CHAPTER 1

Introduction

The motivation for quantum lattice models is the study of electronic properties of condensed matter systems. It is a fact of life that solids exist, even if the mathematical description of solidification is very wanting. Since nuclei are heavy it is natural to treat them as fixed classical particles. There remains to understand the behaviour of electrons and their spins. They are governed by the laws of quantum mechanics and they interact via Coulomb forces with themselves and with the nuclei. They are impossible to study directly and physicists have been compelled to introduce simplified models.

Models usually involve the "tight-binding approximation" where the motion of electrons are restricted to the sites of the lattice formed by the nuclei. Interactions are also simplified. Starting from the original electrons with Coulomb interactions, the simplification leads to one of the fundamental models of condensed matter physics, the Hubbard model.

The discovery and study of certain phenomena has led to further models such as Ising, Heisenberg, XY, t-J, as well as models involving bosons. There are few situations where these models describe exactly the properties of the system, but their understanding help physicists to have much better understanding.

The existence of all these models, even if they do not describe the physical systems directly, is a blessing for mathematicians. They have a rich and attractive mathematical structure. Their study is possible, even though it is often difficult and it requires much creative thinking. Several arguments are indirect and convoluted — but eminently interesting. We hope these notes will help convince the reader that this field is indeed attractive and intellectually rewarding.

The main topic addressed in these notes is *symmetry breaking*, where low temperature equilibrium states have less symmetry than the system. We review the general theory, which can be presented in a rigorous fashion for a large class of models. Namely, we show that the set of Gibbs states (that is, the equilibrium states) has the structure of a "Choquet simplex" where each state is given by a convex combination of *extremal states*. These extremal states enjoy special properties, such as clustering and 0-1 law.

Once the general theory is understood, there remains to identity the set of extremal states for given systems. This turns out to be an extremely difficult task, and rigorous results are few and partial. These notes discuss the following situations:

- At high temperatures, the Gibbs state is unique. This result is actually very general.
- In the Ising model, there are exactly two translation-invariant extremal Gibbs states at low temperature in dimension two and higher. This model is not quantum but it fits our setting nonetheless.
- For models with continuous symmetry in two dimensions, one can prove that all Gibbs states retain this symmetry, even at low temperatures.
- In some cases, such as the antiferromagnetic Heisenberg model, one can prove the occurrence of long-range order at low temperature and at dimension three and higher. This implies the existence of several distinct extremal Gibbs states.

These notes explain the setting and these results. The mathematical proofs are given in details. It should be pointed out that physicists have achieved much greater understanding using non-rigorous approaches and calculations. We hope that the topics covered here will interest readers and prompt them to bring further creative and elegant solutions to the many open problems.

Finally, a word on the literature. For the general theory, we mainly rely on the books of Ruelle [1969], Israel [1979] and Simon [1993]. A deeper dive into the theory of C^* algebra can be found in the books of Bratteli and Robinson [1987]. The theory of classical spin systems is beautifully introduced in Friedli and Velenik [2017]. Notes by Sims and Nachtergaele [2022] are useful. The book of Tasaki [2000] describes explicit quantum lattice systems and their properties, mainly in the ground state.

Acknowledgments: We are grateful to Sam Olesker-Taylor, and especially Vedran Sohinger, for pointing out many typos.