Introduction

The motivation for quantum lattice models is the study of electronic properties of condensed matter systems. It is a fact of life that solids exist, even if the mathematical description of solidification is very wanting. Since nuclei are heavy it is natural to treat them as fixed. There remains to understand the behaviour of electrons and their spins. They are governed by the laws of quantum mechanics and they interact via Coulomb forces with each other and with the nuclei. They are impossible to study directly and physicists have been compelled to introduce simplified models.

Models usually involve the "tight-binding approximation" where the motion of electrons are restricted to the sites of the lattice formed by the nuclei. Interactions are also simplified. Starting from the original electrons with Coulomb interactions, the simplification leads to one of the fundamental models of condensed matter physics, the Hubbard model.

The discovery and study of certain phenomena has led to further models such as Ising, Heisenberg, XY, Hubbard, t-J, as well as models involving bosons. There are few situations where these models describe exactly the properties of the system, but they help physicists to gain much better understanding.

The existence of all these models, even if they do not describe the physical systems directly, is a blessing for mathematicians. They have a rich and attractive mathematical structure. Their study is achievable, even though it is often difficult and it requires much creative thinking. Several arguments are indirect and convoluted — but eminently interesting. We hope these notes will help convince the reader that this field is indeed attractive and intellectually rewarding.

These notes introduce the basic setting for quantum spin systems and for lattice fermion systems. We review the notion of states, and discuss Gibbs states (that is, the equilibrium states) in finite and infinite domains. The set of Gibbs states has the structure of a "Choquet simplex" where each state is given by a convex combination of *extremal states*. These extremal states enjoy special properties, such as clustering and 0-1 law.

Once the general theory is understood, there remains to identity the set of extremal states for given systems. This turns out to be an extremely difficult task, and rigorous results are few and partial. These notes discuss the following situations:

- At high temperatures, the Gibbs state is unique. This result is very general.
- In the XXZ-model where the "Z" interactions are stronger, the discrete spin-flip symmetry is broken in dimension two and higher. This is similar to the classical Ising model, where there are exactly two translation-invariant extremal Gibbs states at low temperature.
- For models with continuous symmetry in two dimensions, all Gibbs states retain this symmetry, even at low temperatures.
- In some cases with continuous symmetry, such as the antiferromagnetic Heisenberg model, one can use the theory of reflection positivity to prove the occurrence of long-range order at low temperature and at dimension three and higher. This implies the existence of several distinct extremal Gibbs states.

The setting and properties of quantum lattice systems have been discovered and clarified over several decades by physicists and mathematicians. Physicists have contributed great insights and perspectives, that have led to the development of beautiful mathematics. Mathematicians have brought in creative and elegant methods to make studies possible. We hope the readers will share our fascination for this topic, and that this will prompt them to bring further creative and elegant solutions to the many open problems!

Finally, a word on the literature. For the general theory, we mainly rely on the books of Ruelle [**26**], Israel [**20**] and Simon [**27**]. A deeper dive into the theory of C^* algebra can be found in the books of Bratteli and Robinson [**4**]. The theory of classical spin systems is beautifully introduced in Friedli and Velenik [**10**]. The book of Tasaki [**28**] describes explicit quantum lattice systems and their properties, mainly in the ground state. Further useful reading include the book of Naaijkens [**24**] and the notes of Nachtergaele and Sims [**25**].

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