CHAPTER 6

2D systems with continuous symmetry

We consider translation-invariant Gibbs states for two-dimensional models, on $\mathbb{Z}^2$, for an interaction Φ which is invariant under a continuous group of rotations. We prove the absence of symmetry breaking in the sense that all translation-invariant infinite-volume Gibbs states retain the continuous symmetry.

Let $\Phi = (\Phi_X : X \Subset \mathbb{Z}^2) \in \mathcal{I}$ be an interaction such that there are Hermitian operators S_x, $x \in \mathbb{Z}^2$, with $[\Phi_X, \sum_{x\in X} S_x] = 0$ for all $X \Subset \mathbb{Z}^2$. Here S_x acts on the tensor factor associated with $x \in \mathbb{Z}^2$; in most examples they are copies of the same operator, though in principle they could be different. We assume that $\|S_x\| \leq 1$ for all x.

For an angle $\theta \in \mathbb{R}$ define the unitary rotation $U_X(\theta) = \exp(\sum_{x\in X} \mathrm{i}\theta S_x)$. The condition $[\Phi_X, \sum_{x\in X} S_x] = 0$ means that $U_X^*(\theta)\Phi_X U_X(\theta) = \Phi_X$ for any angle θ, thus the interaction is invariant under rotations.

Given a state $\langle\cdot\rangle \in \mathcal{G}^{\Phi}_{\text{t.i.}}$ we can define another Gibbs state $\langle\cdot\rangle'$ by setting, for all $\Lambda \Subset \mathbb{Z}^2$ and all $A \in \mathcal{A}_\Lambda$, $\langle A\rangle' = \langle U_\Lambda^*(\theta) A U_\Lambda(\theta)\rangle$ (compare Exercise 3.14). In this sense the *set* $\mathcal{G}^{\Phi}_{\text{t.i.}}$ is invariant under the symmetry of the interaction. The following result shows that the Gibbs states are actually *individually* invariant under the symmetry:

THEOREM 6.1. *Let the interaction Φ satisfy*

$$\sum_{X\ni 0} \|\Phi_X\| (\operatorname{diam} X)^2 \, \mathrm{e}^{\varepsilon|X|} < \infty$$

for some $\varepsilon > 0$. Let $\langle\cdot\rangle \in \mathcal{G}^{\Phi}_{\text{t.i.}}$ is a translation-invariant Gibbs state. Then for all $\Lambda \Subset \mathbb{Z}^2$ and all $A \in \mathcal{A}_\Lambda$ we have that

$$\langle U_\Lambda^*(\theta) A U_\Lambda(\theta)\rangle = \langle A\rangle. \tag{6.1}$$

To put Theorem 6.1 in context, consider the XXZ-model ($J^{(1)} = J^{(2)}$ in (1.37)) in $\mathbb{Z}^2$. The assumption on continuous symmetry holds, e.g. with $S_x = S_x^{(3)}$. The model also has a discrete (spin-flip) symmetry, represented e.g. by $V = \mathrm{e}^{\mathrm{i}\pi S^{(1)}}$. Theorem 4.4 shows that, for $J^{(3)} = \Delta > 1$ and β large enough, there exists a Gibbs state $\langle\cdot\rangle$ satisfying $\langle S_0^{(3)} S_x^{(3)}\rangle \geq c > 0$ uniformly in all $x \in \mathbb{Z}^d$. One may in fact show that there is a translation-invariant Gibbs state $\langle\cdot\rangle^+$ satisfying $\langle S_0^{(3)}\rangle^+ > 0$. Applying the spin-flip symmetry, the state $\langle\cdot\rangle^- := \langle \otimes_x V_x^* \cdot \otimes_x V_x\rangle^+$ is a also translation-invariant Gibbs state; it satisfies $\langle S_0^{(3)}\rangle^- = -\langle S_0^{(3)}\rangle^+ < 0$. In particular, $\langle\cdot\rangle^- \neq \langle\cdot\rangle^+$. In this sense, the discrete symmetry is broken: we have two distinct Gibbs states, related by the discrete symmetry. Theorem 6.1 shows that the corresponding conclusion does not hold for the continuous symmetry: the 'rotated' state $\langle \otimes_x U_x^*(\theta) \cdot \otimes_x U_x(\theta)\rangle$ coincides with $\langle\cdot\rangle$.

The underlying reason for this difference is the absence of *contours*. Recall that the key method for the proof of Theorem 4.4 was to analyse the contours separating + and − spins: the spins at 0 and at x are likely to be the same, because if they differ then there is a contour separating them, and contours are 'costly'. When there is a continuous symmetry, however, distant spins can differ due to a gradual rotation as one travels from one site to the other. Indeed, the proof of Theorem 6.1 uses this idea of gradual rotations.

Also note that this result applies to 2-dimensional systems. The situation in higher dimensions is different. The breaking of continuous symmetries is possible and it can be proved in some cases, see Theorem 4.7.

For the proof of Theorem 6.1 we rely on two lemmas. The first tells us that we can approximate the restriction of $\langle\cdot\rangle$ to $\mathcal{A}_\Lambda$ using finite-volume states. This holds without assumptions on the dimension d or symmetries of the interaction Φ. The second lemma is the quantum Pinsker inequality, that estimates the distance between states using the relative entropy.

LEMMA 6.2. *Let $\langle\cdot\rangle \in \mathcal{G}^{\Phi}_{\text{t.i.}}$ be a translation-invariant Gibbs state for an interaction $\Phi \in \mathcal{I}$. Let $\Lambda \Subset \mathbb{Z}^d$ and let $\Lambda_n \uparrow \mathbb{Z}^d$. Then there is a sequence of interactions Ψ_n such that $\|\Psi_n\| \to 0$, such that for all $A \in \mathcal{A}_\Lambda$ we have*

$$\langle A\rangle = \lim_{n\to\infty} \langle A\rangle^{\Phi+\Psi_n}_{\Lambda_n}.$$

Note that we do not quite get an approximation of the whole state $\langle\cdot\rangle$, only its restriction to $\mathcal{A}_\Lambda$. The sequence Ψ_n depends on Λ and on Λ_n.

PROOF. Let $A_1,\dots,A_k$ be a basis of $\mathcal{A}_\Lambda$, where $k = \dim \mathcal{A}_\Lambda = N^{2|\Lambda|}$. It suffices to show that the lemma holds for A being any of the A_j. We use the tangent-functional characterisation where the free energy satisfies

$$f(\Phi + \textstyle\sum_{j=1}^N t_j \Psi_{A_j}) \le f(\Phi) + \sum_{j=1}^N t_j \langle A_j\rangle \tag{6.2}$$

for all $t_1,\dots,t_N \in \mathbb{R}$. Moreover, f is the pointwise limit of f_{Λ_n} and the latter are smooth, convex functions. Using Lemma A.13, one can choose sequences $t_j^{(n)}$, each converging to 0, such that

$$\langle A_j\rangle = \lim_{n\to\infty} \tfrac{\partial}{\partial t_j} f_{\Lambda_n}(t_1^{(n)},\dots,t_N^{(n)}) = \lim_{n\to\infty} \langle A_j\rangle^{\Phi+\sum_{j=1}^N t_j^{(n)}\Psi_{A_j}}_{\Lambda_n}. \tag{6.3}$$

This gives the relevant approximation with $\Psi_n = \sum_{j=1}^N t_j^{(n)} \Psi_{A_j}$. □

The next lemma plays an important role in quantum information theory. The proof can be found in Section A.8.

LEMMA 6.3 (Quantum Pinsker's inequality). *For two density-matrices $\rho,\sigma \in \mathcal{B}(\mathcal{H})$ on a finite-dimensional Hilbert space $\mathcal{H}$, define the* relative entropy

$$S(\rho\|\sigma) := \operatorname{Tr}\rho(\log\rho - \log\sigma). \tag{6.4}$$

Then $S(\rho\|\sigma) \ge \frac{1}{2}\|\rho-\sigma\|_1^2$. In particular $S(\rho\|\sigma) \ge 0$.

Here are the steps of the proof of Theorem 6.1:

(1) The key idea of the proof is to introduce 'gradual' rotations and then to transfer the rotations from the observable to the interactions. Let $\boldsymbol{\theta}^{(m)} = (\theta_x^{(m)})$ be angles such that $\theta_x^{(m)} = \theta$ for all $x \in \Lambda$ and all m. Let

$$U_m = \sum_x \theta_x^{(m)} S_x. \tag{6.5}$$

(2) Using Lemma 6.2, it is enough to show that for all $A \in \mathcal{A}_\Lambda$, we have

$$\lim_{m\to\infty} \lim_{n\to\infty} \Big(\langle A \rangle_{\Lambda_n}^{\Phi+\Psi_n} - \langle U_m^* A U_m \rangle_{\Lambda_n}^{\Phi+\Psi_n} \Big) = 0. \tag{6.6}$$

(3) We estimate the difference above in terms of the relative entropy. This in turns gives an estimate involving the difference of hamiltonians $H_{\Lambda_n}^{\Phi+\Psi_n}$ and $H_\Lambda^{U_m(\Phi+\Psi_n)U_m^*}$.
(4) The difference above can be bounded by $C \sum_{\|x-y\|=1} (\theta_x^{(m)} - \theta_y^{(m)})^2$.
(5) In two dimensions, we can find $\boldsymbol{\theta}^{(m)}$ such that $\theta_x^{(m)} = \theta$ for all $x \in \Lambda$ and all m, such that the gradient above goes to 0 as $m \to \infty$.

For Step (1) we introduce the angles $\boldsymbol{\theta}^{(m)}$. For now we only assume that $\sum_{\|x-y\|=1} |\theta_x^{(m)} - \theta_y^{(m)}|$ is finite for each m (not necessarily uniformly). The next lemma gives Step (3).

LEMMA 6.4. *We have the bound*

$$\Big| \langle A \rangle_{\Lambda_n}^{\Phi+\Psi_n} - \langle U_m^* A U_m \rangle_{\Lambda_n}^{\Phi+\Psi_n} \Big|^2 \leq 2 \big\| 2H_{\Lambda_n}^{\Phi} - H_{\Lambda_n}^{U_m \Phi U_m^*} - H_{\Lambda_n}^{U_m^* \Phi U_m} \big\| + 2 \| \Psi_n \| \| \boldsymbol{\theta}^{(m)} \|_1 \|A\|.$$

PROOF. Let n large enough that $\Lambda_n \supseteq \Lambda$. Recall that $\|S_x\| \leq 1$ for all x. Recall the definition of U_m in (6.5).

Let us write $Z_{\Lambda_n}^{\Phi+\Psi_n} = \operatorname{Tr} \mathrm{e}^{-H_{\Lambda_n}^{\Phi} - H_{\Lambda_n}^{\Psi_n}}$ and

$$\rho_{\Lambda_n} = \frac{\mathrm{e}^{-H_{\Lambda_n}^{\Phi} - H_{\Lambda_n}^{\Psi_n}}}{Z_{\Lambda_n}^{\Phi+\Psi_n}} \tag{6.7}$$

for the density-matrix associated with the finite-volume Gibbs-state $\langle \cdot \rangle_{\Lambda_n}^{\Phi+\Psi_n}$. Let us also write

$$\rho_{\Lambda_n}^{U_m} = \frac{\mathrm{e}^{-U_m H_{\Lambda_n}^{\Phi} U_m^* - U_m H_{\Lambda_n}^{\Psi_n} U_m^*}}{Z_n^{\Phi+\Psi_n}} \tag{6.8}$$

for the density-matrix associated with $\langle U_m^* \cdot U_m \rangle_{\Lambda_n}^{\Phi+\Psi_n}$. (The partition-functions are the same as the trace is invariant under conjugation.) We have that

$$|\langle A \rangle_{\Lambda_n}^{\Phi+\Psi_n} - \langle U_\Lambda^*(\theta) A U_\Lambda(\theta) \rangle_{\Lambda_n}^{\Phi+\Psi_n}| = |\langle A \rangle_{\Lambda_n}^{\Phi+\Psi_n} - \langle U_m^* A U_m \rangle_{\Lambda_n}^{\Phi+\Psi_n}| = |\operatorname{Tr} A(\rho_{\Lambda_n} - \rho_{\Lambda_n}^{U_m})| \tag{6.9}$$

Applying first Hölder's inequality followed by Pinsker's inequality we find that

$$|\operatorname{Tr} A(\rho_{\Lambda_n} - \rho_{\Lambda_n}^{U_m})| \leq \|A\|_\infty \|\rho_{\Lambda_n} - \rho_{\Lambda_n}^{U_m}\|_1 \leq \|A\|_\infty \sqrt{2S(\rho_{\Lambda_n} \mid \rho_{\Lambda_n}^{U_m})}. \tag{6.10}$$

Now we apply a strange-looking trick which turns out to be crucial. By the non-negativity of the relative entropy,

$$S(\rho_{\Lambda_n} \mid \rho_{\Lambda_n}^{U_m}) \leq S(\rho_{\Lambda_n} \mid \rho_{\Lambda_n}^{U_m}) + S(\rho_{\Lambda_n} \mid \rho_{\Lambda_n}^{U_m^*}) \tag{6.11}$$

where $\rho_{\Lambda_n}^{U_m^*}$ is the density-matrix as in (6.8) but with U_m^* replacing U_m, or equivalently, with rotations $-\theta_x$ replacing θ_x. The reason for doing this is that it leads to a cancellation of terms which are linear in the angle differences $\theta_x - \theta_y$, leaving only second-order terms.

Now note that

$$\begin{aligned} S(\rho_{\Lambda_n} \mid \rho_{\Lambda_n}^{U_m}) + S(\rho_{\Lambda_n} \mid \rho_{\Lambda_n}^{U_m^{-1}}) =& \big\langle -2H_{\Lambda_n}^{\Phi} + H_{\Lambda_n}^{U_m^*\Phi U_m} + H_{\Lambda_n}^{U_m\Phi U_m^*}\big\rangle_{\Lambda_n}^{\Phi+\Psi_n} \\ &+ \big\langle -2H_{\Lambda_n}^{\Psi_n} + H_{\Lambda_n}^{U_m^*\Psi_n U_m} + H_{\Lambda_n}^{V\Psi_n U_m^*}\big\rangle_{\Lambda_n}^{\Phi+\Psi_n}. \end{aligned} \tag{6.12}$$

The absolute value of the right side is bounded above by

$$\|2H_{\Lambda_n}^{\Phi} - H_{\Lambda_n}^{U_m^*\Phi U_m} - H_{\Lambda_n}^{U_m\Phi U_m^*}\| + \|2H_{\Lambda_n}^{\Psi_n} - H_{\Lambda_n}^{V^*\Psi_n U_m} - H_{\Lambda_n}^{U_m\Psi_n U_m^*}\|. \tag{6.13}$$

Since we assumed that $\theta_x = 0$ when $\|x\|_1 > m$, we have $U_m^*(\Psi_n)_X U_m = (\Psi_n)_X$ whenever all sites of X are at distance more than m from the origin. Thus

$$\|2H_{\Lambda_n}^{\Psi_n} - H_{\Lambda_n}^{U_m^*\Psi_n U_m} - H_{\Lambda_n}^{U_m\Psi_n U_m^*}\| \leq 4\Big\| \sum_{x:\|x\|_1\leq m} \sum_{\substack{X\subseteq\Lambda_n \\ X\ni x}} (\Psi_n)_X \Big\| \leq 4(2m+1)^2 \|\!|\Psi_n|\!\|. \tag{6.14}$$

□

Now we get Step (4). For $X \Subset \mathbb{Z}^2$

$$\overline{S}_{m,X} = \sum_{x\in X} (\theta_x^{(m)} - \bar{\theta}^{(m)}) S_x, \tag{6.15}$$

where $\bar{\theta}^{(m)}$ is the average of $\theta_x^{(m)}$ on X (it depends on X, although we do not indicate it). We have $U_m \Phi_X U_m^* = \mathrm{e}^{\mathrm{i}\overline{S}_{m,X}}\, \Phi_X\, \mathrm{e}^{\mathrm{i}\overline{S}_{m,X}^*}$.

Lemma 6.5. *We have*

$$\Big\|2H_{\Lambda_n}^{\Phi} - H_{\Lambda_n}^{U_m\Phi U_m^*} - H_{\Lambda_n}^{U_m^*\Phi U_m}\Big\| \leq 2 \sum_{X\subset\Lambda_n} \|\Phi_X\|\, \|\overline{S}_{m,X}\|^2\, \mathrm{e}^{2\|\overline{S}_{m,X}\|}.$$

Proof. Recall the Lie–Schwinger multicommutator expansion, Lemma 3.16: $\mathrm{e}^A\, B\, \mathrm{e}^{-A} = \sum_{j\geq 0} \frac{1}{j!}\mathrm{ad}_A^j(B)$ where $\mathrm{ad}_A(\cdot) = [A,\cdot]$. Using this we get

$$\begin{aligned} H_{\Lambda_n}^{U_m^*\Phi U_m} &= \sum_{X\subseteq\Lambda_n} \Phi_X + \sum_{j\geq 1}\sum_{X\subseteq\Lambda_n} \frac{(-\mathrm{i})^j}{j!}\,\mathrm{ad}_{\overline{S}_{m,X}(\boldsymbol{\theta})}^j(\Phi_X) \\ &= H_\Lambda^\Phi + B(\boldsymbol{\theta}) + C(\boldsymbol{\theta}), \end{aligned} \tag{6.16}$$

where

$$B(\boldsymbol{\theta}) = \sum_{\substack{j\geq 1 \\ j\text{ odd}}} \sum_{X\subseteq\Lambda_n} \frac{(-\mathrm{i})^j}{j!}\mathrm{ad}_{\overline{S}_{m,X}(\boldsymbol{\theta})}^j(\Phi_X), \tag{6.17}$$

and

$$C(\boldsymbol{\theta}) = - \sum_{\substack{j \geq 2 \\ j \text{ even}}} \sum_{X \subseteq \Lambda_n} \frac{(-\mathrm{i})^j}{j!} \mathrm{ad}^j_{\overline{S}_{m,X}(\boldsymbol{\theta})}(\Phi_X). \tag{6.18}$$

Similarly

$$H^{U_m \Phi U_m^*}_{\Lambda_n} = H^\Phi_\Lambda + B(-\boldsymbol{\theta}) + C(-\boldsymbol{\theta}). \tag{6.19}$$

Note that $\overline{S}_{m,X}(-\boldsymbol{\theta}) = -\overline{S}_{m,X}(\boldsymbol{\theta})$ and $\mathrm{ad}_{(-\overline{S}_{m,X})}(\cdot) = -\mathrm{ad}_{\overline{S}_{m,X}}(\cdot)$. This means that $B(-\boldsymbol{\theta}) = -B(\boldsymbol{\theta})$ and $C(-\boldsymbol{\theta}) = C(\boldsymbol{\theta})$. Thus

$$2H^\Phi_{\Lambda_n} - H^{U_m^* \Phi U_m}_{\Lambda_n} - H^{U_m \Phi U_m^*}_{\Lambda_n} = 2C(\boldsymbol{\theta}). \tag{6.20}$$

We now estimate $\|C(\boldsymbol{\theta})\|$. Using $\|[A,B]\| \leq 2\|A\|\,\|B\|$ we obtain

$$\begin{aligned} \|C(\boldsymbol{\theta})\| &\leq \sum_{j \geq 1} \sum_{X \subseteq \Lambda} \frac{2^{2j}}{(2j)!} \|\overline{S}_{m,X}\|^{2j} \|\Phi_X\| \\ &= \sum_{X \subseteq \Lambda} \|\Phi_X\| \big(\cosh\big(2\|\overline{S}_{m,X}\|\big) - 1\big) \\ &\leq 2 \sum_{X \subseteq \Lambda_n} \|\Phi_X\| \|\overline{S}_{m,X}\|^2 \, \mathrm{e}^{2\|\overline{S}_{m,X}\|}\,. \end{aligned} \tag{6.21}$$

We used the inequality $\cosh u - 1 \leq \frac{1}{2} u^2 \, \mathrm{e}^u$, which is easily verified for all $u \geq 0$. □

Then we get Step (4):

Lemma 6.6. *We have*

$$\|\overline{S}_{m,X}\|^2 \leq C_2 |X| (\mathrm{diam}\, X)^2 \sum_{\substack{\{x,y\} \subseteq X \\ \|x-y\|=1}} |\theta^{(m)}_x - \theta^{(m)}_y|^2.$$

Proof. We have $\|\overline{S}_{m,X}\| \leq \|\theta^{(m)} - \bar{\theta}^{(m)}\|_1$ since $\|S_x\| = 1$. The result then follows from the discrete Poincaré inequality of Corollary A.12. □

For Step (5) we give an explicit definition of $\boldsymbol{\theta}^{(m)}$:

Lemma 6.7. *The following choice for $\boldsymbol{\theta}^{(m)}$ gives the desired rotation on Λ and its gradient vanishes when $m \to \infty$: With m_0 large enough so that all sites in Λ are at distance at most m_0 from the origin, let*

$$\theta^{(m)}_x = \begin{cases} \theta & \text{if } \|x\|_1 \leq m_0, \\ \theta\big(1 - \frac{\log(\|x\|_1 - m_0)}{\log m}\big) & \text{if } m_0 < \|x\|_1 < m_0 + m, \\ 0 & \text{if } \|x\|_1 \geq m. \end{cases}$$

Then

$$\sum_{\substack{\{x,y\} \subseteq \Lambda_n \\ \|x-y\|=1}} |\theta^{(m)}_x - \theta^{(m)}_y|^2 \leq \frac{\mathrm{const}}{\log m}.$$

PROOF. We can bound

$$\begin{aligned}\sum_{\substack{\{x,y\}\subseteq\Lambda_n\\ \|x-y\|=1}} |\theta_x^{(m)} - \theta_y^{(m)}|^2 &= \sum_{r=0}^{n} \sum_{x:\|x\|_1=r} \sum_{y:\|y\|_1=r+1} |\theta_x^{(m)} - \theta_y^{(m)}|^2 \\ &\le \sum_{r=m_0}^{m} 4\cdot 8r\theta^2 \Big(\frac{\log(r+1)-\log r}{\log m}\Big)^2 \\ &\le \frac{\text{const}}{(\log m)^2} \sum_{r=1}^{m} r(\log(1+\tfrac{1}{r}))^2 \le \frac{\text{const}}{\log m}.\end{aligned} \tag{6.22}$$

□

PROOF OF THEOREM 6.1. It is enough to prove the theorem for small θ since we can then iterate to get the result for $2\theta, 4\theta, \dots$ From the lemmas above, and also using $\|\overline{S}_{m,X}\| \le |\theta||X| \le \frac{1}{2}\varepsilon|X|$ and $|X| \le C\,\mathrm{e}^{\frac{1}{2}\varepsilon|X|}$ we get

$$\begin{aligned}&\lim_{n\to\infty} \Big| \langle A\rangle_{\Lambda_n}^{\Phi+\Psi_n} - \langle U_m^* A U_m\rangle_{\Lambda_n}^{\Phi+\Psi_n} \Big|^2 \\ &\qquad \le 2CC_2 \sum_{X\Subset\mathbb{Z}^2} \|\Phi_X\|(\operatorname{diam} X)^2\,\mathrm{e}^{\varepsilon|X|} \sum_{\substack{\{x,y\}\subseteq X\\ \|x-y\|=1}} |\theta_x^{(m)} - \theta_y^{(m)}|^2 \\ &\qquad = 2CC_2 \sum_{\substack{\{x,y\}\subset\mathbb{Z}^2\\ \|x-y\|=1}} |\theta_x^{(m)} - \theta_y^{(m)}|^2 \sum_{X\supset\{x,y\}} \|\Phi_X\|(\operatorname{diam} X)^2\,\mathrm{e}^{\varepsilon|X|}.\end{aligned} \tag{6.23}$$

The latter sum is bounded uniformly in $\{x,y\}$. Then everything goes to 0 as $m\to\infty$ because of Lemma 6.7. □

BIBLIOGRAPHICAL REFERENCES

Mermin and Wagner [1966] proved that the quantum Heisenberg has no spontaneous magnetisation at all positive temperatures. Such is the importance of this result that the name "Mermin-Wagner theorem" has come to designate *all* results about absence of continuous symmetry breaking in two dimensions, although the mathematical setting and the methods of proofs of further results are very different. The fact that all Gibbs states remain symmetric was first proved in classical systems by Dobrushin and Shlosman [1975]. The extension to quantum systems was achieved by Fröhlich and Pfister [1981]; the present proof uses similar ideas but is more elementary.