CHAPTER 2

Fermionic and bosonic systems

These systems are naturally defined in the continuum space but they also makes sense on lattices, where the setting is simpler and relevant for our purpose. Excellent introductory textbooks include Martin and Rothen [2004], ... A thorough description of systems in the continuum can be found in Bratteli and Robinson [1987].

2.1. Fock spaces

The Hilbert space for a single particle in $\Lambda \Subset \mathbb{Z}^d$ is $\ell^2(\Lambda)$. Recall that $\ell^2(\Lambda)$ is the vector space $\mathbb{C}^\Lambda$ with inner product

$$\langle \varphi | \psi \rangle = \sum_{x \in \Lambda} \overline{\varphi(x)}\, \psi(x), \qquad \varphi, \psi \in \ell^2(\Lambda). \tag{2.1}$$

A natural basis is $\{e_x\}_{x \in \Lambda}$ where these functions are defined by $e_x(y) = \delta_{x,y}$. The dimension of $\ell^2(\Lambda)$ is $|\Lambda|$.

The Hilbert space $\mathcal{H}_{\Lambda,n}$ for n *distinguishable* particles is the tensor product $\otimes_{i=1}^n \ell^2(\Lambda)$. Its dimension is $|\Lambda|^n$ and it can be identified with the linear space $\ell^2(\Lambda^n)$ of functions of n sites. Then

$$\mathcal{H}_{\Lambda,n} = \otimes_{i=1}^n \ell^2(\Lambda) \cong \ell^2(\Lambda^n). \tag{2.2}$$

A basis for $\otimes_{i=1}^n \ell^2(\Lambda)$ consists of the functions

$$\Big\{ \bigotimes_{i=1}^n e_{x_i} \Big\}_{x_1,\dots,x_n \in \Lambda}, \tag{2.3}$$

where the functions e_{x_i} are as above. A basis for $\ell^2(\Lambda^n)$ consists of the functions $e_{x_1,\dots,x_n}$ that satisfy

$$e_{x_1,\dots,x_n}(y_1,\dots,y_n) = \prod_{i=1}^n \delta_{x_i,y_i}. \tag{2.4}$$

As physicists have progressively understood in the early days of Quantum Mechanics, the Hilbert space for *indistinguishable* particles is different. Particles fall in two kinds of species: the symmetric **bosons** and the antisymmetric **fermions**. The latter include the electrons and are therefore very relevant to condensed matter physics. The former are also relevant in an indirect way, as they can describe composite particles (made of an even number of fermions) or virtual particles (such as the phonons that describe lattice vibrations). The correct Hilbert spaces are the symmetric and antisymmetric subspaces of $\mathcal{H}_{\Lambda,n}$. To define them we introduce the **symmetrisation operator** P_+ and the **antisymmetrisation operator** P_-. They can be defined both on $\otimes_{i=1}^n \ell^2(\Lambda)$ and $\ell^2(\Lambda^n)$. First, the action of P_+ is

$$\begin{aligned} \big(P_+\varphi\big)(x_1,\dots,x_n) &= \frac{1}{n!}\sum_{\sigma\in\mathfrak{S}_n}\varphi(x_{\sigma(1)},\dots,x_{\sigma(n)}), \qquad \varphi\in\ell^2(\Lambda^n) \\ P_+\bigotimes_{i=1}^n\varphi_i &= \frac{1}{n!}\sum_{\sigma\in\mathfrak{S}_n}\bigotimes_{i=1}^n\varphi_{\sigma(i)}, \qquad \varphi_i\in\ell^2(\Lambda) \text{ for } i=1,\dots,n. \end{aligned} \tag{2.5}$$

Here, $\mathfrak{S}_n$ denotes the symmetric group and the sum is over permutations of n elements. Second, the action of P_- is

$$\begin{aligned} \big(P_-\varphi\big)(x_1,\dots,x_n) &= \frac{1}{n!}\sum_{\sigma\in\mathfrak{S}_n}(-1)^\sigma\varphi(x_{\sigma(1)},\dots,x_{\sigma(n)}), \qquad \varphi\in\ell^2(\Lambda^n) \\ P_-\bigotimes_{i=1}^n\varphi_i &= \frac{1}{n!}\sum_{\sigma\in\mathfrak{S}_n}(-1)^\sigma\bigotimes_{i=1}^n\varphi_{\sigma(i)}, \qquad \varphi_i\in\ell^2(\Lambda) \text{ for } i=1,\dots,n, \end{aligned} \tag{2.6}$$

where $(-1)^\sigma$ is the signature of the permutation σ (it is equal to $+1$ if σ can be written as the product of an even number of transpositions; it is -1 if the number of transpositions is odd). Note that $P_\pm$ are Hermitian projection operators in the sense that

$$P_\pm^2 = P_\pm, \qquad P_\pm^* = P_\pm. \tag{2.7}$$

The symmetric subspace $\mathcal{H}_{\Lambda,n}^{(+)}$, resp. antisymmetric subspace $\mathcal{H}_{\Lambda,n}^{(-)}$, are then

$$\mathcal{H}_{\Lambda,n}^{(\pm)} \cong P_\pm\ell^2(\Lambda^n) \cong P_\pm\bigotimes_{i=1}^n\ell^2(\Lambda). \tag{2.8}$$

These spaces consist of symmetric or antisymmetric functions. We can identify

$$\begin{aligned} \mathcal{H}_{\Lambda,n}^{(+)} &= \Big\{\varphi\in\mathbb{C}^\Lambda : \varphi(x_{\sigma(1)},\dots,x_{\sigma(n)}) = \varphi(x_1,\dots,x_n)\ \forall\sigma\in\mathfrak{S}_n\Big\}; \\ \mathcal{H}_{\Lambda,n}^{(-)} &= \Big\{\varphi\in\mathbb{C}^\Lambda : \varphi(x_{\sigma(1)},\dots,x_{\sigma(n)}) = (-1)^\sigma\varphi(x_1,\dots,x_n)\ \forall\sigma\in\mathfrak{S}_n\Big\}. \end{aligned} \tag{2.9}$$

We now introduce the notion of **occupation numbers**. They are a convenient way to describe the spaces of symmetric and antisymmetric functions. Let

$$\begin{aligned} \mathcal{N}_{\Lambda,n}^{(+)} &= \big\{(n_x)_{x\in\Lambda} : n_x\in\mathbb{N} \text{ and } \sum_{x\in\Lambda} n_x = n\big\}; \\ \mathcal{N}_{\Lambda,n}^{(-)} &= \big\{(n_x)_{x\in\Lambda} : n_x\in\{0,1\} \text{ and } \sum_{x\in\Lambda} n_x = n\big\}. \end{aligned} \tag{2.10}$$

The set $\mathcal{N}^{(-)}_{\Lambda,n}$ is nonempty only if $n \leq |\Lambda|$. The goal now is to check that

$$\mathcal{H}^{(\pm)}_{\Lambda,n} \cong \ell^2(\mathcal{N}^{(\pm)}_{\Lambda,n}). \tag{2.11}$$

To see this, we define the vector $|\boldsymbol{n}\rangle$ in $\mathcal{H}^{\pm}_{\Lambda,n}$, for $\boldsymbol{n} = (n_x) \in \mathcal{N}^{(\pm)}_{\Lambda,n}$:

$$\begin{aligned} |\boldsymbol{n}\rangle &= \Big(\frac{n!}{\prod_x n_x!}\Big)^{1/2} P_+ e_{x_1,\dots,x_n} \qquad \text{in } \mathcal{H}^{(+)}_{\Lambda,n}; \\ |\boldsymbol{n}\rangle &= (n!)^{1/2} P_+ e_{x_1,\dots,x_n} \qquad \text{in } \mathcal{H}^{(-)}_{\Lambda,n}. \end{aligned} \tag{2.12}$$

The vectors $e_{x_1,\dots,x_n}$ above are the basis vectors defined in (2.4); the sites $x_1,\dots,x_n$ are chosen so that $\#\{i = 1,\dots,n : x_i = x\} = n_x$ for all $x \in \Lambda$. The order of $(x_1,\dots,x_n)$ does not matter for $P_+ e_{x_1,\dots,x_n}$. The order affects the sign for $P_- e_{x_1,\dots,x_n}$ so the sites should satisfy $x_1 \prec \dots \prec x_n$ where $\prec$ is some fixed total order on Λ. One can check that the prefactors have been chosen so that $|\boldsymbol{n}\rangle$ is normalised, see Exercise 2.3. It is not too hard to check that $\langle \boldsymbol{n}'|\boldsymbol{n}\rangle = 0$ if $\boldsymbol{n}' \neq \boldsymbol{n}$. Since the vectors $e_{x_1,\dots,x_n}$ span $\mathcal{H}_{\Lambda,n}$, it follows that $\{|\boldsymbol{n}\rangle\}_{\boldsymbol{n}\in\mathcal{N}^{(\pm)}_{\Lambda,n}}$ is an orthonormal basis for $\mathcal{H}^{(\pm)}_{\Lambda,n}$. The dimensions of $\mathcal{H}^{(+)}_{\Lambda,n}$ and $\mathcal{H}^{(-)}_{\Lambda,n}$ are then equal to the cardinalities of $\mathcal{N}^{(\pm)}_{\Lambda,n}$; we get

$$\begin{aligned} \dim \mathcal{H}^{(+)}_{\Lambda,n} &= |\mathcal{N}^{(+)}_{\Lambda,n}| = \binom{n + |\Lambda| - 1}{|\Lambda| - 1}, \\ \dim \mathcal{H}^{(-)}_{\Lambda,n} &= |\mathcal{N}^{(-)}_{\Lambda,n}| = \binom{|\Lambda|}{n} \qquad \text{if } n \leq |\Lambda|; \end{aligned} \tag{2.13}$$

this is verified in Exercise 2.4.

Next we introduce the Fock spaces that describe systems with variable numbers of particles. Let

$$\mathcal{F}^{(+)}_{\Lambda} = \bigoplus_{n=0}^{\infty} \mathcal{H}^{(+)}_{\Lambda,n}, \qquad \mathcal{F}^{(-)}_{\Lambda} = \bigoplus_{n=0}^{|\Lambda|} \mathcal{H}^{(-)}_{\Lambda,n}. \tag{2.14}$$

Here $\mathcal{H}^{(\pm)}_{\Lambda,0} \cong \mathbb{C}$ by definition. An element of $\mathcal{F}^{(+)}_{\Lambda}$ is an ∞-tuple $(\varphi_0, \varphi_1, \dots)$ where each φ_n is a vector in $\mathcal{H}^{(+)}_{\Lambda,n}$. The inner product in $\mathcal{F}^{(+)}_{\Lambda}$ is defined by

$$\langle \varphi, \psi \rangle = \sum_{n\geq 0} \langle \varphi_n, \psi_n \rangle_{\mathcal{H}^{(+)}_{\Lambda,n}}. \tag{2.15}$$

The dimension of $\mathcal{F}^{(+)}_{\Lambda}$ is infinite. In terms of occupation numbers, we have

$$\mathcal{F}^{(+)}_{\Lambda} \cong \ell^2(\mathcal{N}^{(+)}_{\Lambda}) \tag{2.16}$$

where

$$\mathcal{N}^{(+)}_{\Lambda} = \bigcup_{n\geq 0} \mathcal{N}^{(+)}_{\Lambda,n} = \mathbb{N}^{\Lambda}. \tag{2.17}$$

An element of $\mathcal{F}_\Lambda^{(-)}$ is an $|\Lambda|$-tuple $(\varphi_0, \varphi_1, \dots, \varphi_{|\Lambda|})$ where φ_n is a vector in $\mathcal{H}_{\Lambda,n}^{(-)}$; the inner product is defined by

$$\langle \varphi, \psi \rangle = \sum_{n=0}^{|\Lambda|} \langle \varphi_n, \psi_n \rangle_{\mathcal{H}_{\Lambda,n}^{(-)}}. \tag{2.18}$$

The dimension of $\mathcal{F}_\Lambda^{(-)}$ is $2^{|\Lambda|}$. In terms of occupation numbers, we have

$$\mathcal{F}_\Lambda^{(-)} \cong \ell^2(\mathcal{N}_\Lambda^{(-)}) \tag{2.19}$$

where

$$\mathcal{N}_\Lambda^{(-)} = \bigcup_{n \geq 0} \mathcal{N}_{\Lambda,n}^{(+)} = \{0,1\}^\Lambda. \tag{2.20}$$

2.2. Creation and annihilation operators

We define annihilation operators a_x and creation operators a_x^* in $\ell^2(\mathcal{N}_{\Lambda,n}^{(\pm)})$ or $\ell^2(\mathcal{N}_\Lambda^{(\pm)})$; this immediately extends to $\mathcal{H}_{\Lambda,n}^{(\pm)}$ and $\mathcal{F}_\Lambda^{(\pm)}$.

$$\begin{aligned} \text{Bosons:} \qquad & a_x : \ell^2(\mathcal{N}_{\Lambda,n}^{(+)}) \to \ell^2(\mathcal{N}_{\Lambda,n-1}^{(+)}) \\ & \quad a_x|\boldsymbol{n}\rangle = \begin{cases} \sqrt{n_x}\,|\boldsymbol{n} - \delta_x\rangle & \text{if } n_x \geq 1, \\ 0 & \text{if } n_x = 0. \end{cases} \\ & a_x^* : \ell^2(\mathcal{N}_{\Lambda,n}^{(+)}) \to \ell^2(\mathcal{N}_{\Lambda,n+1}^{(+)}) \\ & \quad a_x^*|\boldsymbol{n}\rangle = \sqrt{n_x + 1}\,|\boldsymbol{n} + \delta_x\rangle. \end{aligned} \tag{2.21}$$

These definitions extend to $\ell^2(\mathcal{N}_\Lambda^{(+)})$ but we need to specify the domain since these are unbounded operators in an infinite-dimensional space. Consider $\ell^2_{\mathrm{f}}(\mathcal{N}_\Lambda^{(+)})$, the linear space of finite linear combinations of $\{|\boldsymbol{n}\rangle : \boldsymbol{n} \in \mathcal{N}_\Lambda^{(+)}\}$. This domain is dense in $\ell^2(\mathcal{N}_\Lambda^{(+)})$, and a_x, a_x^* can be defined as operators $\ell^2_{\mathrm{f}}(\mathcal{N}_\Lambda^{(+)}) \to \ell^2(\mathcal{N}_\Lambda^{(+)})$. The operators can be closed by taking the closure of their graphs.

In the exercises (Exercise 2.5) you can check that a_x^* is the adjoint of a_x (and conversely), and that these bosonic operators satisfy the commutation relations

$$[a_x, a_y] = 0; \qquad [a_x^*, a_y^*] = 0; \qquad [a_x, a_y^*] = \delta_{x,y} \mathbb{1}. \tag{2.22}$$

One can also check that

$$a_x^* a_x |\boldsymbol{n}\rangle = n_x\,|\boldsymbol{n}\rangle. \tag{2.23}$$

We now turn to fermions and recall the order $\prec$ on the sites of Λ.

$$\text{Fermions:}\qquad \begin{aligned} a_x &: \ell^2(\mathcal{N}^{(-)}_{\Lambda,n}) \to \ell^2(\mathcal{N}^{(-)}_{\Lambda,n-1}) \\ a_x|\boldsymbol{n}\rangle &= \begin{cases} (-1)^{\sum_{y\prec x} n_y}\,|\boldsymbol{n}-\delta_x\rangle & \text{if } n_x = 1, \\ 0 & \text{if } n_x = 0. \end{cases} \\ a_x^* &: \ell^2(\mathcal{N}^{(-)}_{\Lambda,n}) \to \ell^2(\mathcal{N}^{(-)}_{\Lambda,n+1}) \\ a_x^*|\boldsymbol{n}\rangle &= \begin{cases} (-1)^{\sum_{y\prec x} n_y}\,|\boldsymbol{n}+\delta_x\rangle & \text{if } n_x = 0, \\ 0 & \text{if } n_x = 1. \end{cases} \end{aligned} \tag{2.24}$$

The definitions extend to $\ell^2(\mathcal{N}^{(-)}_{\Lambda})$ and $\mathcal{F}^{(-)}_{\Lambda}$.

These operators are also adjoint of each other. They satisfy the anticommutation relations

$$\{a_x, a_y\} = 0; \qquad \{a_x^*, a_y^*\} = 0; \qquad \{a_x, a_y^*\} = \delta_{x,y}\mathbb{1}. \tag{2.25}$$

Here also we have that

$$a_x^* a_x|\boldsymbol{n}\rangle = n_x\,|\boldsymbol{n}\rangle. \tag{2.26}$$

One-body operators can be conveniently represented by creation and annihilation operators. Let $B = (b_{x,y})_{x,y\in\Lambda}$ be an operator on $\ell^2(\Lambda)$ (i.e. a $\Lambda\times\Lambda$ complex matrix). This yields the following operator on $\mathcal{H}_{\Lambda,n}$:

$$\boldsymbol{B} = \sum_{i=1}^{n} B_i, \tag{2.27}$$

where

$$B_i = \mathbb{1}\otimes\cdots\otimes \underbrace{B}_{i\text{th particle}} \otimes\cdots\otimes\mathbb{1}. \tag{2.28}$$

One easily checks that $[\boldsymbol{B}, P_\pm] = 0$ so $\boldsymbol{B}$ can also be viewed as an operator on $\mathcal{H}^{(+)}_{\Lambda,n}$ or $\mathcal{H}^{(-)}_{\Lambda,n}$.

Lemma 2.1. *On $\mathcal{H}^{(+)}_{\Lambda,n}$ or $\mathcal{H}^{(-)}_{\Lambda,n}$, we have*

$$\boldsymbol{B} = \sum_{x,y\in\Lambda} b_{x,y} a_x^* a_y.$$

Proof. Here we restrict to bosons, fermions are similar. Recalling that $\langle\boldsymbol{m}|a_x^* = \langle a_x\boldsymbol{m}|$, the matrix elements of the right side are

$$\langle\boldsymbol{m}|b_{x,y}a_x^*a_y|\boldsymbol{n}\rangle = \sqrt{m_x n_y}\;\delta_{\boldsymbol{m}-\delta_x,\boldsymbol{n}-\delta_y}. \tag{2.29}$$

Using that $P_+ = P_+^*$ commutes with $\boldsymbol{B}$ we get

$$\begin{aligned}\langle \boldsymbol{m}|\boldsymbol{B}|\boldsymbol{n}\rangle &= \frac{n!}{\sqrt{\prod_z m_z! n_z!}} \langle e_{x_1,\dots,x_n}|P_+^2 \boldsymbol{B}|e_{y_1,\dots,y_n}\rangle \\ &= \frac{n!}{\sqrt{\prod_z m_z! n_z!}} \sum_{i=1}^n \langle e_{x_1,\dots,x_n}|P_+^2 B_i|e_{y_1,\dots,y_n}\rangle.\end{aligned} \tag{2.30}$$

Here the sites $x_1, \dots, x_n$ are compatible with $\boldsymbol{m}$ and the sites $y_1, \dots, y_n$ are compatible with $\boldsymbol{n}$. It suffices to consider a matrix B with a single nonzero entry, $b_{x,y} = 1$ for some fixed $x, y \in \Lambda$. The general case follows by linearity. For this B, we have that

$$B_i|e_{y_1,\dots,y_n}\rangle = \delta_{y_i,y}|e_{y_1,\dots,x,\dots,y_n}\rangle \tag{2.31}$$

where the vector on the right has an x in position i. Thus

$$\begin{aligned}\langle \boldsymbol{m}|\boldsymbol{B}|\boldsymbol{n}\rangle &= \Bigl(\frac{n!}{\prod_z n_z!}\Bigr)^{1/2} \sum_{i=1}^n \delta_{y_i,y} \langle \boldsymbol{m}|P_+|e_{y_1,\dots,x,\dots,y_n}\rangle \\ &= \Bigl(\frac{n!}{\prod_z n_z!}\Bigr)^{1/2} n_y \Bigl(\frac{n!}{\prod_z (n_z - \delta_y + \delta_x)!}\Bigr)^{-1/2} \langle \boldsymbol{m}|\boldsymbol{n} - \delta_y + \delta_x\rangle \\ &= n_y \bigl(\tfrac{n_x+1}{n_y}\bigr)^{1/2} \, \delta_{\boldsymbol{m}-\delta_x, \boldsymbol{n}-\delta_y}.\end{aligned} \tag{2.32}$$

This agrees with (2.29). □

One can generalise this lemma to many-body operators. A natural hamiltonian for lattice particles with two-body interactions is

$$H_\Lambda = -\sum_{i=1}^n \Delta_i + \sum_{1 \le i < j \le n} V_{i,j}, \tag{2.33}$$

where $\Delta_i = \mathbb{1} \otimes \dots \otimes \Delta \otimes \dots \otimes \mathbb{1}$ and Δ is the discrete laplacian such that

$$(\Delta\varphi)(x) = \sum_{y \in \Lambda} t_{x,y} \varphi(y). \tag{2.34}$$

Here $t_{x,y} = t_{y,x} \in \mathbb{R}$ is finite-range or fast decaying (the standard case involves same sites and nearest-neighbours). The interactions are given by a multiplication operator

$$V_{i,j}\varphi(x_1, \dots, x_n) = W(x_i - x_j)\varphi(x_1, \dots, x_n). \tag{2.35}$$

Here $W(x)$ is a real function, of finite range or with fast decay. The hamiltonian above represents the energy of n particles, that consists of kinetic energy (the laplacians) and pair interactions (given by W). The hamiltonian above is both symmetric and antisymmetric, in the sense that $[H_\Lambda, P_\pm] = 0$, and its action on

$\mathcal{H}_{\Lambda,n}^{(\pm)}$ can be written as

$$H_\Lambda = -\sum_{x,y\in\Lambda} t_{x,y} a_x^* a_y + \tfrac12 W(0) \sum_{x\in\Lambda} a_x^* a_x (a_x^* a_x - 1) + \tfrac12 \sum_{\substack{x,y\in\Lambda \\ x\neq y}} W(x-y) a_x^* a_x a_y^* a_y. \quad (2.36)$$

As an operator in $\mathcal{F}_\Lambda^{(+)}$ it is unbounded. It is well defined on $\ell^2_{\rm f}(\mathcal{N}_\Lambda^{(+)})$; it is symmetric, and its closure is self-adjoint.

One can take the limit $W(0) \to \infty$, which yields *hard-core bosons*, where at most one particle per site is allowed. The Hilbert space is then identical to that of $S = \frac12$ quantum spins. One can identify $n_x = 0$ with $\sigma_x = -\frac12$, and $n_x = 1$ with $\sigma_x = \frac12$. As for operators we have

$$a_x \equiv S_x^{(-)}, \qquad a_x^* \equiv S_x^{(+)}, \qquad a_x^* a_x \equiv S_x^{(3)} + \tfrac12. \quad (2.37)$$

2.3. Bose–Einstein condensation

We say that a two-body potential W is **stable** if there exists a constant B such for all $n \in \mathbb{N}$ and all $x_1, \dots, x_n \in \mathbb{Z}^d$, we have the lower bound

$$\sum_{1\leq i<j\leq n} W(x_i - x_j) \geq -Bn. \quad (2.38)$$

Typical examples are nonnegative (repulsive) potentials (the inequality is trivial, with $B = 0$) and potentials that are repulsive at short distance but attractive at longer distance. This condition guarantees that the particles of a large system spread everywhere, and do not collapse in a small region. This property is necessary for statistical mechanics to hold.

We define the **free energy** of a particle system by

$$\begin{aligned} f_{\Lambda,n}(\beta) &= -\frac{1}{\beta|\Lambda|} \log \operatorname{Tr}_{\mathcal{H}_{\Lambda,n}^{(\pm)}} \mathrm{e}^{-\beta H_\Lambda}. \\ f(\beta,\rho) &= \lim_{\Lambda\Uparrow\mathbb{Z}^d} f_{\Lambda,\lfloor\rho|\Lambda|\rfloor}(\beta). \end{aligned} \quad (2.39)$$

Here, the parameter ρ is the **density**. Existence of the limit can be proved in a similar fashion as for spin systems (the stability condition gives a lower bound for $f_{\Lambda,n}$ that is necessary for the subadditive argument). Introducing the number operator

$$N_\Lambda|\varphi\rangle = n|\varphi\rangle, \qquad \varphi \in \mathcal{H}_{\Lambda,n}^{(\pm)}, \quad (2.40)$$

we define the **pressure** by

$$\begin{aligned} p_\Lambda(\beta,\mu) &= \frac{1}{|\Lambda|} \log \operatorname{Tr}_{\mathcal{F}_\Lambda^{(\pm)}} \mathrm{e}^{-\beta(H_\Lambda - \mu N_\Lambda)}. \\ p(\beta,\mu) &= \lim_{\Lambda\Uparrow\mathbb{Z}^d} p_\Lambda(\beta,\mu). \end{aligned} \quad (2.41)$$

The functions $f(\beta,\rho)$ and $p(\beta,\mu)$ are related by Legendre transforms.

The hamiltonian commutes with the number of particles in the box, $[H_\Lambda, N_\Lambda] = 0$, which implies the presence of a continuous $U(1)$ symmetry:

$$H_\Lambda = \mathrm{e}^{\mathrm{i}\theta N_\Lambda} H_\Lambda \, \mathrm{e}^{-\mathrm{i}\theta N_\Lambda}, \qquad \theta \in [0, 2\pi). \tag{2.42}$$

The corresponding order parameter is the **off-diagonal long range order** proposed by Penrose and Onsager [1956]: the correlation function $\langle a_x^* a_y \rangle_{\Lambda,\beta}$ (in either the canonical or gran canonical ensemble). The question is whether it remains positive in the infinite volume limit, and as $\|x - y\| \to \infty$.

In the hard-core Bose, which is equivalent to the quantum XY model, off-diagonal long range order is equivalent to spontaneous magnetisation in the XY plane. The latter can be proved using reflection positivity (Dyson, Lieb, Simon [1978], see previous chapter). This is the only known proof of Bose–Einstein condensation in an interacting Bose gas, in the standard setting.

We conclude the chapter by describing the Bose–Einstein condensation of the ideal gas (no interactions) on the lattice.

Let $\Lambda_\ell^{\mathrm{per}} = \{1, \dots, \ell\}^d$ with periodic boundary conditions. We consider the model (2.36) with $W \equiv 0$.

Theorem 2.2.

$$\lim_{\ell \to \infty} \frac{1}{|\Lambda_\ell^{\mathrm{per}}|} \sum_{x \in \Lambda_\ell^{\mathrm{per}}} \langle a_0^* a_x \rangle_{\Lambda_\ell^{\mathrm{per}}, \beta, \lfloor \rho \ell^d \rfloor} = \max(0, \rho - \rho_{\mathrm{c}})$$

where the critical density is equal to

$$\rho_{\mathrm{c}} = \frac{1}{(2\pi)^d} \int_{[-\pi,\pi]^d} \frac{1}{\mathrm{e}^{\beta\varepsilon(k)} - 1} \mathrm{d}k.$$

Recall that $\varepsilon(k) = \sum_x t_{0,x} \, \mathrm{e}^{-\mathrm{i}kx}$. The critical density is finite when $d \geq 3$. In the continuum we have $\varepsilon(k) = k^2$; one can expand the fraction as geometric series, integrate the gaussians, and one gets the well-known formula of Einstein.

Proof. Let us introduce the creation and annihilation operators of the Fourier modes, namely

$$\hat{a}_k = \frac{1}{\ell^{d/2}} \sum_{x \in \Lambda_\ell^{\mathrm{per}}} \mathrm{e}^{-\mathrm{i}kx} \, a_x, \qquad k \in \Lambda_\ell^*. \tag{2.43}$$

Then we have

$$a_x = \frac{1}{\ell^{d/2}} \sum_{k \in \Lambda_\ell^*} \mathrm{e}^{\mathrm{i}kx} \, \hat{a}_k \tag{2.44}$$

and

$$\sum_{x,y \in \Lambda_\ell^{\mathrm{per}}} t_{x,y} a_x^* a_y = \sum_{k \in \Lambda_\ell^*} \varepsilon(k) \hat{a}_k^* \hat{a}_k. \tag{2.45}$$

One can also check that the eigenvalues of $\hat{a}_k^* \hat{a}_k$ are $0, 1, 2, \ldots$. We also have

$$\frac{1}{\ell^d} \sum_{x \in \Lambda_\ell^{\rm per}} \langle a_0^* a_x \rangle_{\Lambda_\ell^{\rm per},\beta,n} = \frac{1}{\ell^{2d}} \sum_{x,y \in \Lambda_\ell^{\rm per}} \langle a_x^* a_y \rangle_{\Lambda_\ell^{\rm per},\beta,n} = \frac{1}{\ell^d} \langle \hat{a}_0^* \hat{a}_0 \rangle_{\Lambda_\ell^{\rm per},\beta,n}. \tag{2.46}$$

The relevant expectation can then be written using random partitions $(n_k)_{k \in \Lambda_\ell^*}$ indexed by Λ_ℓ^* and satisfying $\sum_k n_k = n$. Namely,

$$\frac{1}{\ell^d} \sum_{x \in \Lambda_\ell^{\rm per}} \langle a_0^* a_x \rangle_{\Lambda_\ell^{\rm per},\beta,n} = \frac{1}{\ell^d} \langle \hat{a}_0^* \hat{a}_0 \rangle_{\Lambda_\ell^{\rm per},\beta,n} = \frac{1}{Z_{\Lambda_\ell^{\rm per},\beta,\rho}} \sum_{(n_k)_{k \in \Lambda_\ell^*} : \sum_k n_k = n} \frac{n_0}{\ell^k} \, {\rm e}^{-\beta \sum_k \varepsilon(k) n_k}. \tag{2.47}$$

We denote $\mathbb{P}, \mathbb{E}$ the corresponding probability and expectation where a partition (n_k) has probability proportional to ${\rm e}^{-\beta \sum_k \varepsilon(k) n_k}$. We have

$$\begin{aligned}
\frac{1}{\ell^d} \sum_{x \in \Lambda_\ell^{\rm per}} \langle a_0^* a_x \rangle_{\Lambda_\ell^{\rm per},\beta,n} &= \frac{1}{\ell^d} \mathbb{E}[n_0] = \frac{n}{\ell^d} - \frac{1}{\ell^d} \sum_{k \neq 0} \mathbb{E}[n_k] \\
&= \frac{n}{\ell^d} - \frac{1}{\ell^d} \sum_{k \neq 0} \sum_{i \geq 1} \mathbb{P}[n_k \geq i] \\
&= \frac{n}{\ell^d} - \frac{1}{\ell^d} \sum_{k \neq 0} \frac{1}{Z_{\Lambda_\ell^{\rm per},\beta,n}} \sum_{i \geq 1} \sum_{(n_{k'}) : \sum_{k'} n_k = n, n_k \geq i} {\rm e}^{-\beta \sum_{k'} \varepsilon(k') n_{k'}} \\
&= \frac{n}{\ell^d} - \frac{1}{\ell^d} \sum_{k \neq 0} \sum_{i \geq 1} {\rm e}^{-\beta \varepsilon(k) i} \, \frac{Z_{\Lambda_\ell^{\rm per},\beta,n-i}}{Z_{\Lambda_\ell^{\rm per},\beta,n}} \\
&\geq \frac{n}{\ell^d} - \frac{1}{\ell^d} \sum_{k \neq 0} \frac{1}{{\rm e}^{\beta \varepsilon(k)} - 1}.
\end{aligned} \tag{2.48}$$

Notice that the ratio of partition functions is equal to $\mathbb{P}[n_0 \geq i]$ which is less than 1. As $\ell \to \infty$, the last term converges to $\rho - \rho_{\rm c}$.

It is perhaps worth noting the infrared bound $\mathbb{E}[n_k] \leq ({\rm e}^{\beta \varepsilon(k)} - 1)^{-1}$, which implies long-range order as in the case of spin systems.

In order to prove the converse bound, let us observe that the pressure of the ideal Bose gas can be computed exactly, yielding (for $\mu < 0$)

$$p(\beta, \mu) = \lim_{\ell \to \infty} \frac{1}{\ell^d} \sum_{(n_k)} {\rm e}^{-\beta \sum_k (\varepsilon(k) - \mu) n_k} = -\frac{1}{(2\pi)^d} \int_{[-\pi,\pi]^d} \log\Big(1 - {\rm e}^{-\beta(\varepsilon(k) - \mu)}\Big) {\rm d}k. \tag{2.49}$$

The density is

$$\rho(\beta, \mu) = \frac{1}{\beta} \frac{\partial}{\partial \mu} p(\beta, \mu) = \frac{1}{(2\pi)^d} \int_{[-\pi,\pi]^d} \frac{1}{{\rm e}^{\beta(\varepsilon(k) - \mu)} - 1} {\rm d}k. \tag{2.50}$$

The critical density is equal to the limit $\mu \to 0-$ of $\rho(\beta,\mu)$. The free energy is given by the Legendre transform

$$f(\beta,\rho) = \sup_{\mu<0} \big(\rho\mu - \tfrac{1}{\beta} p(\beta,\mu) \big). \tag{2.51}$$

The plot of the pressure and its Legendre transform can be found in Fig. 2.1.

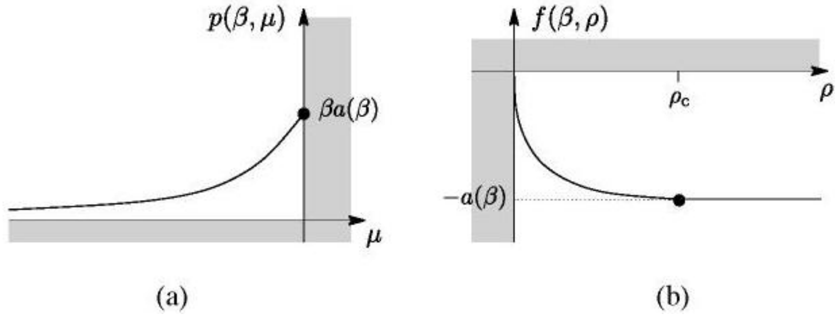

FIGURE 2.1. (a) The pressure of the ideal Bose gas; (b) its Legendre transform, the free energy.

For any $\eta \geq 0$, we have that

$$\lim_{\ell\to\infty} \frac{1}{\beta\ell^d} \log \mathbb{P}[n_0 \geq \ell^d \eta] = \lim_{\ell\to\infty} \frac{1}{\beta\ell^d} \log \frac{Z_{\Lambda_\ell^{\rm per},\beta,n-\ell^d\eta}}{Z_{\Lambda_\ell^{\rm per},\beta,n}} = f(\beta,\rho) - f(\beta,\rho-\eta). \tag{2.52}$$

If $\eta > \max(0, \rho - \rho_{\rm c})$, we have

$$\mathbb{P}[n_0 \geq \ell^d \eta] \leq \mathrm{e}^{-\ell^d \delta} \tag{2.53}$$

for some $\delta > 0$. It follows that $\frac{1}{\ell^d}\mathbb{E}[n_0] \leq \max(0, \rho - \rho_{\rm c})$, which completes the proof. □

2.4. The Hubbard model

The Hubbard model is a lattice model that involves itinerant electrons, which are fermions carrying $S = \frac{1}{2}$ spins. It is the most important model in condensed matter physics. It is a gross simplification of the description of a large system of electrons that interact among themselves and with fixed nuclei. It is nonetheless a difficult model to study, and people expect it to have very rich phase diagrams. At low temperatures, and depending on the lattice and on the particle density, the system may exhibit spontaneous magnetisation, antiferromagnetic long-range order, or superconductivity. We recommend the recent book of Tasaki [**29**] for an excellent introduction to the Hubbard model. Here we give a minimal introduction to the setting and we show a connection to the Heisenberg antiferromagnet using a suitable perturbative expansion.

Let $\Lambda \Subset \mathbb{Z}^d$. A configuration of electrons carrying spins is denoted $\eta = (\eta_{x,\sigma})$ where $\eta_{x,\sigma} \in \{0,1\}$ with $x \in \Lambda$ and $\sigma \in \{\uparrow,\downarrow\}$. Let $\mathfrak{H}_\Lambda$ denote the set of configurations:

$$\mathfrak{H}_\Lambda = \{0,1\}^{\Lambda\times\{\uparrow,\downarrow\}} \simeq \{0,\uparrow,\downarrow,2\}^\Lambda. \tag{2.54}$$

We have $|\mathfrak{H}_\Lambda| = 4^{|\Lambda|}$. The Hilbert space is $\mathcal{H}_\Lambda = \ell^2(\mathfrak{H}_\Lambda)$ whose dimension is $4^{\Lambda|}$.

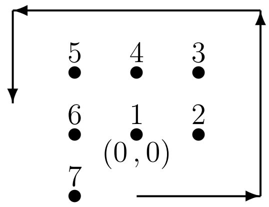

FIGURE 2.2. The spiral order in $\mathbb{Z}^2$ (figure taken from [**6**]).

We need to choose an order on $\Lambda \times \{\uparrow,\downarrow\}$. It is convenient to consider the spiral order in $\mathbb{Z}^d$ (it is illustrated in Fig. 2.2), which has the advantage that the number of preceding sites is always finite. We also decide that $\downarrow<\uparrow$. Then

$$(x,\sigma) < (y,\sigma') \iff x < y \text{ or } \big(x = y \text{ and } \sigma < \sigma'\big). \tag{2.55}$$

This allows to define the creation and annihilation operators as follows. Given $\eta \in \mathfrak{H}_\Lambda$ and $x \in \Lambda$, $\sigma \in \{\uparrow,\downarrow\}$,

$$\begin{aligned} c^*_{x,\sigma}|\eta\rangle &= (-1)^{\sum_{(y,\sigma')<(x,\sigma)} \eta_{y,\sigma'}} \begin{cases} |\eta + \delta_{x,\sigma}\rangle & \text{if } \eta_{x,\sigma} = 0, \\ 0 & \text{otherwise.} \end{cases} \\ c_{x,\sigma}|\eta\rangle &= (-1)^{\sum_{(y,\sigma')<(x,\sigma)} \eta_{y,\sigma'}} \begin{cases} |\eta - \delta_{x,\sigma}\rangle & \text{if } \eta_{x,\sigma} = 1, \\ 0 & \text{otherwise.} \end{cases} \end{aligned} \tag{2.56}$$

The prefactor looks cumbersome but one rarely invokes it in calculations. One should rather keep in mind that $c^*_{x,\sigma}$ and $c_{x,\sigma}$ are adjoint of each other, and that we have the following anticommutation relations, for any $x, y \in \Lambda$ and $\sigma,\sigma' \in \{\uparrow,\downarrow\}$:

$$\begin{aligned} \{c^*_{x,\sigma}, c^*_{y,\sigma'}\} &= \{c_{x,\sigma}, c_{y,\sigma'}\} = 0, \\ \{c^*_{x,\sigma}, c_{y,\sigma'}\} &= \delta_{x,y}\delta_{\sigma,\sigma'}\mathbb{1}. \end{aligned} \tag{2.57}$$

We also introduce the number operators $n_{x,\sigma} = c^*_{x,\sigma}c_{x,\sigma}$ that satisfy

$$n_{x,\sigma}|\eta\rangle = \eta_{x,\sigma}|\eta\rangle. \tag{2.58}$$

Let $n_x = n_{x,\uparrow} + n_{x,\downarrow}$. Further, all basis vectors can be expressed using a product a creation operators that act on the "vacuum state" $|0\rangle$, which is the vector that corresponds to the empty configuration $\eta \equiv 0$. Namely,

$$|\eta\rangle = c^*_{x_1,\sigma_1}c^*_{x_2,\sigma_2}\dots c^*_{x_N,\sigma_N}|0\rangle. \tag{2.59}$$

Here $(x_1, \sigma_1) < \cdots < (x_N, \sigma_N)$ are such that $\eta_{x_i,\sigma_i} = 1$ and the order is according to (2.55). N is the number of particles in the state $|\eta\rangle$, it is equal to $\sum_{x,\sigma} \eta_{x,\sigma}$. The relations (2.57)–(2.59) are verified in Exercise 2.9.

The Hubbard model depends on three real parameters. t is in front of the kinetic energy of the electrons and is related to their mass; U is the strength of the onsite interactions (whose physical origin is the Coulomb interactions between electrons); μ is the chemical potential that allows to vary the density.

Hubbard hamiltonian:

$$H_\Lambda = -t \sum_{xy \in \mathcal{E}(\Lambda)} \sum_{\sigma=\downarrow,\uparrow} \big(c^*_{x,\sigma} c_{y,\sigma} + c^*_{y,\sigma} c_{x,\sigma} \big) + U \sum_{x\in\Lambda} n_{x,\uparrow} n_{x,\downarrow} - \mu \sum_{x\in\Lambda} (n_{x,\uparrow} + n_{x,\downarrow}). \tag{2.60}$$

Partition functions and Gibbs states are defined exactly as for spin systems. The terms in the Hubbard model are made of *even* numbers of creation and annihilation operators, and they therefore commute when their supports are disjoint. The construction of the evolution operator, Proposition 3.14, remains valid, and KMS states are well defined. The other characterisations of infinite volume Gibbs states also hold.

2.5. Spin operators and symmetries

Using creation and annihilation operators, we define

$$S^{(+)}_x = c^*_{x,\uparrow} c_{x,\downarrow}, \qquad S^{(-)}_x = c^*_{x,\downarrow} c_{x,\uparrow}, \qquad S^{(3)}_x = \tfrac12 (n_{x,\uparrow} - n_{x,\downarrow}). \tag{2.61}$$

We further set $S^{(1)}_x = \frac12 (S^{(+)}_x + S^{(-)}_x)$ and $S^{(2)}_x = \frac12 (S^{(+)}_x - S^{(-)}_x)$. Then we have (Exercise 2.10)

$$[S^{(1)}_x, S^{(2)}_y] = \mathrm{i}\delta_{x,y} S^{(3)}_x, \quad [S^{(2)}_x, S^{(3)}_y] = \mathrm{i}\delta_{x,y} S^{(1)}_x, \quad [S^{(3)}_x, S^{(1)}_y] = \mathrm{i}\delta_{x,y} S^{(2)}_x. \tag{2.62}$$

Let us list some symmetries of the Hubbard hamiltonian.

PROPOSITION 2.3.

(a) SU(2) symmetry: $[H_\Lambda, \sum_{x\in\Lambda} S^{(i)}_x] = 0$ *for* $i = 1, 2, 3$.

(b) Gauge invariance: $[H_\Lambda, \sum_{x\in\Lambda} n_x] = 0$.

The proof is in Exercise 2.11. This allow to construct the unitary operators $U^{(\vec a)}_\Lambda = \mathrm{e}^{\mathrm{i} \sum_{x\in\Lambda} \vec a \cdot \vec S_x}$, $\vec a \in \mathbb{R}^3$, and $U^{(\theta)}_\Lambda = \mathrm{e}^{\mathrm{i}\theta \sum_{x\in\Lambda} n_x}$ that allow to establish that finite-volume Gibbs states satisfy

$$\langle S^{(i)}_x \rangle_{\Lambda,\beta} = 0, \qquad \langle c^*_{x,\uparrow} c^*_{y,\downarrow} \rangle_{\Lambda,\beta} = 0. \tag{2.63}$$

But one cannot rule out the existence of infinite-volume Gibbs states where the above expectations differ from 0; the first is a state displaying magnetic properties, the second is a superconductive state with Cooper pairs.

Next we consider particle-hole transformations. This is not a symmetry of the model, but it allows to relate hamiltonians with different parameters. Let $U^{\rm ph}_{x,\sigma} = {\rm i}(c^*_{x,\sigma} + c_{x,\sigma})$. We then have

$$(U^{\rm ph}_{x,\sigma})^{-1} c_{x,\sigma} U^{\rm ph}_{x,\sigma} = c^*_{x,\sigma}, \qquad (U^{\rm ph}_{x,\sigma})^{-1} c^*_{x,\sigma} U^{\rm ph}_{x,\sigma} = c_{x,\sigma}. \tag{2.64}$$

Let $U^{\rm ph}_\Lambda = \prod_{x\in\Lambda} U^{\rm ph}_{x,\uparrow} U^{\rm ph}_{x,\downarrow}$. Then we have

$$(U^{\rm ph}_\Lambda)^{-1} H^{t,U,\mu}_\Lambda U^{\rm ph}_\Lambda = H^{-t,U,U-\mu}_\Lambda + (U-\mu)|\Lambda|. \tag{2.65}$$

This is checked in Exercise 2.13. This shows that the sign of t does not matter.

2.6. Relation to the antiferromagnetic Heisenberg model

We now explain a perturbation expansion that applies to half-filling (density 1) and U large. The particular feature of the expansion is to deal with interactions rather than operators; it was first developed and used in [**6**].

We consider the hamiltonian $H_\Lambda = H^{(0)}_\Lambda + tT_\Lambda$ where

$$\begin{aligned} H^{(0)}_\Lambda &= U \sum_{x\in\Lambda} (n_{x,\uparrow} - \tfrac12)(n_{x,\downarrow} - \tfrac12), \\ T_\Lambda &= \sum_{xy\in\mathcal{E}(\Lambda)} \sum_{\sigma=\uparrow,\downarrow} \big(c^*_{x,\sigma} c_{y,\sigma} + c^*_{y,\sigma} c_{x,\sigma} \big). \end{aligned} \tag{2.66}$$

This the Hubbard model at half-filling, where a particle-hole symmetry guarantees that $\langle n_x\rangle_{\Lambda,\beta} = 1$ for any bipartite lattice Λ and any β (see Exercise 2.14).

In order to introduce the expansion, recall the definition of the adjoint operation and of its inverse: For A, B hermitian matrices, we let

$$\begin{aligned} {\rm ad}_A B &= [A,B], \\ {\rm ad}_A^{-1} B &= \sum_{\substack{a,a'\in{\rm Spec}(A) \\ a\neq a'}} P_a \frac{B}{a-a'} P_{a'}. \end{aligned} \tag{2.67}$$

Here P_a is the projector onto the eigensubspace of A with eigenvalue a. One can check (Exercise 2.15) that ${\rm ad}_A$ and ${\rm ad}_A^{-1}$ are inverse operations, in the sense that

$${\rm ad}_A {\rm ad}_A^{-1} B = {\rm ad}_A^{-1} {\rm ad}_A B = B^{\rm od}, \tag{2.68}$$

where $B^{\rm od} = \sum_{a\neq a'} P_a B P_{a'}$ is the off-diagonal part of B in the basis of eigenvectors of A.

Let S_Λ such that $S^*_\Lambda = -S_\Lambda$ and let $U_\Lambda = {\rm e}^{tS_\Lambda}$. By the Lie-Schwinger expansion (Lemma 3.15) we have

$$U_\Lambda H_\Lambda U_\Lambda^{-1} = H^{(0)}_\Lambda + tT_\Lambda + t{\rm ad}_{S_\Lambda} H^{(0)}_\Lambda + t^2 {\rm ad}_{S_\Lambda} T_\Lambda + \tfrac12 t^2 {\rm ad}^2_{S_\Lambda} H^{(0)}_\Lambda + O(t^3). \tag{2.69}$$

We require that $T_\Lambda = -\mathrm{ad}_{S_\Lambda} H_\Lambda^{(0)} = \mathrm{ad}_{H_\Lambda^{(0)}} S_\Lambda$, so we choose

$$S_\Lambda = \mathrm{ad}^{-1}_{H_\Lambda^{(0)}} T_\Lambda = \sum_{xy\in\mathcal{E}(\Lambda)} \mathrm{ad}^{-1}_{H_\Lambda^{(0)}} T_{xy} \equiv \sum_{xy\in\mathcal{E}(\Lambda)} S_{xy}. \tag{2.70}$$

We also have $\mathrm{ad}^2_{S_\Lambda} H_\Lambda^{(0)} = -\mathrm{ad}_{S_\Lambda} T_\Lambda$. We obtain the expansion

$$U_\Lambda H_\Lambda U_\Lambda^{-1} = H_\Lambda^{(0)} + \tfrac12 t^2 \mathrm{ad}_{S_\Lambda} T_\Lambda + O(t^3). \tag{2.71}$$

We can check that

$$\begin{aligned}\langle 0,2|S_{xy}|\uparrow,\downarrow\rangle = \langle 2,0|S_{xy}|\uparrow,\downarrow\rangle = -\frac{1}{U} = -\langle\uparrow,\downarrow|S_{xy}|0,2\rangle = -\langle\uparrow,\downarrow|S_{xy}|2,0\rangle,\\ \langle 0,2|S_{xy}|\downarrow,\uparrow\rangle = \langle 2,0|S_{xy}|\downarrow,\uparrow\rangle = +\frac{1}{U} = -\langle\downarrow,\uparrow|S_{xy}|0,2\rangle = -\langle\downarrow,\uparrow|S_{xy}|2,0\rangle.\end{aligned} \tag{2.72}$$

We now restrict on the subspace with exactly one particle per site. Let $P_\Lambda^{(1)}$ be the corresponding projector, namely

$$P_\Lambda^{(1)} = \otimes_{x\in\Lambda}\big(|\uparrow\rangle\langle\uparrow| + |\downarrow\rangle\langle\downarrow|\big). \tag{2.73}$$

Notice that $P_\Lambda^{(1)} H_\Lambda P_\Lambda^{(1)} = -\frac14 U|\Lambda| P_\Lambda^{(1)}$; in order to obtain a nontrivial perturbation result, we need to expand before we apply the projector! One can check that $P_\Lambda^{(1)}(\mathrm{ad}_{S_\Lambda} T_\Lambda) P_\Lambda^{(1)}$ is a sum of nearest-neighbour terms only. We have

$$\begin{aligned}\langle\uparrow,\downarrow|S_{xy}T_{xy}|\uparrow,\downarrow\rangle = \langle\downarrow,\uparrow|S_{xy}T_{xy}|\downarrow,\uparrow\rangle = -\frac{2}{U},\\ \langle\uparrow,\downarrow|S_{xy}T_{xy}|\downarrow,\uparrow\rangle = \langle\downarrow,\uparrow|S_{xy}T_{xy}|\uparrow,\downarrow\rangle = +\frac{2}{U}.\end{aligned} \tag{2.74}$$

The matrix elements involving $|\uparrow,\uparrow\rangle$ and $|\downarrow,\downarrow\rangle$ are zero. One gets (Exercise 2.16)

$$P_{xy}^{(1)}[S_{xy}, T_{xy}] P_\Lambda^{(1)} = \frac{8}{U} P_\Lambda^{(1)} \big(\vec S_x \cdot \vec S_y - \tfrac14\big) P_\Lambda^{(1)}. \tag{2.75}$$

We have obtained:

Theorem 2.4.

$$P_\Lambda^{(1)} U_\Lambda H_\Lambda U_\Lambda^{-1} P_\Lambda^{(1)} = \frac{4t^2}{U} \sum_{xy\in\Lambda} P_\Lambda^{(1)}\, \vec S_x \cdot \vec S_y\, P_\Lambda^{(1)} - \tfrac14 U|\Lambda| P_\Lambda^{(1)} + O(t^3).$$

The $O(t^3)$ term is a sum of local interactions with exponential decay. The first term is indeed the antiferromagnetic Heisenberg model. This expansion suggests the existence of a phase with antiferromagnetic long-range order. But the Hubbard hamiltonian is not reflection positive and such a phase has not been mathematically proved.

EXERCISE 2.1. *Verify that the operators $P_\pm$ defined in (2.5)–(2.6) are indeed projectors.*

EXERCISE 2.2. *Let $x_1, \dots, x_n \in \Lambda$ such that $x_i = x_j$ for some $i \neq j$. Check that $P_- e_{x_1,\dots,x_n} = 0$.*

EXERCISE 2.3. *Let $(n_x) \in \mathcal{N}_{\Lambda,n}^{(\pm)}$ and $(x_1, \dots, x_n)$ such that $\#\{i = 1, \dots, n : x_i = x\} = n_x$ for all $x \in \Lambda$. Verify that*

$$\|P_\pm e_{x_1,\dots,x_n}\| = \left(\frac{\prod_{x\in\Lambda} n_x!}{n!} \right)^{1/2}.$$

EXERCISE 2.4. *Verify Eq. (2.13) about the dimensions of the symmetric and antisymmetric spaces.*

EXERCISE 2.5. *Verify that that a_x and a_x^* are adjoint of one another. In the bosonic case, this involves their domains.*

EXERCISE 2.6. *Verify the commutation relations (2.22) and (2.25).*

EXERCISE 2.7. *Give the proof of Lemma 2.1 in the fermionic case.*

EXERCISE 2.8. *In this exercise we outline a variant of the proof of Theorem 2.2, starting from the probabilistic representation in (2.47) and (2.48).*

(1) *Show that we can write*

$$\frac{1}{\ell^d} \sum_{k\neq 0} \mathbb{E}[n_k] = \mathbb{E}[X_\ell \mid X_\ell \leq \rho] \tag{2.76}$$

where

$$X_\ell = \frac{1}{\ell^d} \sum_{k\in\Lambda_\ell^*\setminus\{0\}} N_k \tag{2.77}$$

and the N_k are independent geometric random variables:

$$\mathbb{P}(N_k = r) = (\,\mathrm{e}^{-\beta\varepsilon(k)}\,)^r (1 - \,\mathrm{e}^{-\beta\varepsilon(k)}\,), \qquad r \geq 0. \tag{2.78}$$

The goal is thus to show that

$$\lim_{\ell\to\infty} \mathbb{E}[X_\ell \mid X_\ell \leq \rho] = \begin{cases} \rho & \text{if } \rho \leq \rho_\mathrm{c}, \\ \rho_\mathrm{c} & \text{if } \rho \geq \rho_\mathrm{c}. \end{cases} \tag{2.79}$$

(2) *Clearly $\lim_{\ell\to\infty} \mathbb{E}[X_\ell \mid X_\ell \leq \rho] \leq \rho$. Show that $\lim_{\ell\to\infty} \mathbb{E}[X_\ell] = \rho_\mathrm{c}$.*

(3) *Show that*

$$\mathbb{E}[(X_\ell - \rho_\mathrm{c})^2] \to 0, \qquad \text{as } \ell \to \infty. \tag{2.80}$$

(4) *Use Markov's inequality to deduce that*

$$\lim_{\ell\to\infty} \mathbb{E}[X_\ell \mid X_\ell \leq \rho] = \rho_\mathrm{c} \tag{2.81}$$

whenever $\rho > \rho_\mathrm{c}$.

(5) *It remains to show that* $\lim_{\ell\to\infty} \mathbb{E}[X_\ell \mid X_\ell \leq \rho] \geq \rho$ *when* $\rho \leq \rho_{\mathrm{c}}$, *and this is the hardest part. It uses ideas from large deviations theory.*
 (a) *Show that*
$$\Lambda(t) := \lim_{\ell\to\infty} \frac{1}{\ell^d} \log \mathbb{E}[\, \mathrm{e}^{t\ell^d X_\ell}\,] \tag{2.82}$$
 exists in $[-\infty, \infty]$ *for all* $t \in \mathbb{R}$.
 (b) *Deduce that for any* $x < \rho_{\mathrm{c}}$
$$\lim_{\ell\to\infty} -\frac{1}{\ell^d} \log \mathbb{P}(X_\ell \leq x) = \Lambda^*(x) := \sup_{t\in\mathbb{R}} \big(xt - \Lambda(t)\big) \tag{2.83}$$
 (c) *For any* $\rho \leq \rho_{\mathrm{c}}$ *and* $\delta > 0$,
$$\mathbb{E}[X_\ell \mid X_\ell \leq \rho] \geq (\rho - \delta)\Big(1 - \frac{\mathbb{P}(X_\ell \leq \rho - \delta)}{\mathbb{P}(X_\ell \leq \rho - \delta/2)}\Big). \tag{2.84}$$
 (d) *Deduce the result.*

Exercise 2.9.
 (a) *From the definition (2.56) of the creation and annihilation operators, check that the relations (2.57) and (2.58) hold true.*
 (b) *Check that the vectors* $|\eta\rangle$ *defined in (2.59) form an orthonormal basis of* $\mathcal{H}_\Lambda$.
 (c) *Consider a vector of the form (2.59) where* $(x_i, \sigma_i) = (x_j, \sigma_j)$ *for some* $i \neq j$ *(such a vector does not correspond to a configurattion). Check that such a vector is 0.*

Exercise 2.10. *Check the spin commutation relations in (2.62).*

Exercise 2.11. *Prove Proposition 2.3. For this, you may find it convenient to prove*
 (a) $[H_\Lambda, \sum_{x\in\Lambda} S_x^{(\pm)}] = 0$, $[H_\Lambda, \sum_{x\in\Lambda} S_x^{(3)}] = 0$.
 (b) $\mathrm{e}^{\mathrm{i}\theta n_x}\, c^*_{x,\sigma}\, \mathrm{e}^{-\mathrm{i}\theta n_x} = \mathrm{e}^{\mathrm{i}\theta}\, c^*_{x,\sigma}$ *and* $\mathrm{e}^{\mathrm{i}\theta n_x}\, c_{x,\sigma}\, \mathrm{e}^{-\mathrm{i}\theta n_x} = \mathrm{e}^{-\mathrm{i}\theta}\, c^*_{x,\sigma}$.

Exercise 2.12. *Use the symmetries of the Hubbard model to prove the relations (2.63).*

Exercise 2.13. *Check that the particle-hole symmetry operator* $U^{\mathrm{ph}}_{x,\sigma}$ *is unitary and check the relations (2.64) and (2.65).*

Exercise 2.14. *Show that* $\langle n_x \rangle_{\Lambda,\beta} = 1$ *where the hamiltonian is that of the half-filled Hubbard model defined in (2.66). The lattice is assumed here to be bipartite, that is,* $\Lambda = \Lambda_A \cup \Lambda_B$ *and* $\mathcal{E}(\Lambda)$ *only involve edges with one endpoint in* Λ_A *and the other in* Λ_B.

Exercise 2.15. *Prove the relations (2.68).*

EXERCISE 2.16. *Check Eqs (2.74) and (2.75). You may use* $\vec{S}_x \cdot \vec{S}_y = \frac{1}{2}(S_x^{(+)} S_y^{(-)} + S_x^{(-)} S_y^{(+)}) + S_x^{(3)} S_y^{(3)}$.