CHAPTER 6

2D systems with continuous symmetry

We consider translation-invariant Gibbs states for two-dimensional models, on $\mathbb{Z}^2$, for an interaction Φ which is invariant under a continuous group of rotations. We prove the absence of symmetry breaking in the sense that all infinite-volume Gibbs states retain the continuous symmetry.

Let $\Phi = (\Phi_X : X \Subset \mathbb{Z}^2) \in \mathcal{I}$ be an interaction such that there are Hermitian operators S_x, $x \in \mathbb{Z}^2$, with $[\Phi_X, \sum_{x\in X} S_x] = 0$ for all $X \Subset \mathbb{Z}^2$. Here S_x acts on the tensor factor associated with $x \in \mathbb{Z}^2$; in most examples they are copies of the same operator, though in principle they could be different. We assume that the operator norms of the S_x are uniformly bounded. We also assume that $\|\Phi\|_{8\pi+3} < \infty$ where $\|\Phi\|_r = \sum_{X\ni 0} \mathrm{e}^{r|X|}\, \|\Phi_X\|$ is the r-norm in (3.26), and that $\Phi_X = 0$ unless X is connected (if this does not hold we can always modify Φ to make it hold, however this may change the norm $\|\Phi\|_r$ and in particular whether or not $\|\Phi\|_r < \infty$).

For an angle $\theta \in [0, 2\pi]$ define the unitary rotation $U_X(\theta) = \exp(\sum_{x\in X} \mathrm{i}\theta S_x)$. The condition $[\Phi_X, \sum_{x\in X} S_x] = 0$ means that $U_X^*(\theta)\Phi_X U_X(\theta) = \Phi_X$ for any angle θ, thus the interaction is invariant under rotations.

Given a state $\langle\cdot\rangle \in \mathcal{G}^{\Phi}_{\mathrm{t.i.}}$ we can define another Gibbs state $\langle\cdot\rangle'$ by setting, for all $\Lambda \Subset \mathbb{Z}^2$ and all $A \in \mathcal{A}_\Lambda$, $\langle A\rangle' = \langle U_\Lambda^*(\theta) A U_\Lambda(\theta)\rangle$ (see Exercise 3.14). In this sense the *set* $\mathcal{G}^{\Phi}_{\mathrm{t.i.}}$ is invariant under the symmetry of the interaction. The following result shows that the Gibbs states are actually *individually* invariant under the symmetry:

Theorem 6.1. *Under the assumptions above, assume that* $\langle\cdot\rangle = \lim_{\Lambda_n \Uparrow \mathbb{Z}^2} \langle\cdot\rangle^{\Phi+\Psi_n}_{\Lambda_n} \in \mathcal{G}^{\Phi}_{\mathrm{t.i.}}$ *is a limiting Gibbs state, where the interactions* $\Psi_n \in \mathcal{I}$ *satisfy* $\|\Psi_n\| \to 0$. *Then for all* $\Lambda \Subset \mathbb{Z}^2$ *and all* $A \in \mathcal{A}_\Lambda$ *we have that*

$$\langle U_\Lambda^*(\theta) A U_\Lambda(\theta)\rangle = \langle A\rangle. \tag{6.1}$$

To put Theorem 6.1 in context, consider the XXZ-model ($J^{(1)} = J^{(2)}$ in (1.37)) in $\mathbb{Z}^2$. The assumption on continuous symmetry holds, e.g. with $S_x = S_x^{(3)}$. The model also has a discrete (spin-flip) symmetry, represented e.g. by $V = \mathrm{e}^{\mathrm{i}\pi S^{(1)}}$. Theorem 4.4 shows that, for $J^{(3)} = \Delta > 1$ and β large enough, there exists a Gibbs state $\langle\cdot\rangle$ satisfying $\langle S_0^{(3)} S_x^{(3)}\rangle \geq c > 0$ uniformly in all $x \in \mathbb{Z}^d$. One may in fact show that there is a translation-invariant Gibbs state $\langle\cdot\rangle^+$ satisfying

$\langle S_0^{(3)}\rangle^+ > 0$. Applying the spin-flip symmetry, the state $\langle\cdot\rangle^- := \langle\otimes_x V_x^* \cdot \otimes_x V_x\rangle^+$ is a also translation-invariant Gibbs state; it satisfies $\langle S_0^{(3)}\rangle^- = -\langle S_0^{(3)}\rangle^+ < 0$. In particular, $\langle\cdot\rangle^- \neq \langle\cdot\rangle^+$. In this sense, the discrete symmetry is broken: we have two distinct Gibbs states, related by the discrete symmetry. Theorem 6.1 shows that the corresponding conclusion does not hold for the continuous symmetry: the 'rotated' state $\langle\otimes_x U_x^*(\theta) \cdot \otimes_x U_x(\theta)\rangle$ coincides with $\langle\cdot\rangle$.

The underlying reason for this difference is the absence of *contours*. Recall that the key method for the proof of Theorem 4.4 was to analyse the contours separating $+$ and $-$ spins: the spins at 0 and at x are likely to be the same, because if they differ then there is a contour separating them, and contours are 'costly'. When there is a continuous symmetry, however, distant spins can differ due to a gradual rotation as one travels from one site to the other. Indeed, the proof of Theorem 6.1 uses this idea of gradual rotations.

Also note that this result applies to 2-dimensional systems. In higher dimensions, one can use infrared bounds and results such as Theorem 4.7 to prove breaking also of continuous symmetris.

For the proof of Theorem 6.1 we rely on the following lemma, which also plays an important role in quantum information theory. The proof can be found in e.g. [**30**].

LEMMA 6.2 (Quantum Pinsker's inequality). *For two density-matrices $\rho, \sigma \in \mathcal{B}(\mathcal{H})$ on a finite-dimensional Hilbert space $\mathcal{H}$, define the* relative entropy

$$S(\rho\|\sigma) := \operatorname{Tr} \rho(\log\rho - \log\sigma). \tag{6.2}$$

Then $S(\rho\|\sigma) \geq \frac{1}{2}\|\rho - \sigma\|_1^2$. In particular $S(\rho\|\sigma) \geq 0$.

PROOF OF THEOREM 6.1. Let n large enough that $\Lambda_n \supseteq \Lambda$. Since the $\|S_x\|$ are assumed to be uniformly bounded, we can assume $\|S_x\| \leq 1$ for all x.

The key idea of the proof is to introduce 'gradual' rotations and then to transfer the rotations from the observable to the interactions. Let $\boldsymbol{\theta} = (\theta_x)_{x\in\mathbb{Z}^2}$ be angles such that (i) $\theta_x = \theta$ for $x \in \Lambda$, (ii) $|\theta_x| \leq \theta$ for all $x \in \mathbb{Z}^2$, and $\theta_x = 0$ for $\|x\|_1 > m$. Our precise choice of angles is given below in (6.25). Define

$$V(\boldsymbol{\theta}) = \exp\big(\mathrm{i} \textstyle\sum_{x\in\Lambda_n} \theta_x S_x\big). \tag{6.3}$$

Note that $U_\Lambda^*(\theta) A U_\Lambda(\theta) = V^*(\boldsymbol{\theta}) A V(\boldsymbol{\theta})$ since $A \in \mathcal{A}_\Lambda$. In what follows we simply write V for $V(\boldsymbol{\theta})$ to lighten the notation.

Let us write $Z_{\Lambda_n}^{\Phi+\Psi_n} = \operatorname{Tr} \mathrm{e}^{-H^\Phi - H^{\Psi_n}}$ and

$$\rho_{\Lambda_n} = \frac{\mathrm{e}^{-H_{\Lambda_n}^\Phi - H_{\Lambda_n}^{\Psi_n}}}{Z_n^{\Phi+\Psi_n}} \tag{6.4}$$

for the density-matrix associated with the finite-volume Gibbs-state $\langle\cdot\rangle_{\Lambda_n}^{\Phi+\Psi_n}$. Let us also write

$$\rho_{\Lambda_n}^V = \frac{\mathrm{e}^{-VH_{\Lambda_n}^{\Phi}V^*-VH_{\Lambda_n}^{\Psi_n}V^*}}{Z_n^{\Phi+\Psi_n}} \tag{6.5}$$

for the density-matrix associated with $\langle V^*\cdot V\rangle_{\Lambda_n}^{\Phi+\Psi_n}$. (The partition-functions are the same as the trace is invariant under conjugation.) We have that

$$|\langle A\rangle_{\Lambda_n}^{\Phi+\Psi_n} - \langle U_\Lambda^*(\theta)AU_\Lambda(\theta)\rangle_{\Lambda_n}^{\Phi+\Psi_n}| = |\langle A\rangle_{\Lambda_n}^{\Phi+\Psi_n} - \langle V^*AV\rangle_{\Lambda_n}^{\Phi+\Psi_n}| = |\mathrm{Tr}\, A(\rho_{\Lambda_n} - \rho_{\Lambda_n}^V)| \tag{6.6}$$

Applying first Hölder's inequality followed by Pinsker's inequality we find that

$$|\mathrm{Tr}\, A(\rho_{\Lambda_n} - \rho_{\Lambda_n}^V)| \le \|A\|_\infty \|\rho_{\Lambda_n} - \rho_{\Lambda_n}^V\|_1 \le \|A\|_\infty \sqrt{2S(\rho_{\Lambda_n} \mid \rho_{\Lambda_n}^V)}. \tag{6.7}$$

Now we apply a strange-looking trick which turns out to be crucial. By the non-negativity of the relative entropy,

$$S(\rho_{\Lambda_n} \mid \rho_{\Lambda_n}^V) \le S(\rho_{\Lambda_n} \mid \rho_{\Lambda_n}^V) + S(\rho_{\Lambda_n} \mid \rho_{\Lambda_n}^{V^*}) \tag{6.8}$$

where $\rho_{\Lambda_n}^{V^*}$ is the density-matrix as in (6.5) but with V^* replacing V, or equivalently, with rotations $-\theta_x$ replacing θ_x. The reason for doing this is that it leads to a cancellation of terms which are linear in the angle differences $\theta_x - \theta_y$, leaving only second-order terms.

Now note that

$$\begin{aligned} S(\rho_{\Lambda_n} \mid \rho_{\Lambda_n}^V) + S(\rho_{\Lambda_n} \mid \rho_{\Lambda_n}^{V^{-1}}) =& \Big\langle -2H_{\Lambda_n}^{\Phi} + H_{\Lambda_n}^{V^*\Phi V} + H_{\Lambda_n}^{V\Phi V^*}\Big\rangle_{\Lambda_n}^{\Phi+\Psi_n} \\ &+ \Big\langle -2H_{\Lambda_n}^{\Psi_n} + H_{\Lambda_n}^{V^*\Psi_n V} + H_{\Lambda_n}^{V\Psi_n V^*}\Big\rangle_{\Lambda_n}^{\Phi+\Psi_n}. \end{aligned} \tag{6.9}$$

The absolute value of the right side is bounded above by

$$\|2H_{\Lambda_n}^{\Phi} - H_{\Lambda_n}^{V^*\Phi V} - H_{\Lambda_n}^{V\Phi V^*}\| + \|2H_{\Lambda_n}^{\Psi_n} - H_{\Lambda_n}^{V^*\Psi_n V} - H_{\Lambda_n}^{V\Psi_n V^*}\|. \tag{6.10}$$

Since we assumed that $\theta_x = 0$ when $\|x\|_1 > m$, we have $V^*(\Psi_n)_X V = (\Psi_n)_X$ whenever all sites of X are at distance more than m from the origin. Thus

$$\|2H_{\Lambda_n}^{\Psi_n} - H_{\Lambda_n}^{V^*\Psi_n V} - H_{\Lambda_n}^{V\Psi_n V^*}\| \le 4\Big\| \sum_{x:\|x\|_1\le m} \sum_{\substack{X\subseteq\Lambda_n\\ X\ni x}} (\Psi_n)_X \Big\| \le 4(2m+1)^2 \|\!|\Psi_n|\!\|. \tag{6.11}$$

For fixed m, this goes to 0 as $n \to \infty$. We now check that the first term is as small as we wish by taking m large.

For each set $X \subseteq \Lambda_n$, we fix $x_0(X)$ to be one of its elements. We have

$$V^*\Phi_X V = \mathrm{e}^{-\mathrm{i}T_X}\, \Phi_X\, \mathrm{e}^{\mathrm{i}T_X}\,, \tag{6.12}$$

where

$$T_X = T_X(\boldsymbol{\theta}) = \sum_{x\in X} (\theta_x - \theta_{x_0}) S_x. \tag{6.13}$$

The term with x_0 commutes with Φ_X and actually cancels; however, we include it to help with later estimates.

Recall the Lie–Schwinger multicommutator expansion, Lemma 3.15: $e^A B e^{-A} = \sum_{j\geq 0} \frac{1}{j!}\mathrm{ad}_A^j(B)$ where $\mathrm{ad}_A(\cdot) = [A,\cdot]$. Using this we get

$$\begin{aligned} H_{\Lambda_n}^{V^*\Phi V} &= \sum_{X\subseteq\Lambda_n} \Phi_X + \sum_{j\geq 1}\sum_{X\subseteq\Lambda_n} \frac{(-\mathrm{i})^j}{j!}\,\mathrm{ad}^j_{T_X(\boldsymbol{\theta})}(\Phi_X) \\ &= H_\Lambda^\Phi + B(\boldsymbol{\theta}) + C(\boldsymbol{\theta}), \end{aligned} \tag{6.14}$$

where

$$B(\boldsymbol{\theta}) = \sum_{j\geq 0}\sum_{X\subseteq\Lambda_n} \frac{\mathrm{i}}{(2j+1)!}\mathrm{ad}^{2j+1}_{T_X(\boldsymbol{\theta})}(\Phi_X), \tag{6.15}$$

and

$$C(\boldsymbol{\theta}) = -\sum_{j\geq 1}\sum_{X\subseteq\Lambda_n} \frac{1}{(2j)!}\mathrm{ad}^{2j}_{T_X(\boldsymbol{\theta})}(\Phi_X). \tag{6.16}$$

Similarly

$$H_{\Lambda_n}^{V\Phi V^*} = H_\Lambda^\Phi + B(-\boldsymbol{\theta}) + C(-\boldsymbol{\theta}). \tag{6.17}$$

Note that $T_X(-\boldsymbol{\theta}) = -T_X(\boldsymbol{\theta})$ and $\mathrm{ad}_{(-T_X)}(\cdot) = -\mathrm{ad}_{T_X}(\cdot)$. This means that $B(-\boldsymbol{\theta}) = -B(\boldsymbol{\theta})$ and $C(-\boldsymbol{\theta}) = C(\boldsymbol{\theta})$. Thus

$$2H_{\Lambda_n}^\Phi - H_{\Lambda_n}^{V^*\Phi V} - H_{\Lambda_n}^{V\Phi V^*} = 2C(\boldsymbol{\theta}). \tag{6.18}$$

We now estimate $\|C(\boldsymbol{\theta})\|$. Using $\|[A,B]\| \leq 2\|A\|\,\|B\|$ we obtain

$$\begin{aligned} \|C(\boldsymbol{\theta})\| &\leq \sum_{j\geq 1}\sum_{X\subseteq\Lambda} \frac{2^{2j}}{(2j)!}\|T_X\|^{2j}\|\Phi_X\| \\ &= \sum_{X\subseteq\Lambda} \|\Phi_X\|\big(\cosh(2\|T_X\|) - 1\big) \\ &\leq 2\sum_{X\subseteq\Lambda_n} \|\Phi_X\|\|T_X\|^2\, e^{2\|T_X\|}. \end{aligned} \tag{6.19}$$

We used the inequality $\cosh u - 1 \leq \frac{1}{2}u^2 e^u$, which is easily verified for all $u \geq 0$.

We now bound $\|T_X\|$. Since we assumed $\|S_x\| \leq 1$ for all x, we have

$$\|T_X\| \leq \sum_{x\in X} |\theta_x - \theta_{x_0}|. \tag{6.20}$$

Since we have $|\theta_x| \leq |\theta|$ for all x, a first upper bound is $\|T_X\| \leq 2|\theta|$. We use this for the last factor in (6.19): $e^{2\|T_X\|} \leq e^{4|\theta||X|}$. Another bound, which allows us to exploit a gradual change in the angles θ_x, is based on:

$$|\theta_x - \theta_{x_0}| \leq |\theta_{x_0} - \theta_{x_1}| + |\theta_{x_1} - \theta_{x_2}| + \cdots + |\theta_{x_{k-1}} - \theta_x| \tag{6.21}$$

where $x_0, x_1, x_2, \ldots, x_{k-1}, x_k = x$ is a sequence of nearest-neighbour sites in X forming a (non-repeating) path from x_0 to x (this is where we use that X is assumed to be connected). This gives

$$\|T_X\| \le |X| \sum_{\substack{\{x,y\}\subseteq X\\ \|x-y\|=1}} |\theta_x - \theta_y|, \tag{6.22}$$

where we used that a given edge is traversed at most X times. We now combine this with Cauchy–Schwarz:

$$\|T_X\|^2 \le |X|^2 \Big(\sum_{\substack{\{x,y\}\subseteq X\\ \|x-y\|=1}} |\theta_x - \theta_y| \Big)^2 \le 2|X|^3 \sum_{\substack{\{x,y\}\subseteq X\\ \|x-y\|=1}} |\theta_x - \theta_y|^2 \tag{6.23}$$

where we used that X contains at most $2|X|$ edges. We obtain

$$\begin{aligned} \|C(\boldsymbol{\theta})\| &\le 4 \sum_{X\subseteq\Lambda_n} \|\Phi_X\| |X|^3 \,\mathrm{e}^{4|\theta||X|} \sum_{\substack{\{x,y\}\subseteq Z\\ \|x-y\|=1}} |\theta_x - \theta_y|^2 \\ &\le 4 \sum_{\substack{\{x,y\}\subseteq\Lambda_n\\ \|x-y\|=1}} |\theta_x - \theta_y|^2 \sum_{X\supseteq\{x,y\}} \|\Phi_X\| |X|^3 \,\mathrm{e}^{4|\theta||X|} \\ &\le 8\|\Phi\|_{4|\theta|+3} \sum_{\substack{\{x,y\}\subseteq\Lambda_n\\ \|x-y\|=1}} |\theta_x - \theta_y|^2. \end{aligned} \tag{6.24}$$

We used $x \le \mathrm{e}^x$. Note that $\|\Phi\|_{4|\theta|+3} \le \|\Phi\|_{8\pi+3} < \infty$.

We now choose the angles $\boldsymbol{\theta} = (\theta_x)$. Let m_0 be large enough so that all sites in Λ are at distance at most m_0 from the origin. Then we take

$$\theta_x = \begin{cases} \theta & \text{if } \|x\|_1 \le m_0, \\ \theta(1 - \frac{\log(\|x\|_1 - m_0)}{\log m}) & \text{if } m_0 < \|x\|_1 < m_0 + m, \\ 0 & \text{if } \|x\|_1 \ge m. \end{cases} \tag{6.25}$$

We can then bound

$$\begin{aligned} \sum_{\substack{\{x,y\}\subseteq\Lambda_n\\ \|x-y\|=1}} |\theta_x - \theta_y|^2 &= \sum_{r=0}^{n} \sum_{x:\|x\|_1=r} \sum_{y:\|y\|_1=r+1} |\theta_x - \theta_y|^2 \\ &\le \sum_{r=m_0}^{m} 4\cdot 8r\theta^2 \Big(\frac{\log(r+1) - \log r}{\log m} \Big)^2 \\ &\le \frac{\text{const}}{(\log m)^2} \sum_{r=1}^{m} r(\log(1+\tfrac{1}{r}))^2 \le \frac{\text{const}}{\log m}. \end{aligned} \tag{6.26}$$

Using our bound in (6.24), (6.18) and then (6.10), we see that the difference of the expectation of the local observable A in (6.6) is vanishingly small. □

Bibliographical references

Mermin and Wagner [1966] proved that the quantum Heisenberg has no spontaneous magnetisation at all positive temperatures. Such is the importance of this result that the name "Mermin-Wagner theorem" has come to designate *all* results about absence of continuous symmetry breaking in two dimensions, although the mathematical setting and the methods of proofs of further results are very different. The fact that all Gibbs states remain symmetric was first proved in classical systems by Dobrushin and Shlosman [1975]. The extension to quantum systems was obtained by Fröhlich and Pfister [1981]; the present proof uses similar ideas but is more elementary.