APPENDIX B

Solutions to some exercises

B.1. Spin systems

EXERCISE 1.1 Unitaries act on operators by conjugation ($S \mapsto U^*SU$), so we appeal to Lemma 1.4.

(a) $S^{(1)}, S^{(2)}, S^{(3)}$ are the same as $S^{\mathbf{e}_1}, S^{\mathbf{e}_2}, S^{\mathbf{e}_3}$ where the $\mathbf{e}_i$ are the standard basis of $\mathbb{R}^3$. Rotation around $\mathbf{e}_1$ by $\frac{\pi}{2} = 90°$ indeed has the effect described.
(b) Rotation around $\vec{a} = \frac{\pi}{2}(0,1,0)$ maps $S^{(1)} \mapsto -S^{(3)}$, $S^{(3)} \mapsto S^{(1)}$, $S^{(2)} \mapsto S^{(2)}$. Rotation around $\vec{a} = \frac{\pi}{2}(0,0,1)$ maps $S^{(1)} \mapsto S^{(2)}$, $S^{(2)} \mapsto -S^{(1)}$, $S^{(3)} \mapsto S^{(3)}$.
(c) Since $\|\vec{a}\| = 2\pi/3$ this is indeed the case.
(d) $U = \mathrm{e}^{\mathrm{i}aS^{(3)}}$ acts by rotation of the $1,2$-plane by angle a, so we can use the rotation matrix:
$$\begin{pmatrix} \cos a & -\sin a \\ \sin a & \cos a \end{pmatrix}.$$
We have
$$\begin{aligned} \mathrm{e}^{-\mathrm{i}aS^{(3)}} S^{(+)} \mathrm{e}^{\mathrm{i}aS^{(3)}} &= \mathrm{e}^{-\mathrm{i}aS^{(3)}} (S^{(1)} + \mathrm{i}S^{(2)}) \mathrm{e}^{\mathrm{i}aS^{(3)}} \\ &= (\cos a - \mathrm{i}\sin a)S^{(1)} + (\sin a + \mathrm{i}\cos a)S^{(2)} \\ &= (\cos(-a) + \mathrm{i}\sin(-a))S^{(1)} + \mathrm{i}(\cos(-a) + \mathrm{i}\sin(-a))S^{(2)} = \mathrm{e}^{-\mathrm{i}a} S^{(+)}. \end{aligned}$$
The calculation for $S^{(-)}$ is similar.

EXERCISE 1.2 One can find such vectors provided $n_1, n_2 \geq 2$ (if one of them, say $\mathcal{H}^{(1)}$, is one-dimensional then it is $= \mathbb{C}$ so $\mathcal{H}^{(1)} \otimes \mathcal{H}^{(2)} = \mathcal{H}^{(2)}$ and tensoring with a scalar is just multiplication with that scalar). Assuming $n_1, n_2 \geq 2$, take a basis $\mathbf{e}_1, \mathbf{e}_2, \dots$ for $\mathcal{H}^{(1)}$ and a basis $\mathbf{f}_1, \mathbf{f}_2, \dots$ for $\mathcal{H}^{(2)}$, and consider the vector
$$x = \mathbf{e}_1 \otimes \mathbf{f}_2 + \mathbf{e}_2 \otimes \mathbf{f}_1.$$
Assuming we could write $x = v \otimes w$, expand $v = \sum_{i\geq 1} a_i \mathbf{e}_i$ and $w = \sum_{i\geq 1} b_i \mathbf{f}_i$, then by multi-linearity $x = \sum_{i,j\geq 1} a_i b_j \mathbf{e}_i \otimes \mathbf{f}_j$. The definition of x requires that $a_1 b_2 = a_2 b_1 = 1 \neq 0$ (so $a_1, b_1, a_2, b_2 \neq 0$) but also $a_1 b_1 = a_2 b_2 = 0$, which is a contradiction.

EXERCISE 1.3 Perhaps this is mainly an exercise in remembering the definition of an inner product: we need $\langle x, y\rangle = \overline{\langle y, x\rangle}$, $\langle \alpha x + \beta y, z\rangle = \overline{\alpha}\langle x, z\rangle + \overline{\beta}\langle y, z\rangle$, and $\langle x, x\rangle \geq 0$ with $= 0$ if and only if $x = 0$. All these properties are inherited for the tensor product.

EXERCISE 1.4 Since (1.35) is the special case $J = \frac{1}{2}$ of (1.36), it suffices to do the latter. Expanding the commutator we get

$$[S^{(1)}, S^{(2)}] = \tfrac{1}{2\mathrm{i}}[S^{(-)}, S^{(+)}].$$

We also get

$$(S^{(1)})^2 + (S^{(2)})^2 + (S^{(3)})^2 = \tfrac{1}{2}(S^{(+)}S^{(-)} + S^{(-)}S^{(+)}) + (S^{(3)})^2$$

so we should consider the products $S^{(+)}S^{(-)}$ and $S^{(-)}S^{(+)}$. We get:

$$S^{(-)}S^{(+)}|\omega\rangle = \begin{cases} 0 & \text{if } \omega_x = J, \\ \big(J(J+1) - \omega_x(\omega_x+1)\big)|\omega\rangle, & \text{if } \omega_x < J \end{cases}$$
$$S^{(+)}S^{(-)}|\omega\rangle = \begin{cases} 0 & \text{if } \omega_x = -J, \\ \big(J(J+1) - \omega_x(\omega_x-1)\big)|\omega\rangle, & \text{if } \omega_x > -J \end{cases}$$

Checking all the cases we get

$$[S^{(1)}, S^{(2)}]|\omega\rangle = \tfrac{1}{2\mathrm{i}}[S^{(-)}, S^{(+)}]|\omega\rangle = \tfrac{1}{2\mathrm{i}}(-2\omega_x)|\omega\rangle = \mathrm{i}S^{(3)}|\omega\rangle,$$

as required. One should similarly check the other commutation relations. As to the Casimir operator, we have

$$(S^{(-)}S^{(+)} + S^{(+)}S^{(-)})|\omega\rangle = \begin{cases} 2J|\omega\rangle & \text{if } \omega_x = \pm J, \\ 2\big(J(J+1) - \omega_x^2\big)|\omega\rangle, & \text{otherwise} \end{cases}$$

which in all cases gives

$$\big((S^{(1)})^2 + (S^{(2)})^2 + (S^{(3)})^2\big)|\omega\rangle = \big(\tfrac{1}{2}(S^{(+)}S^{(-)} + S^{(-)}S^{(+)}) + (S^{(3)})^2\big)|\omega\rangle = J(J+1)|\omega\rangle,$$

as required.

EXERCISE 1.5 The basic fact we need for computation is that

$$x_i \tfrac{\partial}{\partial x_j} x_k \tfrac{\partial}{\partial x_l} = \begin{cases} x_i x_k \frac{\partial^2}{\partial x_j x_l} & \text{if } j \neq k, \\ x_i\big(\frac{\partial}{\partial x_l} + x_j \frac{\partial^2}{\partial x_j x_l}\big) & \text{if } j = k. \end{cases}$$

With this one may check that

$$L^{(1)}L^{(2)} = (-1)\Big[x_2\big(\tfrac{\partial}{\partial x_1} + x_3 \tfrac{\partial^2}{\partial x_1 x_3}\big) - x_1x_2\tfrac{\partial^2}{\partial x_3^2} - x_3^2 \tfrac{\partial^2}{\partial x_1 x_2} + x_1x_3\tfrac{\partial^2}{\partial x_2 x_3}\Big]$$
$$L^{(2)}L^{(1)} = (-1)\Big[x_1\big(\tfrac{\partial}{\partial x_2} + x_3 \tfrac{\partial^2}{\partial x_2 x_3}\big) - x_1x_2\tfrac{\partial^2}{\partial x_3^2} - x_3^2 \tfrac{\partial^2}{\partial x_1 x_2} + x_2x_3\tfrac{\partial^2}{\partial x_1 x_3}\Big]$$

so the commutator

$$[L^{(1)}, L^{(2)}] = (-1)\big[x_2\tfrac{\partial}{\partial x_1} - x_1\tfrac{\partial}{\partial x_2}\big] = \mathrm{i}L^{(3)}.$$

Similarly, the other commutation relations hold. We also need to check the Casimir operator: somewhat lengthy calculations give

$$\sum_{i=1}^{3}(L^{(i)})^2 = (-1)\Big[x_1^2\big(\tfrac{\partial^2}{\partial x_2^2} + \tfrac{\partial^2}{\partial x_3^2}\big) + x_2^2\big(\tfrac{\partial^2}{\partial x_1^2} + \tfrac{\partial^2}{\partial x_3^2}\big) + x_3^2\big(\tfrac{\partial^2}{\partial x_1^2} + \tfrac{\partial^2}{\partial x_2^2}\big)$$
$$- 2\big(x_1x_2\tfrac{\partial^2}{\partial x_1 x_2} + x_1\tfrac{\partial}{\partial x_1} + x_1x_3\tfrac{\partial^2}{\partial x_1 x_3} + x_3\tfrac{\partial}{\partial x_3} + x_2x_3\tfrac{\partial^2}{\partial x_2 x_3} + x_2\tfrac{\partial}{\partial x_2}\big)\Big].$$

Using that the Laplacian is 0 on $\mathcal{H}_\ell$ we can write this as

$$x_1^2\frac{\partial^2}{\partial x_1^2}+x_2^2\frac{\partial^2}{\partial x_2^2}+x_3^2\frac{\partial^2}{\partial x_3^2}+2\big(x_1x_2\frac{\partial^2}{\partial x_1x_2}+x_1\frac{\partial}{\partial x_1}+x_1x_3\frac{\partial^2}{\partial x_1x_3}+x_3\frac{\partial}{\partial x_3}+x_2x_3\frac{\partial^2}{\partial x_2x_3}+x_2\frac{\partial}{\partial x_2}\big),$$

which in turn we can recognize as

$$\Big(1+\sum_{i=1}^3 x_i\frac{\partial}{\partial x_i}\Big)\Big(\sum_{i=1}^3 x_i\frac{\partial}{\partial x_i}\Big).$$

Since the operator $\sum_{i=1}^3 x_i\frac{\partial}{\partial x_i}$ acts as mutliplication by ℓ on degree ℓ monomials, the last display is indeed $\ell(\ell+1)\mathbb{1}$ on $\mathcal{H}_\ell$.

Checking that $(x_1+\mathrm{i}x_2)^\ell$ is a highest weight vector is straightforward, just apply $L^{(3)}$. To construct the other eigenvectors for $L^{(3)}$, we can use the lowering operator:

$$L^{(-)}=L^{(1)}-\mathrm{i}L^{(2)}=x_1\frac{\partial}{\partial x_3}-x_3\frac{\partial}{\partial x_1}-\mathrm{i}x_2\frac{\partial}{\partial x_3}+\mathrm{i}x_3\frac{\partial}{\partial x_2}.$$

Then $(L^{(-)})^k(x_1+\mathrm{i}x_2)^\ell$ will give eigenvectors for all eigenvalues. (Since we checked that the L's are spin operators this is guaranteed to work, see Lemma 1.3.) For $\ell=1$ it is easy to compute that

$$L^{(-)}(x_1+\mathrm{i}x_2)=-2x_3,\qquad L^{(-)}x_3=x_1-\mathrm{i}x_2$$

are such eigenvectors (not normalized, indeed we did not define a norm on $\mathcal{H}_\ell$).

Exercise 1.6 First expand M_Λ:

$$[H_{\Lambda,h},M_\Lambda]=\sum_{x\in\Lambda}[H_{\Lambda,h},S_x^{(3)}].$$

Since $S_x^{(3)}$ commutes with the external-field term, and with any term not involving the site x, we get

$$[H_{\Lambda,h},S_x^{(3)}]=-\sum_{y\sim x}[J^{(1)}S_x^{(1)}S_y^{(1)}+J^{(2)}S_x^{(2)}S_y^{(2)}+J^{(3)}S_x^{(3)}S_y^{(3)},S_x^{(3)}]$$

where the sum is over all sites y neighbouring x. Since $S_x^{(3)}$ commutes with itself and using the commutation relations,

$$\begin{aligned}[H_{\Lambda,h},S_x^{(3)}]&=-\sum_{y\sim x}J^{(1)}[S_x^{(1)},S_x^{(3)}]S_y^{(1)}+J^{(2)}[S_x^{(2)},S_x^{(2)}]S_y^{(2)}\\&=-\mathrm{i}\sum_{y\sim x}(J^{(2)}S_x^{(1)}S_y^{(2)}-J^{(1)}S_x^{(2)}S_y^{(1)}).\end{aligned}$$

Summing over x we get

$$[H_{\Lambda,h},M_\Lambda]=\mathrm{i}\sum_{xy\in\mathcal{E}(\Lambda)}(J^{(1)}-J^{(2)})(S_x^{(1)}S_y^{(2)}+S_x^{(2)}S_y^{(1)})$$

where the sum is over the edges. The commutator vanishes if and only if $J^{(1)}=J^{(2)}$.

Exercise 1.7

(a) To verify that $\vec{S}_x \cdot \vec{S}_y = \frac{1}{2}T - \frac{1}{4}$, check it on the product basis $|\pm, \mp\rangle$. It is convenient to work with $4\vec{S}_x \cdot \vec{S}_y = \vec{\sigma}_x \cdot \vec{\sigma}_y$. Expanding $\vec{\sigma}_x \cdot \vec{\sigma}_y + \mathbb{1} = \sigma^{(1)} \otimes \sigma^{(1)} + \sigma^{(2)} \otimes \sigma^{(2)} + \sigma^{(3)} \otimes \sigma^{(3)} + \mathbb{1}$ we can compute:

$$\big(\vec{\sigma}_x \cdot \vec{\sigma}_y + \mathbb{1}\big) \begin{cases} |+,+\rangle \\ |+,-\rangle \\ |-,+\rangle \\ |-,-\rangle \end{cases} = \begin{cases} |-,-\rangle + \mathrm{i}^2|-,-\rangle + |+,+\rangle + |+,+\rangle \\ |-,+\rangle + \mathrm{i}(-\mathrm{i})|-,+\rangle - |+,-\rangle + |+,-\rangle \\ |+,-\rangle + \mathrm{i}(-\mathrm{i})|+,-\rangle - |-,+\rangle + |-,+\rangle \\ |+,+\rangle + (-\mathrm{i})^2|-,-\rangle + |-,-\rangle + |-,-\rangle \end{cases} = \begin{cases} 2|+,+\rangle \\ 2|-,+\rangle \\ 2|+,-\rangle \\ 2|-,-\rangle \end{cases}$$

Thus $\vec{\sigma}_x \cdot \vec{\sigma}_y + \mathbb{1} = 2T$ as claimed.

Another way to see it is to start from $S^{(+)}$ and $S^{(-)}$: we have

$$\big(S_x^{(+)}S_y^{(-)} + S_x^{(-)}S_y^{(+)}\big) \begin{cases} |+,+\rangle \\ |+,-\rangle \\ |-,+\rangle \\ |-,-\rangle \end{cases} = \begin{cases} 0 \\ |-,+\rangle \\ |+,-\rangle \\ 0 \end{cases}$$

so $S_x^{(+)}S_y^{(-)} + S_x^{(-)}S_y^{(+)}$ is 'almost' the transposition-operator. At the same time

$$\big(4S_x^{(3)}S_y^{(3)} + \mathbb{1}\big) \begin{cases} |+,+\rangle \\ |+,-\rangle \\ |-,+\rangle \\ |-,-\rangle \end{cases} = \begin{cases} 2|+,+\rangle \\ 0 \\ 0 \\ 2|-,-\rangle \end{cases}$$

so $4S_x^{(3)}S_y^{(3)} + \mathbb{1}$ acts like a scalar perpendicular to $S_x^{(+)}S_y^{(-)} + S_x^{(-)}S_y^{(+)}$. Combining these, we see that

$$2\big(S_x^{(+)}S_y^{(-)} + S_x^{(-)}S_y^{(+)}\big) + \big(4S_x^{(3)}S_y^{(3)} + \mathbb{1}\big) = 2T,$$

which after expanding $S^{(\pm)}$ in $S^{(1)}$ and $S^{(2)}$ gives the claim.

(b) Use $\frac{1}{4} - \vec{S}_x \cdot \vec{S}_y = \frac{1}{2}(\mathbb{1} - T)$. Since $T^2 = \mathbb{1}$, the eigenvalues of T are ± 1. Then $\frac{1}{2}(\mathbb{1} - T)$ is non-negative since its eigenvalues are 0 and 1.

(c) We have $T = (\vec{S}_x \cdot \vec{S}_y)^2 + \vec{S}_x \cdot \vec{S}_y - \mathbb{1}$. This can be verified through brute force.

Exercise 1.8

(a) We compute

$$\begin{aligned}(A \otimes A)\psi &= \sum_{j=1}^n (A\mathbf{e}_j) \otimes (A\mathbf{e}_j) = \sum_{j=1}^n \Big(\sum_{k=1}^n A_{k,j}\mathbf{e}_j\Big) \otimes \Big(\sum_{\ell=1}^n A_{\ell,j}\mathbf{e}_j\Big) \\ &= \sum_{k,\ell=1}^n \Big(\sum_{j=1}^n A_{k,j}A_{\ell,j}\Big)\mathbf{e}_k \otimes \mathbf{e}_\ell = \sum_{k,\ell=1}^n \Big(\sum_{j=1}^n A_{k,j}A^{\mathsf{T}}_{j,\ell}\Big)\mathbf{e}_k \otimes \mathbf{e}_\ell \\ &= \sum_{k,\ell=1}^n \delta_{k,\ell}\mathbf{e}_k \otimes \mathbf{e}_\ell = \psi.\end{aligned}$$

Any vector φ can be decomposed as $\varphi = a\psi + \psi'$ with $a \in \mathbb{C}$ and $\langle \psi, \psi' \rangle = 0$. Then (using the shorthand A for $A \otimes A$) we have $\langle \psi, A\psi' \rangle = \langle A^*\psi, \psi' \rangle$ and $A^*\psi = \overline{A^{-1}}\psi = \overline{A^{-1}\psi} = \psi$ since $A^{-1}\psi = \psi$ and $\overline{\psi} = \psi$. Then $AQ\varphi = A(a\psi) = a\psi$ while $QA\varphi = Q(a\psi) + Q(A\psi') = a\psi$ also so Q and A commute.

(b) Write $\psi' = \frac{1}{\sqrt{2}}\big(|+,-\rangle - |-,+\rangle\big)$. Then $\psi' = (\sigma^{(3)} \otimes \mathbb{1})\psi$. So $U = \sigma^{(3)} \otimes \mathbb{1} = U^*$ works.

B.2. Fermionic systems

Exercise 2.14: In addition to the operator $U^{\rm ph}_{x,\sigma}$, let us introduce $\tilde U^{\rm ph}_{x,\sigma} = c_{x,\sigma} - c^*_{x,\sigma}$. We then have

$$(\tilde U^{\rm ph}_{x,\sigma})^{-1} c_{x,\sigma} \tilde U^{\rm ph}_{x,\sigma} = -c^*_{x,\sigma}, \qquad (\tilde U^{\rm ph}_{x,\sigma})^{-1} c^*_{x,\sigma} \tilde U^{\rm ph}_{x,\sigma} = -c_{x,\sigma}.$$

Let $U_\Lambda = \prod_{x\in\Lambda_A} (U^{\rm ph}_{x,\uparrow} U^{\rm ph}_{x,\downarrow}) \prod_{x\in\Lambda_B} (\tilde U^{\rm ph}_{x,\uparrow} \tilde U^{\rm ph}_{x,\downarrow})$. One can check that $U_\Lambda^{-1} T_\Lambda U_\Lambda = T_\Lambda$ and that $U_\Lambda^{-1} n_{x,\sigma} U_\Lambda = 1 - n_{x,\sigma}$. Then $U_\Lambda^{-1} H^{(0)}_\Lambda U_\Lambda = H^{(0)}_\Lambda$. The hamiltonian is then invariant, so that

$$\langle n_x \rangle_{\Lambda,\beta} = \langle U_\Lambda^{-1} n_x U_\Lambda \rangle_{\Lambda,\beta} = 2 - \langle n_x \rangle_{\Lambda,\beta}.$$

Then $\langle n_x \rangle_{\Lambda,\beta} = 1$.

B.3. Equilibrium states

Exercise 3.1 (a) We define $B = A + A^*$ and $C = \mathrm{i}(A - A^*)$. (b) We have

$$\|AB\| = \sup_{v\in\mathcal{H}} \frac{\|A(Bv)\|}{\|Bv\|} \frac{\|Bv\|}{\|v\|} \leq \sup_{v\in\mathcal{H}} \frac{\|A(Bv)\|}{\|Bv\|} \sup_{v\in\mathcal{H}} \frac{\|Bv\|}{\|v\|} \leq \|A\| \|B\|.$$

Exercise 3.2

(a) Consider $\langle (A+\mathbb{1})^*(A+\mathbb{1})\rangle$ and $\langle (A+\mathrm{i}\mathbb{1})^*(A+\mathrm{i}\mathbb{1})\rangle$. Both are non-negative, in particular real, and expanding gives identities for the real and imaginary parts of $\langle A\rangle$ and $\langle A^*\rangle$.

(b) Let $t,\theta \in \mathbb{R}$. Then, using the previous part,

$$0 \leq \langle (A + t\,\mathrm{e}^{\mathrm{i}\theta}\, B)^*(A + t\,\mathrm{e}^{\mathrm{i}\theta}\, B)\rangle = \langle A^*A\rangle + t\big(\mathrm{e}^{\mathrm{i}\theta}\, \langle A^*B\rangle + \overline{\mathrm{e}^{\mathrm{i}\theta}\, \langle A^*B\rangle}\big) + t^2 \langle B^*B\rangle.$$

Choose θ so that $\mathrm{e}^{\mathrm{i}\theta}\, \langle A^*B\rangle \in \mathbb{R}$, then this gives

$$\langle A^*A\rangle + 2t\,\mathrm{e}^{\mathrm{i}\theta}\, \langle A^*B\rangle + t^2\langle B^*B\rangle \geq 0, \qquad \text{for all } t \in \mathbb{R}.$$

Then the discriminant is ≤ 0, i.e.

$$0 \geq (2\,\mathrm{e}^{\mathrm{i}\theta}\, \langle A^*B\rangle)^2 - 4\langle A^*A\rangle\langle B^*B\rangle = 4|\langle A^*B\rangle|^2 - 4\langle A^*A\rangle\langle B^*B\rangle$$

as claimed.

(c) We have

$$|\langle A\rangle|^2 = |\langle \mathbb{1}^*A\rangle|^2 \leq \langle \mathbb{1}^*\mathbb{1}\rangle\langle A^*A\rangle = \langle A^*A\rangle.$$

Now assume $\|A\| = 1$. Note that $\mathbb{1} - A^*A \geq 0$, because for any $v \in \mathcal{H}$:

$$\langle v, (\mathbb{1} - A^*A)v\rangle = \|v\|^2 - \|Av\|^2 \geq \|v\|^2 - \|v\|^2 = 0.$$

Then

$$0 \leq \langle \mathbb{1} - A^*A\rangle = 1 - \langle A^*A\rangle,$$

so for all A with $\|A\| = 1$ we have

$$|\langle A\rangle|^2 \leq \langle A^*A\rangle \leq 1.$$

I.e. $\|\langle\cdot\rangle\| \leq 1$. For the opposite inequality, take $A = \mathbb{1}$: then $\|\langle\cdot\rangle\| \geq |\langle\mathbb{1}\rangle| = 1$.

Exercise 3.3

(a)

Exercise 3.4

(a) We can write out X explicitly in terms of a_1, a_2, a_3:

$$X = \frac{1}{2}\begin{pmatrix} 1 + a_3 & a_1 - \mathrm{i}a_2 \\ a_1 + \mathrm{i}a_2 & 1 - a_3 \end{pmatrix}.$$

Then we see that

$$\det(X - \lambda\mathbb{1}) = \lambda^2 - \lambda - \tfrac{1}{4}(1 - \|\vec{a}\|^2).$$

Thus the eigenvalues are

$$\lambda = \frac{1 \pm \|\vec{a}\|}{2}$$

which means that $X \geq 0$ if and only if $\|\vec{a}\| \leq 1$.

(b) It is clear from the explicit form of X that any Hermitian matrix can be written in this form.

(c) A short computation shows that

$$XY = \tfrac{1}{4}\big((1 + \vec{a}\cdot\vec{b})\mathbb{1} + (\vec{a} + \vec{b} + \vec{a}\times\vec{b})\cdot\vec{\sigma}\big).$$

Thus

$$\operatorname{Tr} XY = \tfrac{1}{2}(1 + \vec{a}\cdot\vec{b}).$$

Note that $x_1 = \frac{1+\|\vec{a}\|}{2}$, $x_2 = \frac{1-\|\vec{a}\|}{2}$, $y_1 = \frac{1+\|\vec{b}\|}{2}$, $y_2 = \frac{1-\|\vec{b}\|}{2}$. Thus

$$x_1y_1 + x_2y_2 = \tfrac{1}{2}(1 + \|\vec{a}\|\|\vec{b}\|), \qquad x_1y_2 + x_2y_1 = \tfrac{1}{2}(1 - \|\vec{a}\|\|\vec{b}\|).$$

The claimed inequality is

$$\tfrac{1}{2}(1 - \|\vec{a}\|\|\vec{b}\|) \leq \tfrac{1}{2}(1 + \vec{a}\cdot\vec{b}) \leq \tfrac{1}{2}(1 + \|\vec{a}\|\|\vec{b}\|),$$

which holds since $\vec{a}\cdot\vec{b} = \|\vec{a}\|\|\vec{b}\|\cos\theta$ and $-1 \leq \cos\theta \leq 1$.

Exercise 3.5 We check the positivity by computing:

$$\operatorname{Tr} A^*A\,\mathrm{e}^{-\beta H} = \operatorname{Tr}\,\mathrm{e}^{-\frac{1}{2}\beta H} A^*A\,\mathrm{e}^{-\frac{1}{2}\beta H} = \operatorname{Tr}\,(A\,\mathrm{e}^{-\frac{1}{2}\beta H})^*A\,\mathrm{e}^{-\frac{1}{2}\beta H} \geq 0,$$

since the trace of a positive semidefinite operator is non-negative.

Exercise 3.6 Let H and A be fixed self-adjoint operators and consider the function $f : s \mapsto F(H + sA)$. It is concave and the derivative at $s = 0$ is equal to $\langle A\rangle_{H,\beta}$, which shows that $F_\beta(H + A) \leq F_\beta(A) + \langle A\rangle_{H,\beta}$. Uniqueness follows from the fact that the function is differentiable.

Exercise 3.7 We first check that the minimiser of $\mathcal{F}_\beta(\cdot)$ is in the interior of the set of density operators. Indeed, let ρ belong to its boundary. Then its kernel has positive dimension and there exists a density operator ρ' that lives in the kernel. Since $\rho \perp \rho'$ we can check that

$$\mathcal{F}_\beta\big((1-\varepsilon)\rho + \varepsilon\rho'\big) = (1-\varepsilon)\mathcal{F}_\beta(\rho) + \varepsilon\mathcal{F}_\beta(\rho') + \tfrac{1}{\beta}(1-\varepsilon)\log(1-\varepsilon) + \tfrac{1}{\beta}\varepsilon\log\varepsilon. \tag{B.1}$$

It is clear that $\varepsilon = 0$ is not a minimum, as the last term is negative and stronger than linear.

We now know that any minimiser ρ_0 is in the interior of the set of density operators. Further, for any operator η such that $\operatorname{Tr}\eta = 0$, the stationary condition is

$$0 = \frac{\mathrm{d}}{\mathrm{d}s}\mathcal{F}_\beta(\rho_0 + s\eta)\Big|_{s=0} = \operatorname{Tr}\eta(H + \tfrac{1}{\beta}\log\rho_0). \tag{B.2}$$

It follows that $H + \frac{1}{\beta}\log\rho_0$ is proportional to the identity, so that $\rho_0 = \mathrm{const}\,\mathrm{e}^{-\beta H}$ is the only stationary point (so it is the minimiser). The constant is $1/\mathrm{Tr}\ \mathrm{e}^{-\beta H}$ in order for ρ_0 to be a density operator.

Although this is redundant, one can check that

$$\frac{\mathrm{d}^2}{\mathrm{d}s^2}\mathcal{F}_\beta(\rho_0 + s\eta)\Big|_{s=0} = \tfrac{1}{2\beta}\operatorname{Tr}\eta\rho_0^{-1}\eta \geq 0, \tag{B.3}$$

which confirms that ρ_0 is a minimiser.

Exercise 3.8 If $\langle\cdot\rangle = \langle\cdot\rangle_{H,\beta}$ is the Gibbs state, for $s \in \mathbb{R}$ define the inner product

$$A, B \mapsto (A,B)_s = \langle\alpha_{-\mathrm{i}s}(A^*)B\rangle = \langle A^*\alpha_{\mathrm{i}s}(B)\rangle \tag{B.4}$$

and introduce the function $f(s)$ on $[0,\beta]$ by

$$f(s) = (A,A) = \langle\alpha_{-\mathrm{i}s}(A^*)A\rangle = \langle A^*\alpha_{\mathrm{i}s}(A)\rangle. \tag{B.5}$$

We then have

$$\begin{aligned} f'(s) &= -\langle A^*[H,\alpha_{\mathrm{i}s}(A)]\rangle = -\langle\alpha_{-\mathrm{i}s}(A^*)[H,A]\rangle. \\ f''(s) &= \langle[\alpha_{-\mathrm{i}s}(A^*),H]\,[H,A]\rangle = \langle[\alpha_{-\frac{1}{2}\mathrm{i}s}(A^*),H]\,[H,\alpha_{\frac{1}{2}\mathrm{i}s}(A)]\rangle. \end{aligned} \tag{B.6}$$

The last expression shows that $f''(s) \geq 0$ so that f is convex. Using the Cauchy-Schwarz inequality of the inner product on $f'(s) = -(A,[H,A])$, we find that

$$f'(s)^2 \leq (A,A)\,([H,A],[H,A]) = f(s)f''(s). \tag{B.7}$$

It follows that $\log f$ is convex. Then $(\log f)'(0) \leq \frac{1}{\beta}(\log f(\beta) - \log f(0))$. We get the RAS inequality since $f(0) = \langle A^*A\rangle$, $f(\beta) = \langle AA^*\rangle$ and $f'(0) = -\langle A^*[H,A]\rangle$.

For the other direction, take $A = 1 + tB$, then

$$\langle A^*[H,A]\rangle = t\langle[H,B]\rangle + t^2\langle B^*[H,B]\rangle$$

and

$$\log\frac{\langle A^*A\rangle}{\langle AA^*\rangle} = \log\frac{1 + t\langle B+B^*\rangle + t^2\langle B^*B\rangle}{1 + t\langle B+B^*\rangle + t^2\langle BB^*\rangle} = O(t^2).$$

So if $\langle\cdot\rangle$ satisfies the RAS-inequality, since the right side is $O(t^2)$ it follows that the density matrix ρ satisfies

$$\langle[H,B]\rangle = \operatorname{Tr}B[\rho,H] = 0.$$

This being true for all B, we must have

$$[\rho, H] = 0.$$

Since they commute, they have a common eigenbasis e_i:

$$\rho|e_i\rangle = \rho_i|e_i\rangle, \qquad H|e_i\rangle = h_i|e_i\rangle.$$

Now use the RAS-inequality again for the off-diagonal matrices $A = |e_i\rangle\langle e_j|$, with $i \neq j$:

$$(h_i - h_j)\rho_j \geq \frac{1}{\beta}\rho_j \log \frac{\rho_j}{\rho_i}.$$

(A consequence is that all $\rho_i > 0$.) Taking exponentials we get

$$\mathrm{e}^{\beta h_i}\,\rho_i \geq \mathrm{e}^{\beta h_j}\,\rho_j \qquad \text{for all } i \neq j.$$

Since I and j were arbitrary indices, it follows that

$$\mathrm{e}^{\beta h_i}\,\rho_i = \mathrm{e}^{\beta h_j}\,\rho_j \qquad \text{for all } i \neq j.$$

Then $\rho = c\,\mathrm{e}^{-\beta H}$ for some c, which is in turn fixed by the normalization.

EXERCISE 3.9 We use the function $\mathcal{F}_\beta(A) = \mathrm{Tr}_\Lambda A H_\Lambda^\Phi + \frac{1}{\beta}\mathrm{Tr}_\Lambda A \log A$ from the Proposition. To use the result, we need to plug in a density matrix for Tr_Λ, which in the notation of Theorem 3.13 can be $\rho_\Lambda/\dim \mathcal{H}_\Lambda$. We get

$$\mathcal{F}_\beta(\rho_\Lambda/\dim \mathcal{H}_\Lambda) \geq \mathcal{F}_\beta(\,\mathrm{e}^{-\beta H_\Lambda^\Phi}/\mathrm{Tr}_\Lambda\,\mathrm{e}^{-\beta H_\Lambda^\Phi}\,).$$

Reorganizing this and using that

$$\boldsymbol{\rho}(\frac{1}{|\Lambda|}H_\Lambda^\Phi) \to \boldsymbol{\rho}(A_\Phi)$$

due to translation-invariance, the result follows.

EXERCISE 3.10 Clearly tr is a state, and cyclicity is precisely the KMS condition at $\beta = 0$. Now take $\Lambda \Subset \mathbb{Z}^d$ and assume that $\langle AB\rangle = \langle BA\rangle$ for all $A, B \in \mathcal{A}_\Lambda$. Let ρ_Λ be the density matrix for the restriction to $\mathcal{A}_\Lambda$. Taking $A = |i\rangle\langle j|$ and $B = |k\rangle\langle \ell|$ for various combinations of i, j, k, ℓ shows that $\rho_\Lambda(i,j) = 0$ for $i \neq j$ and $\rho_\Lambda(i,i) = \rho_\Lambda(j,j)$ for all i, j. Thus $\rho_\Lambda = \mathbb{1}$ for all $\Lambda \Subset \mathbb{Z}^d$ which means that $\langle\cdot\rangle = \mathrm{tr}\,\cdot$.

EXERCISE 3.11: It is clear that if the KMS condition holds for any observables A, B, then it holds for A, A^*. To prove the converse, consider $A, B \in \tilde{\mathcal{A}}$. We have

$$\langle (A^* + B)(A + B^*)\rangle = \langle (A + B^*)\alpha_{\mathrm{i}\beta}(A^* + B)\rangle. \tag{B.8}$$

Expanding and simplifying, we get

$$\langle A^*B^*\rangle + \langle BA\rangle = \langle B^*\alpha_{\mathrm{i}\beta}(A^*)\rangle + \langle A\alpha_{\mathrm{i}\beta}(B)\rangle. \tag{B.9}$$

Repeating with $A + \mathrm{i}B^*$ we get

$$\langle A^*B^*\rangle - \langle BA\rangle = \langle B^*\alpha_{\mathrm{i}\beta}(A^*)\rangle - \langle A\alpha_{\mathrm{i}\beta}(B)\rangle. \tag{B.10}$$

Then $\langle BA\rangle = \langle A\alpha_{\mathrm{i}\beta}(B)\rangle$ indeed.

EXERCISE 3.12 Let $B \in \tilde{\mathcal{A}}$ and observe that the complex function $\langle \alpha_z^\Phi(B)\rangle$ is entire and bounded in the strip $0 \leq \mathrm{Im}\, z \leq \beta$. The function $F_{\mathbb{1},B}(z)$ in the KMS condition (b) must be equal to $\langle \alpha_z^\Phi(B)\rangle$. Further, the KMS condition states that $F_{\mathbb{1},B}(t+\mathrm{i}\beta) = F_{\mathbb{1},B}(t)$, so this function is periodic in the imaginary direction. Then $F_{\mathbb{1},B}(z)$ is bounded in the whole complex plane, and is therefore constant by Liouville's theorem. Then $\langle \alpha_t^\Phi(B)\rangle = F_{\mathbb{1},B}(t)$ is constant.

Exercise 3.13 Following the hint, let A be local, say $A \in \mathcal{A}_\Lambda$, and use the positive semi-definite square root of B_n i.e. $B_n = \sqrt{B_n}\sqrt{B_n}$ with $\sqrt{B_n} \geq 0$. We have $[H^\Phi, A] = [H_\Lambda^\Phi, A]$ so for n large enough B_n commutes with all terms in the commutator, and then from the RAS condition for $\langle\cdot\rangle$:

$$\begin{aligned}\langle A^*[H_\Lambda^\Phi, A]B_n\rangle &= \langle (A\sqrt{B_n})^*[H_\Lambda^\Phi, A\sqrt{B_n}]\rangle \\ &\geq \frac{1}{\beta}\langle (A\sqrt{B_n})^* A\sqrt{B_n}\rangle \log \frac{\langle (A\sqrt{B_n})^* A\sqrt{B_n}\rangle}{\langle A\sqrt{B_n}(A\sqrt{B_n})^*\rangle} \\ &= \frac{1}{\beta}\langle A^*AB_n\rangle \log \frac{\langle A^*AB_n\rangle}{\langle AA^*B_n\rangle}\end{aligned}$$

which gives the RAS-inequality for $\langle\cdot\rangle'$.

Exercise 3.14 Here you can use whichever characterization of Gibbs states you prefer – but since the question does not assume translation-invariance, a complete solution would check the KMS or RAS conditions. Checking the RAS-condition is quite convenient: thanks to the invariance of Φ, RAS for $\langle\cdot\rangle'$ is the inequality

$$\langle (U^*AU)^*[H_\Lambda^\Phi, (U^*AU)]\rangle \geq \frac{1}{\beta}\langle (U^*AU)^*(U^*AU)\rangle \log \frac{\langle (U^*AU)^*(U^*AU)\rangle}{\langle (U^*AU)(U^*AU)^*\rangle}$$

which indeed holds since $\langle\cdot\rangle$ satisfies RAS. (For translation-invariant states, the variational characterization is also easy to check.)

B.4. Uniqueness and non-uniqueness of Gibbs states

Exercise 4.1: We have

$$\begin{aligned}\|A\|_2^2 &= \mathrm{tr}_{\{x\}\cup\Lambda} A^*A = \sum_{i,j} \mathrm{tr}_{\{x\}\cup\Lambda} e_i e_j \otimes C_i^* C_j = \sum_j \mathrm{tr}_{\{x\}\cup\Lambda} e_j^2 \otimes C_j^* C_j \\ &= \frac{2}{N}\sum_j \mathrm{tr}_\Lambda C_j^* C_j = \frac{2}{N}\sum_j \|C_j\|_2^2.\end{aligned}$$

We now use Schwarz inequality to get

$$\Big(\sum_{j=0}^{N^2-1} \|C_j\|_2\Big)^2 \leq N^2 \sum_j \|C_j\|_2^2 = \frac{N^3}{2}\|A\|_2^2.$$

We take the square root and get the result.

B.5. Mean-field systems

EXERCISE 5.1: We have

$$H_\rho = -\vec{a}\cdot\vec{S} - \tfrac{1}{4}\mathbb{1}. \tag{B.11}$$

Then

$$\rho H_\rho = -\tfrac{1}{8}\mathbb{1} - \tfrac{3}{4}\vec{a}\cdot\vec{S} - (\vec{a}\cdot\vec{S})^2, \qquad \operatorname{Tr}\rho H_\rho = -\tfrac{1}{4} - \tfrac{\|\vec{a}\|^2}{2}.$$

Further, we saw in a previous exercse that ρ has eigenvalues $\frac{1+\|\vec{a}\|}{2}$, $\frac{1-\|\vec{a}\|}{2}$, so

$$\operatorname{Tr}\rho\log\rho = \tfrac{1+\|\vec{a}\|}{2}\log\tfrac{1+\|\vec{a}\|}{2} + \tfrac{1-\|\vec{a}\|}{2}\log\tfrac{1-\|\vec{a}\|}{2}.$$

Thus

$$f(\rho) = -\tfrac{1}{4} - \tfrac{1}{2}\|\vec{a}\|^2 + \tfrac{1}{\beta}\big(\tfrac{1+\|\vec{a}\|}{2}\log\tfrac{1+\|\vec{a}\|}{2} + \tfrac{1-\|\vec{a}\|}{2}\log\tfrac{1-\|\vec{a}\|}{2}\big).$$

With the parameterization $t = \frac{1-\|\vec{a}\|}{2}$ this becomes

$$f(\rho) = 2t(1-t) - \tfrac{3}{4} + \tfrac{1}{\beta}\big(t\log t + (1-t)\log(1-t)\big).$$

Note that, with $x = \|\vec{a}\|$ we have

$$f'(t) = 2 - 4t + \tfrac{1}{\beta}\log\tfrac{t}{1-t} = 2x + \tfrac{1}{\beta}\log\tfrac{1-x}{1+x}.$$

Thus

$$f'(t) = 0 \quad\Leftrightarrow\quad \mathrm{e}^{2\beta x} = \frac{1+x}{1-x} \quad\Leftrightarrow\quad x = \tanh(\beta x).$$

Then for $\beta > 1$, we see that $f(\rho)$ is minimized for any $\vec{a}$ such that $x = \|\vec{a}\|$ is a positive solution. In particular, the minimizers have SO(3)-symmetry.

EXERCISE 5.2: Consider the case when $b_3 \geq b_2 \geq b_1 \geq 0$ and

$$\Phi_{\{x,y\}} = -(b_1 S_x^{(1)} S_y^{(1)} + b_2 S_x^{(2)} S_y^{(2)} + b_3 S_x^{(3)} S_y^{(3)}).$$

We again parameterize ρ as $\rho = \frac{1}{2}\mathbb{1} + \vec{a}\cdot\vec{S}$, and arguing as for (B.11) we obtain

$$H_\rho = -\sum_{j=1}^{3} a_j b_j S^{(j)} = -\vec{ab}\cdot\vec{S}, \tag{B.12}$$

where we set $\vec{ab} = (a_1b_1, a_2b_2, a_3b_3)$. The mean-field equation is then

$$\rho = \tfrac{1}{2}\mathbb{1} + \vec{a}\cdot\vec{S} = \frac{\mathrm{e}^{2\beta\vec{ab}\cdot\vec{S}}}{\operatorname{Tr}\,\mathrm{e}^{2\beta\vec{ab}\cdot\vec{S}}}. \tag{B.13}$$

Now we recall that the mean-field equation is necessary but not sufficient. Among the solutions $\vec{a}$, we should select those such that ρ minimizes $f(\rho)$. Using (B.13) we get that

$$f(\rho) = -\tfrac{1}{\beta}\log\operatorname{Tr}\,\mathrm{e}^{2\beta\vec{ab}\cdot\vec{S}} = -\tfrac{1}{\beta}\log\cosh(\beta\|\vec{ab}\|). \tag{B.14}$$

Since $\cosh(\cdot)$ is increasing, setting $x = \|\vec{a}\|$ we see that $\vec{a} = x\,\mathbf{e}_3$ gives the best solution, as well as any rotation of $x\,\mathbf{e}_3$ such that $\vec{ab}$ has the same length.

In particular, in the XXZ-case $\Delta = b_3 > b_2 = b_1 = 1$, we should take $\vec{a} = \pm x\,\mathbf{e}_3$. Then

$$f(\rho) = -\tfrac{1}{\beta}\log\cosh(\tfrac{1}{2}\Delta\beta x).$$