CHAPTER 7

At high temperatures

We give here a sufficient condition (high temperature) that guarantees the uniqueness of the infinite-volume Gibbs state. Recall that existence of Gibbs states at any temperature can be established by a compactness argument.

Here we set $\dim \mathcal{H}_0 = N$.

THEOREM 7.1. *Assume that*

$$\|\Phi\|_{N+1} < (2N)^{-1}.$$

Then there exists a unique KMS state for the interaction Φ.

We actually prove the theorem under the more general condition that there exists $s < 1/N$ such that $2\|\Phi\|_{N(1+s)} < s$. The starting point is the following rearrangement of the KMS condition:

$$\langle [a,b] \rangle = \langle b(\alpha_{\mathrm{i}} - \mathbb{1})a \rangle. \tag{7.1}$$

When $\Phi = 0$ the Gibbs state is the tracial state tr. (Recall that tr denotes the normalised trace). For small interactions it is natural to view $\langle \cdot \rangle$ as close to tr; accordingly, we define the linear functional ε by the equation

$$\langle a \rangle = \mathrm{tr}\, a + \varepsilon(a). \tag{7.2}$$

The KMS condition for ε is then

$$\varepsilon\big([a,b]\big) = \mathrm{tr}\,\big(b(\alpha_{\mathrm{i}} - \mathbb{1})a\big) + \varepsilon\big(b(\alpha_{\mathrm{i}} - \mathbb{1})a\big). \tag{7.3}$$

Notice that for small interactions we have $\alpha_{\mathrm{i}} \sim \mathbb{1}$ so that ε should indeed be small. In order to use this equation we need to turn an operator into a sum of commutators. This is the content of the next lemma.

Recall that $\|a\|_2 = \sqrt{\mathrm{tr}\, a^* a}$ is the normalised Hilbert-Schmidt norm. For $a \in \mathcal{A}_\Lambda$ we have $\frac{1}{\sqrt{\dim \mathcal{H}_\Lambda}}\|a\| \leq \|a\|_2 \leq \|a\|$.

Lemma 7.2. *Let A be a hermitian $N \times N$ matrix with the property that $\operatorname{Tr} A = 0$. Then there exist hermitian $N \times N$ matrices $B_1, \dots, B_{N-1}$ and $C_1, \dots, C_{N-1}$ (that depend on A) such that*

$$A = \sum_{i=1}^{N-1} [B_i, C_i]$$

and

$$\sum_{i=1}^{N-1} \|B_i\|_2 \, \|C_i\|_2 \leq \sqrt{N} \, \|A\|_2.$$

Proof. Let $\alpha_1, \dots, \alpha_N$ be the eigenvalues of A (repeated according to their multiplicity). We have that

$$\sum_{i=1}^{N} \alpha_i = 0, \qquad \sum_{i=1}^{N} |\alpha_i|^2 = N \, \|A\|_2^2. \tag{7.4}$$

In particular, each $|\alpha_i|$ is bounded above by $\sqrt{N}\|A\|_2$. Let us order the eigenvalues so that

$$\Big| \sum_{i=1}^{k} \alpha_i \Big| \leq \sqrt{N} \, \|A\|_2 \tag{7.5}$$

for all $1 \leq k \leq N-1$. This is indeed possible, as can be seen by induction using $\sum \alpha_i = 0$: If $0 \leq \sum^k \alpha_i \leq \sqrt{N}\|A\|_2$, we can find $\alpha_{k+1} \leq 0$ among the remaining eigenvalues such that $|\sum^{k+1} \alpha_i| \leq \sqrt{N}\|A\|_2$. And if the partial sum is negative, we can find $\alpha_{k+1} \geq 0$ among the remaining eigenvalues, with the same conclusion.

We work in a basis such that A is diagonal and its eigenvalues are ordered so they satisfy the properties above. Let $\tilde\alpha_k = \sum_{i=1}^k \alpha_i$, and let $\sigma^1_{j,j+1}, \sigma^2_{j,j+1}, \sigma^3_{j,j+1}$ be $N \times N$ matrices that are equal to Pauli matrices on the 2×2 block that contains (j,j) and $(j+1,j+1)$, and that are equal to zero everywhere else. It is not hard to check that

$$A = \sum_{j=1}^{N-1} \tilde\alpha_j \, \sigma^3_{j,j+1}. \tag{7.6}$$

We therefore have that

$$A = \tfrac12 \sum_{j=1}^{N-1} \tilde\alpha_j \, [\sigma^1_{j,j+1}, \sigma^2_{j,j+1}], \tag{7.7}$$

which proves the first claim. The bound follows from $|\tilde\alpha_j| \leq \sqrt{N}\|A\|_2$ and $\|\sigma^i_{j,j+1}\|_2^2 = 2/N$. □

PROOF OF THEOREM 7.1. Let $(e_i)_{i=0}^{N^2-1}$ be a hermitian basis of $\mathcal{M}_N(\mathbb{C})$, with $e_0 = \mathbb{1}$, $\operatorname{Tr} e_i = 0$ if $1 \neq 0$, and $\|e_i\| = 1$, for all i. Let J be the set of multi-indices $j = (j_x)_{x\in\mathbb{Z}^d}$, $0 \leq j_x \leq N^2 - 1$, with finite support

$$\operatorname{supp} j = \{x \in \mathbb{Z}^d | j_x \neq 0\}. \tag{7.8}$$

Given $j \in J$, let $e_j = \otimes_{x\in\operatorname{supp} j} e_{j_x} \in \mathcal{A}_{\operatorname{supp} j}$. The linear span of $\{e_j\}_{j\in J}$ is dense in $\mathcal{A}$. Since $\varepsilon(\mathbb{1}) = 0$ we have

$$\varepsilon(e_j) = \begin{cases} \langle e_j \rangle & \text{if } j \not\equiv 0, \\ 0 & \text{if } j \equiv 0. \end{cases} \tag{7.9}$$

Using Lemma 7.2, we have that

$$e_j = \frac{1}{|\operatorname{supp} j|} \sum_{y\in\operatorname{supp} j} \sum_{i=1}^{N-1} \big[\otimes_{x\neq y} e_{j_x} \otimes b_i^{(j_y)}, \otimes_{x\neq y} \mathbb{1} \otimes c_i^{(j_y)}\big], \tag{7.10}$$

for $j \not\equiv 0$. Here, $b_i^{(k)}, c_i^{(k)}$ are the matrices B_i, C_i of Lemma 7.2 in the case where the matrix A is e_k.

We now use this decomposition and the KMS condition. From Eq. (7.3) we have for $j \not\equiv 0$ that

$$\begin{aligned} \varepsilon(e_j) &= \frac{1}{|\operatorname{supp} j|} \sum_{y\in\operatorname{supp} j} \sum_{i=1}^{N-1} \big\langle \big[\otimes_{x\neq y} e_{j_x} \otimes b_i^{(j_y)}, \otimes_{x\neq y} \mathbb{1} \otimes c_i^{(j_y)}\big] \big\rangle \\ &= \frac{1}{|\operatorname{supp} j|} \sum_{y\in\operatorname{supp} j} \sum_{i=1}^{N-1} \big\langle (\otimes_{x\neq y} e_{j_x} \otimes b_i^{(j_y)} \cdot (\mathbb{1} - \alpha_{\mathrm{i}}) \otimes_{x\neq y} \mathbb{1} \otimes c_i^{(j_y)} \big\rangle \\ &= \delta(e_j) + K\varepsilon(e_j). \end{aligned} \tag{7.11}$$

In the above equation, we set

$$\delta(e_j) = \frac{1}{|\operatorname{supp} j|} \sum_{y\in\operatorname{supp} j} \sum_{i=1}^{N-1} \operatorname{tr} \Big(\otimes_{x\neq y} e_{j_x} \otimes b_i^{(j_y)} \cdot (\mathbb{1} - \alpha_{\mathrm{i}}) \otimes_{x\neq y} \mathbb{1} \otimes c_i^{(j_y)} \Big), \tag{7.12}$$

and the operator K is defined by

$$(K\phi)(e_j) = \frac{1}{|\operatorname{supp} j|} \sum_{y\in\operatorname{supp} j} \sum_{i=1}^{N-1} \phi\Big(\otimes_{x\neq y} e_{j_x} \otimes b_i^{(j_y)} \cdot (\mathbb{1} - \alpha_{\mathrm{i}}) \otimes_{x\neq y} \mathbb{1} \otimes c_i^{(j_y)} \Big). \tag{7.13}$$

Notice that K is a linear operator on the Banach space $\mathcal{L}(\mathcal{A})$ of linear functionals on $\mathcal{A}$. Equation (7.11) can be written as

$$(\mathbb{1} - K)\varepsilon = \delta. \tag{7.14}$$

Let us introduce the following specific norm on $\mathcal{L}(\mathcal{A})$:

$$\|\phi\|_{\mathrm{sp}} = \sup_{j\in J} |\phi(e_j)|. \tag{7.15}$$

Because $\|e_j\| = 1$ for all j, we have $\|\phi|_{\rm sp} \leq \|\phi\|$ and $(\mathcal{L}(\mathcal{A}), \|\cdot\|_{\rm sp})$ is a normed vector space. We consider K as an operator on $(\mathcal{L}(\mathcal{A}), \|cdot\|_{\rm sp})$ and we show that its norm is strictly less than 1; the solution of (7.14) is then unique. The norm of K is equal to

$$\|K\| = \sup_{\|\phi\|_{\rm sp}=1} \sup_{j\in J} |K\phi(e_j)|. \tag{7.16}$$

Recall that $\alpha_{\rm i} = \lim_\Lambda \alpha_{\rm i}^\Lambda$ (with convergence in the operator norm) and that $\alpha_{\rm i}^\Lambda(a)$, $a \in \mathcal{A}$, has an expansion in multiple commutators. From (7.13), we get

$$\begin{split}\bigl|K\phi(e_j)\bigr| \leq \frac{1}{|\operatorname{supp} j|} \sum_{y\in\operatorname{supp} j} \sum_{i=1}^{N-1} \sum_{n\geq1} \frac{1}{n!} \sup_{\Lambda\subset\mathbb{Z}^d} \sum_{X_1,\dots,X_n\subset\Lambda} \\ \Bigl|\phi\Bigl(\otimes_{x\neq y} e_{j_x} \otimes b_i^{(j_y)} \bigl[\Phi_{X_n}, \dots, \bigl[\Phi_{X_1}, \otimes_{x\neq y} \mathbb{1} \otimes c_i^{(j_y)}\bigr] \cdots\bigr]\Bigr)\Bigr|.\end{split} \tag{7.17}$$

Because of the commutators, the sum over the X_k's is restricted to subsets that satisfy the following constraints, as in (5.9):

$$\begin{split} &X_1 \ni y, \\ &X_2 \cap X_1 \neq \emptyset, \\ &\quad\vdots \\ &X_n \cap (X_1 \cup \dots \cup X_{n-1}) \neq \emptyset. \end{split} \tag{7.18}$$

Let $A = \sum_{(j'_x)_{x\in X}} a_{j'} e_{j'}$ be an operator in $\mathcal{A}_X$. For any $(j_x)_{x\notin X}$, we have

$$\begin{split} \bigl|\phi\bigl(\otimes_{x\notin X} e_{j_x} \otimes A\bigr)\bigr| &= \Bigl| \sum_{(j'_x)_{x\in X}} a_{j'} \; \phi\bigl(\otimes_{x\notin X} e_{j_x} \otimes_{x\in X} e_{j'_x}\bigr)\Bigr| \\ &\leq \|\phi\|_{\rm sp} \sum_{(j'_x)_{x\in X}} |a_{j'}| \\ &\leq \|\phi\|_{\rm sp} \|A\|_2 N^{|X|}. \end{split} \tag{7.19}$$

Using Eq. (7.19) with $\|\phi\|_{\rm sp} = 1$, $\|AB\|_2 \leq \|A\| \, \|B\|_2$, and $\|c_i^{(j_y)}\| \leq \sqrt{N} \|c_i^{(j_y)}\|_2$, we get

$$\begin{split} \bigl|K\phi(e_j)\bigr| &\leq \sqrt{N} \sup_{y\in\mathbb{Z}^d} \sum_{n\geq1} \frac{2^n}{n!} \sum_{X_1,\dots,X_n:y} \Bigl(\prod_{k=1}^n \|\Phi_{X_k}\| N^{|X_k|}\Bigr) \sum_{i=1}^{N-1} \|b_i^{(j_y)}\|_2 \|c_i^{(j_y)}\|_2 \\ &\leq N \sup_{y\in\mathbb{Z}^d} \sum_{n\geq1} \frac{2^n}{n!} \sum_{X_1,\dots,X_n:y} \prod_{k=1}^n \|\Phi_{X_k}\| N^{|X_k|}. \end{split} \tag{7.20}$$

We have used Lemma 7.2 to get the last line. The constraint $X_1, \dots, X_n : y$ means that (7.18) must be respected. The final step is to estimate the sum over such subsets. This can be conveniently done with an inductive argument. Namely, let $R_0 = 0$ and, for $m \geq 1$, let

$$R_m = \sup_{y \in \mathbb{Z}^d} \sum_{n=1}^{m} \frac{2^n}{n!} \sum_{X_1,\dots,X_n : y} \prod_{k=1}^{n} \|\Phi_{X_k}\| N^{|X_k|}. \tag{7.21}$$

Summing first over $X_1 \ni y$, then over sets that intersect sites of X_1, we get

$$\begin{aligned} R_m &\leq 2 \sup_y \sum_{X_1 \ni y} \|\Phi_{X_1}\| N^{|X_1|} \prod_{x \in X_1} \Big(\sum_{n=1}^{m} \frac{2^{n-1}}{(n-1)!} \sum_{X_2,\dots,X_n : x} \prod_{k=2}^{n} \|\Phi_{X_k}\| N^{|X_k|} \Big) \\ &\leq 2 \sup_y \sum_{X_1 \ni y} \|\Phi_{X_1}\| N^{|X_1|} (1 + R_{m-1})^{|X_1|}. \end{aligned} \tag{7.22}$$

It follows easily that $R_m \leq r$ for all m, and all r such that $2\|\Phi\|_{N(1+r)} \leq r$. Then $\|K\| \leq Nr$, and the assumption of Theorem 7.1 implies the existence of r such that $Nr < 1$. □

Bibliographical references

O. E. Lanford III [1970] observed that a uniqueness theorem for KMS states follows from an earlier result due to Greenberg [1968]. An extension to general spin systems was proposed in Bratteli, Robinson [1987]. The theorem presented here first appeared in the diploma thesis of S. Dirren supervised by J. Fröhlich in the 1990s; it was eventually published in Fröhlich, Ueltschi [2015].