belong to its boundary. Then its kernel has positive dimension and there exists a density operator ρ' that lives in the kernel. Since $\rho \perp \rho'$ we can check that

$$\mathcal{F}_\beta((1 - \varepsilon)\rho + \varepsilon\rho') = (1 - \varepsilon)\mathcal{F}_\beta(\rho) + \varepsilon\mathcal{F}_\beta(\rho') + \frac{1}{\beta}(1 - \varepsilon)\log(1 - \varepsilon) + \frac{1}{\beta}\varepsilon\log\varepsilon. \quad (\text{B.11})$$

It is clear that $\varepsilon = 0$ is not a minimum, as the last term is negative and stronger than linear.

We now know that any minimiser ρ_0 is in the interior of the set of density operators. Further, for any operator η such that $\text{Tr } \eta = 0$, the stationary condition is

$$0 = \left. \frac{d}{ds} \mathcal{F}_\beta(\rho_0 + s\eta) \right|_{s=0} = \text{Tr } \eta(H + \frac{1}{\beta} \log \rho_0). \quad (\text{B.12})$$

It follows that $H + \frac{1}{\beta} \log \rho_0$ is proportional to the identity, so that $\rho_0 = \text{const } e^{-\beta H}$ is the only stationary point (so it is the minimiser). The constant is $1/\text{Tr } e^{-\beta H}$ in order for ρ_0 to be a density operator.

Although this is redundant, one can check that

$$\left. \frac{d^2}{ds^2} \mathcal{F}_\beta(\rho_0 + s\eta) \right|_{s=0} = \frac{1}{2\beta} \text{Tr } \eta \rho_0^{-1} \eta \geq 0, \quad (\text{B.13})$$

which confirms that ρ_0 is a minimiser.

EXERCISE 3.8 If $\langle \cdot \rangle = \langle \cdot \rangle_{H,\beta}$ is the Gibbs state, for $s \in \mathbb{R}$ define the inner product

$$A, B \mapsto (A, B)_s = \langle \alpha_{-is}(A^*) B \rangle = \langle A^* \alpha_{is}(B) \rangle \quad (\text{B.14})$$

and introduce the function $f(s)$ on $[0, \beta]$ by

$$f(s) = (A, A) = \langle \alpha_{-is}(A^*) A \rangle = \langle A^* \alpha_{is}(A) \rangle. \quad (\text{B.15})$$

We then have

$$\begin{aligned} f'(s) &= -\langle A^*[H, \alpha_{is}(A)] \rangle = -\langle \alpha_{-is}(A^*)[H, A] \rangle. \\ f''(s) &= \langle [\alpha_{-is}(A^*), H][H, A] \rangle = \langle [\alpha_{-\frac{1}{2}is}(A^*), H][H, \alpha_{\frac{1}{2}is}(A)] \rangle. \end{aligned} \quad (\text{B.16})$$

The last expression shows that $f''(s) \geq 0$ so that f is convex. Using the Cauchy-Schwarz inequality of the inner product on $f'(s) = -(A, [H, A])$, we find that

$$f'(s)^2 \leq (A, A) ([H, A], [H, A]) = f(s)f''(s). \quad (\text{B.17})$$

It follows that $\log f$ is convex. Then $(\log f)'(0) \leq \frac{1}{\beta}(\log f(\beta) - \log f(0))$. We get the RAS inequality since $f(0) = \langle A^*A \rangle$, $f(\beta) = \langle AA^* \rangle$ and $f'(0) = -\langle A^*[H, A] \rangle$.

For the other direction, take $A = 1 + tB$, then

$$\langle A^*[H, A] \rangle = t\langle [H, B] \rangle + t^2\langle B^*[H, B] \rangle$$

and

$$\log \frac{\langle A^*A \rangle}{\langle AA^* \rangle} = \log \frac{1 + t\langle B + B^* \rangle + t^2\langle B^*B \rangle}{1 + t\langle B + B^* \rangle + t^2\langle BB^* \rangle} = O(t^2).$$

So if $\langle \cdot \rangle$ satisfies the RAS-inequality, since the right side is $O(t^2)$ it follows that the density matrix ρ satisfies

$$\langle [H, B] \rangle = \text{Tr } B[\rho, H] = 0.$$

This being true for all B , we must have

$$[\rho, H] = 0.$$

Since they commute, they have a common eigenbasis e_i :

$$\rho|e_i\rangle = \rho_i|e_i\rangle, \quad H|e_i\rangle = h_i|e_i\rangle.$$

Now use the RAS-inequality again for the off-diagonal matrices $A = |e_i\rangle\langle e_j|$, with $i \neq j$:

$$(h_i - h_j)\rho_j \geq \frac{1}{\beta}\rho_j \log \frac{\rho_j}{\rho_i}.$$

(A consequence is that all $\rho_i > 0$.) Taking exponentials we get

$$e^{\beta h_i} \rho_i \geq e^{\beta h_j} \rho_j \quad \text{for all } i \neq j.$$

Since i and j were arbitrary indices, it follows that

$$e^{\beta h_i} \rho_i = e^{\beta h_j} \rho_j \quad \text{for all } i \neq j.$$

Then $\rho = c e^{-\beta H}$ for some c , which is in turn fixed by the normalization.

EXERCISE 3.9 We use the function $\mathcal{F}_\beta(A) = \text{Tr}_\Lambda A H_\Lambda^\Phi + \frac{1}{\beta} \text{Tr}_\Lambda A \log A$ from the Proposition. To use the result, we need to plug in a density matrix for Tr_Λ , which in the notation of Theorem 3.14 can be $\rho_\Lambda / \dim \mathcal{H}_\Lambda$. We get

$$\mathcal{F}_\beta(\rho_\Lambda / \dim \mathcal{H}_\Lambda) \geq \mathcal{F}_\beta(e^{-\beta H_\Lambda^\Phi} / \text{Tr}_\Lambda e^{-\beta H_\Lambda^\Phi}).$$

Reorganizing this and using that

$$\rho\left(\frac{1}{|\Lambda|} H_\Lambda^\Phi\right) \rightarrow \rho(A_\Phi)$$

due to translation-invariance, the result follows.

EXERCISE 3.10 Clearly tr is a state, and cyclicity is precisely the KMS condition at $\beta = 0$. Now take $\Lambda \Subset \mathbb{Z}^d$ and assume that $\langle AB \rangle = \langle BA \rangle$ for all $A, B \in \mathcal{A}_\Lambda$. Let ρ_Λ be the density matrix for the restriction to \mathcal{A}_Λ . Taking $A = |i\rangle\langle j|$ and $B = |k\rangle\langle \ell|$ for various combinations of

i, j, k, ℓ shows that $\rho_\Lambda(i, j) = 0$ for $i \neq j$ and $\rho_\Lambda(i, i) = \rho_\Lambda(j, j)$ for all i, j . Thus $\rho_\Lambda = \mathbb{1}$ for all $\Lambda \in \mathbb{Z}^d$ which means that $\langle \cdot \rangle = \text{tr} \cdot$.

EXERCISE 3.11: It is clear that if the KMS condition holds for any observables A, B , then it holds for A, A^* . To prove the converse, consider $A, B \in \tilde{\mathcal{A}}$. We have

$$\langle (A^* + B)(A + B^*) \rangle = \langle (A + B^*)\alpha_{i\beta}(A^* + B) \rangle. \quad (\text{B.18})$$

Expanding and simplifying, we get

$$\langle A^*B^* \rangle + \langle BA \rangle = \langle B^*\alpha_{i\beta}(A^*) \rangle + \langle A\alpha_{i\beta}(B) \rangle. \quad (\text{B.19})$$

Repeating with $A + iB^*$ we get

$$\langle A^*B^* \rangle - \langle BA \rangle = \langle B^*\alpha_{i\beta}(A^*) \rangle - \langle A\alpha_{i\beta}(B) \rangle. \quad (\text{B.20})$$

Then $\langle BA \rangle = \langle A\alpha_{i\beta}(B) \rangle$ indeed.

EXERCISE 3.12 Let $B \in \tilde{\mathcal{A}}$ and observe that the complex function $\langle \alpha_z^\Phi(B) \rangle$ is entire and bounded in the strip $0 \leq \text{Im} z \leq \beta$. The function $F_{\mathbb{1}, B}(z)$ in the KMS condition (b) must be equal to $\langle \alpha_z^\Phi(B) \rangle$. Further, the KMS condition states that $F_{\mathbb{1}, B}(t + i\beta) = F_{\mathbb{1}, B}(t)$, so this function is periodic in the imaginary direction. Then $F_{\mathbb{1}, B}(z)$ is bounded in the whole complex plane, and is therefore constant by Liouville's theorem. Then $\langle \alpha_t^\Phi(B) \rangle = F_{\mathbb{1}, B}(t)$ is constant.

EXERCISE 3.13 Following the hint, let A be local, say $A \in \mathcal{A}_\Lambda$, and use the positive semi-definite square root of B_n i.e. $B_n = \sqrt{B_n}\sqrt{B_n}$ with $\sqrt{B_n} \geq 0$. We have $[H^\Phi, A] = [H_\Lambda^\Phi, A]$ so for n large enough B_n commutes with all terms in the commutator, and then from the RAS condition for $\langle \cdot \rangle$:

$$\begin{aligned} \langle A^*[H_\Lambda^\Phi, A]B_n \rangle &= \langle (A\sqrt{B_n})^*[H_\Lambda^\Phi, A\sqrt{B_n}] \rangle \\ &\geq \frac{1}{\beta} \langle (A\sqrt{B_n})^*A\sqrt{B_n} \rangle \log \frac{\langle (A\sqrt{B_n})^*A\sqrt{B_n} \rangle}{\langle A\sqrt{B_n}(A\sqrt{B_n})^* \rangle} \\ &= \frac{1}{\beta} \langle A^*AB_n \rangle \log \frac{\langle A^*AB_n \rangle}{\langle AA^*B_n \rangle} \end{aligned}$$

which gives the RAS-inequality for $\langle \cdot \rangle'$.

EXERCISE 3.14 Here you can use whichever characterization of Gibbs states you prefer – but since the question does not assume translation-invariance, a complete solution would check the KMS or RAS conditions. Checking the RAS-condition is quite convenient: thanks to the invariance of Φ , RAS for $\langle \cdot \rangle'$ is the inequality

$$\langle (U^*AU)^*[H_\Lambda^\Phi, (U^*AU)] \rangle \geq \frac{1}{\beta} \langle (U^*AU)^*(U^*AU) \rangle \log \frac{\langle (U^*AU)^*(U^*AU) \rangle}{\langle (U^*AU)(U^*AU)^* \rangle}$$

which indeed holds since $\langle \cdot \rangle$ satisfies RAS. (For translation-invariant states, the variational characterization is also easy to check.)

EXERCISE 3.15: Let Λ_n, Λ'_n be two sequences that tend to \mathbb{Z}^d . We define the sequence Λ''_m inductively by choosing $\Lambda''_1 = \Lambda_1$, and

- for m even: choose $\Lambda''_m = \Lambda'_n$ with n large enough so that $\Lambda''_m \supset \Lambda''_{m-1}$;
- for m odd: choose $\Lambda''_m = \Lambda_n$ with n large enough so that $\Lambda''_m \supset \Lambda''_{m-1}$.

Then Λ''_m and Λ_n have a common subsequence. Since a_Λ converges along Λ''_m and along Λ_n , the limits must be the same. Same with Λ''_m and Λ'_n .

B.4. Uniqueness and non-uniqueness of Gibbs states

EXERCISE 4.1: We work in \mathcal{A}_Λ . Summing over an orthonormal basis $\{|i\rangle\}$ of \mathcal{H}_Λ , we have

$$\|AB\|_2^2 = \text{tr } A^*B^*BA = \sum_i \langle i|A^*B^*BA|i\rangle \leq \|B\|^2 \sum_i \langle i|A^*A|i\rangle = \|A\|_2^2 \|B\|^2. \quad (\text{B.21})$$

B.5. Mean-field systems

EXERCISE 5.1: We have

$$H_\rho = -\vec{a} \cdot \vec{S} - \frac{1}{4} \mathbb{1}. \quad (\text{B.22})$$

Then

$$\rho H_\rho = -\frac{1}{8} \mathbb{1} - \frac{3}{4} \vec{a} \cdot \vec{S} - (\vec{a} \cdot \vec{S})^2, \quad \text{Tr } \rho H_\rho = -\frac{1}{4} - \frac{\|\vec{a}\|^2}{2}.$$

Further, we saw in a previous exercise that ρ has eigenvalues $\frac{1+\|\vec{a}\|}{2}$, $\frac{1-\|\vec{a}\|}{2}$, so

$$\text{Tr } \rho \log \rho = \frac{1+\|\vec{a}\|}{2} \log \frac{1+\|\vec{a}\|}{2} + \frac{1-\|\vec{a}\|}{2} \log \frac{1-\|\vec{a}\|}{2}.$$

Thus

$$f(\rho) = -\frac{1}{4} - \frac{1}{2} \|\vec{a}\|^2 + \frac{1}{\beta} \left(\frac{1+\|\vec{a}\|}{2} \log \frac{1+\|\vec{a}\|}{2} + \frac{1-\|\vec{a}\|}{2} \log \frac{1-\|\vec{a}\|}{2} \right).$$

With the parameterization $t = \frac{1-\|\vec{a}\|}{2}$ this becomes

$$f(\rho) = 2t(1-t) - \frac{3}{4} + \frac{1}{\beta} (t \log t + (1-t) \log(1-t)).$$

Note that, with $x = \|\vec{a}\|$ we have

$$f'(t) = 2 - 4t + \frac{1}{\beta} \log \frac{t}{1-t} = 2x + \frac{1}{\beta} \log \frac{1-x}{1+x}.$$

Thus

$$f'(t) = 0 \Leftrightarrow e^{2\beta x} = \frac{1+x}{1-x} \Leftrightarrow x = \tanh(\beta x).$$

Then for $\beta > 1$, we see that $f(\rho)$ is minimized for any \vec{a} such that $x = \|\vec{a}\|$ is a positive solution. In particular, the minimizers have $\text{SO}(3)$ -symmetry.

EXERCISE 5.2: Consider the case when $b_3 \geq b_2 \geq b_1 \geq 0$ and

$$\Phi_{\{x,y\}} = -(b_1 S_x^{(1)} S_y^{(1)} + b_2 S_x^{(2)} S_y^{(2)} + b_3 S_x^{(3)} S_y^{(3)}).$$

We again parameterize ρ as $\rho = \frac{1}{2} \mathbb{1} + \vec{a} \cdot \vec{S}$, and arguing as for (B.22) we obtain

$$H_\rho = -\sum_{j=1}^3 a_j b_j S^{(j)} = -\vec{a}\vec{b} \cdot \vec{S}, \quad (\text{B.23})$$

where we set $\vec{a}\vec{b} = (a_1 b_1, a_2 b_2, a_3 b_3)$. The mean-field equation is then

$$\rho = \frac{1}{2} \mathbb{1} + \vec{a} \cdot \vec{S} = \frac{e^{2\beta \vec{a}\vec{b} \cdot \vec{S}}}{\text{Tr } e^{2\beta \vec{a}\vec{b} \cdot \vec{S}}}. \quad (\text{B.24})$$

Now we recall that the mean-field equation is necessary but not sufficient. Among the solutions \vec{a} , we should select those such that ρ minimizes $f(\rho)$. Using (B.24) we get that

$$f(\rho) = -\frac{1}{\beta} \log \text{Tr } e^{2\beta \vec{a}\vec{b} \cdot \vec{S}} = -\frac{1}{\beta} \log \cosh(\beta \|\vec{a}\vec{b}\|). \quad (\text{B.25})$$

Since $\cosh(\cdot)$ is increasing, setting $x = \|\vec{a}\|$ we see that $\vec{a} = x \mathbf{e}_3$ gives the best solution, as well as any rotation of $x \mathbf{e}_3$ such that $\vec{a}\vec{b}$ has the same length.

In particular, in the XXZ-case $\Delta = b_3 > b_2 = b_1 = 1$, we should take $\vec{a} = \pm x \mathbf{e}_3$. Then

$$f(\rho) = -\frac{1}{\beta} \log \cosh\left(\frac{1}{2} \Delta \beta x\right).$$

B.6. Exercises in the appendix

EXERCISE A.1: The classical Pinsker inequality for Bernoulli random variables amounts to claim that for all $\alpha, \beta \in [0, 1]$ we have

$$\alpha \log \frac{\alpha}{\beta} + (1 - \alpha) \log \frac{1 - \alpha}{1 - \beta} \geq 2(\alpha - \beta)^2.$$

For fixed α let $f(\beta) = \alpha \log \frac{\alpha}{\beta} + (1 - \alpha) \log \frac{1 - \alpha}{1 - \beta} - 2(\alpha - \beta)^2$. We have $f(\alpha) = 0$ and $f'(\beta) = (\beta - \alpha)(1 - 2\beta)^2$, which is negative for $\beta < \alpha$ and positive for $\beta > \alpha$. Then $f(\beta) \geq 0 \forall \beta$.