CHAPTER 6

Extremal Gibbs states

The goal now is to understand better the structure of the set of (translation invariant) infinite-volume Gibbs states for a given interaction. We see in this chapter that it is a convex set, that extremal points enjoy special properties, and that any Gibbs state can be written as a convex combination of extremal states. It is remarkable that these properties can be established in a large class of systems without necessarily identifying the actual Gibbs states.

Some properties can be established straightforwardly; some properties are established using the GNS representation. The main properties and the simple proofs are reviewed in Sections 6.1 and 6.2. The GNS representation and its applications can be found in Section 6.3.

6.1. Extremal Gibbs states

It is easy to see that if $\langle\cdot\rangle^{(1)}$ and $\langle\cdot\rangle^{(2)}$ are two states, then the convex combination $t\langle\cdot\rangle^{(1)} + (1-t)\langle\cdot\rangle^{(2)}$, $t \in [0,1]$, is also a state: the set of states is convex. It is also not hard to check that the set of equilibrium states $\mathcal{G}^{\Phi}_{\text{tr.inv.}}$, for a given interaction $\Phi \in \mathcal{I}$, is convex. Indeed, if $\langle\cdot\rangle^{(1)}, \langle\cdot\rangle^{(2)} \in \mathcal{G}^{\Phi}_{\text{tr.inv}}$, then the convex combination $t\langle\cdot\rangle^{(1)} + (1-t)\langle\cdot\rangle^{(2)}$ satisfies the tangent functional property, the variational principle, and the KMS condition.

Definition 6.1. *A state $\langle\cdot\rangle \in \mathcal{G}^{\Phi}_{\text{tr.inv.}}$ is* **extremal** *if it cannot be written as the convex combination $t\langle\cdot\rangle^{(1)} + (1-t)\langle\cdot\rangle^{(2)}$, $t \in (0,1)$, of distinct states $\langle\cdot\rangle^{(1)}, \langle\cdot\rangle^{(2)} \in \mathcal{G}^{\Phi}_{\text{tr.inv.}}$.*

Another relevant property of states is to have short-range correlations; this is also referred to as "clustering".

Definition 6.2. *A state $\langle\cdot\rangle$ has* **short-range correlations** *if for all $a, b \in \mathcal{A}$ we have*

$$\lim_{\|x\|\to\infty} \big(\langle a\tau_x b\rangle - \langle a\rangle\langle\tau_x b\rangle\big) = 0.$$

When the state is translation-invariant we can of course replace $\langle\tau_x b\rangle$ by $\langle b\rangle$. Finally we introduce a notion that is reminiscent of ergodic measures in dynamical systems, which are equal to the time averages. Here the time evolution is replaced by space translations. We consider the averaging operator m_n whose action on $a \in \mathcal{A}$ is

$$m_n(a) = \frac{1}{n^d} \sum_{x \in \{1,\dots,n\}^d} \tau_x a. \tag{6.1}$$

For translation-invariant states we always have $\langle m_n(a) \rangle = \langle a \rangle$. Notice that $m_n(a) \in \mathcal{A}$ for finite n but the limit $n \to \infty$ does not exist (unless $a = \text{const}\mathbb{1}$). On the other hand, expectations of $m_n(a)^k$ in translation-invariant states converge.

Lemma 6.3. *Let $\langle \cdot \rangle$ be a translation-invariant state. Then for any $a \in \mathcal{A}$ and any $k \in \mathbb{N}$, the limit $n \to \infty$ of $\langle m_n(a)^k \rangle$ exists.*

The proof uses the GNS representation and can be found in Section 6.3. The property that distinguishes states that exhibit ergodicity is that fluctuations of average observables get suppressed, i.e. the their variances vanish.

Definition 6.4. *A translation-invariant state $\langle \cdot \rangle$ is* **ergodic** *if*

$$\lim_{n \to \infty} \Big\langle \big(m_n(a) - \langle a \rangle \big)^2 \Big\rangle = 0$$

for all $a \in \mathcal{A}$.

The limit $n \to \infty$ exists by Lemma 6.3; one could also do without it, by using lim sup or lim inf in the above definition.

A pleasing aspect of the theory is that these properties of states (extremal, short-range correlations, ergodic) are all equivalent. They are also equivalent with having a trivial space of "observables at infinity". The latter concept needs the GNS representation and is defined later in Definition 6.10.

Theorem 6.5. *Let $\langle \cdot \rangle \in \mathcal{G}^{\Phi}_{\text{tr.inv.}}$. The following properties are equivalent.*

(a) *$\langle \cdot \rangle$ is extremal in $\mathcal{G}^{\Phi}_{\text{tr.inv.}}$.*
(b) *$\langle \cdot \rangle$ is extremal in the set of all translation-invariant states on $\mathcal{A}$.*
(c) *$\langle \cdot \rangle$ has short-range correlations.*
(d) *$\langle \cdot \rangle$ is ergodic.*
(e) *The space of observables at infinity $\mathcal{O}$ of $\langle \cdot \rangle$ contains only multiples of the identity.*

We prove that (a) $\Longrightarrow$ (e) $\Longrightarrow$ (c) $\Longrightarrow$ (d) $\Longrightarrow$ (b) $\Longrightarrow$ (a). The two implications involving (e) necessitate the GNS representation and can be found immediately after Definition 6.10.

Partial proof. (b) $\Longrightarrow$ (a): The contrapositive is clear; indeed, if the state can be decomposed in distinct states in $\mathcal{G}^{\Phi}_{\text{tr.inv.}}$, it can also be decomposed among all states.

(c) $\Longrightarrow$ (d): Let $a \in \mathcal{A}$. For every $\varepsilon > 0$ there exists R such that

$$\big|\langle a \tau_x a \rangle - \langle a \rangle^2 \big| < \varepsilon \tag{6.2}$$

for all $\|x\| \geq R$. Then

$$\begin{aligned} \langle m_n(a)^2 \rangle - \langle a \rangle^2 &= \frac{1}{n^{2d}} \sum_{x,y \in \{1,\dots,n\}^d} \big(\langle \tau_x a \tau_y a \rangle - \langle a \rangle^2 \big) \\ &= \frac{1}{n^{2d}} \sum_{\substack{x,y \in \{1,\dots,n\}^d \\ \|x-y\| < R}} \big(\langle \tau_x a \tau_y a \rangle - \langle a \rangle^2 \big) + \frac{1}{n^{2d}} \sum_{\substack{x,y \in \{1,\dots,n\}^d \\ \|x-y\| \geq R}} \big(\langle \tau_x a \tau_y a \rangle - \langle a \rangle^2 \big). \end{aligned} \tag{6.3}$$

The first term is less than $2\|a\|^2 n^{-d} (2R)^d$ and it vanishes in the limit $n \to \infty$. The second term is less than ε. It follows that

$$\lim_{n\to\infty} \big| \langle m_n(a)^2 \rangle - \langle a \rangle^2 \big| \leq \varepsilon \tag{6.4}$$

for any $\varepsilon > 0$.

(d) $\Longrightarrow$ (b): We prove the contrapositive, namely that if the state is not extremal, it is not ergodic. If the state $\langle \cdot \rangle$ is not extremal, it is possible to find $\langle \cdot \rangle^{(1)}$ and $\langle \cdot \rangle^{(2)}$ such that $\langle \cdot \rangle = \frac12 \langle \cdot \rangle^{(1)} + \frac12 \langle \cdot \rangle^{(2)}$ and with $\langle a \rangle^{(1)} \neq \langle a \rangle^{(2)}$ for some hermitian $a \in \mathcal{A}$. Recalling that $(s+t)^2 < 2s^2 + 2t^2$ if $s \neq t$ we have

$$\begin{aligned} \langle a \rangle^2 &= \big(\tfrac12 \langle a \rangle^{(1)} + \tfrac12 \langle a \rangle^{(2)} \big)^2 < \tfrac12 \big(\langle a \rangle^{(1)} \big)^2 + \tfrac12 \big(\langle a \rangle^{(2)} \big)^2 \\ &\leq \lim_{n\to\infty} \Big(\tfrac12 \langle m_n(a)^2 \rangle^{(1)} + \tfrac12 \langle m_n(a)^2 \rangle^{(2)} \Big) = \lim_{n\to\infty} \langle m_n(a)^2 \rangle. \end{aligned} \tag{6.5}$$

The second inequality holds because the variance is nonnegative. Because of the strict inequality the state $\langle \cdot \rangle$ is not ergodic. ☐

6.2. Decomposition of states

We now show that any equilibrium state is a convex combination of extremal states. Since $\mathcal{G}^{\Phi}_{\mathrm{tr.inv}} \subset \mathcal{A}^*$, and $\mathcal{A}^*$ is a Banach space with the usual operator norm, the set of Gibbs states is a metric space. It is also a measurable space with the Borel σ-algebra (the one that is generated by open sets).

THEOREM 6.6. *Let $\Phi \in \mathcal{I}$ and $\langle \cdot \rangle \in \mathcal{G}^{\Phi}_{\mathrm{tr.inv}}$. There exists a unique measure μ on $\mathcal{G}^{\Phi}_{\mathrm{tr.inv}}$, that is concentrated on the extremal states of $\mathcal{G}^{\Phi}_{\mathrm{tr.inv}}$, such that for all $a \in \mathcal{A}$, we have*

$$\langle a \rangle = \int_{\mathcal{G}^{\Phi}_{\mathrm{tr.inv}}} \gamma(a) \, \mu(\mathrm{d}\gamma).$$

Theorem 6.6 is based on Choquet's theory, whose main result can be formulated as follows.

PROPOSITION 6.7 (Choquet). *Let K be a metrisable compact convex set and $\kappa \in K$. Then there exists a probability measure μ on K such that*

(a) *μ is concentrated on the extremal points of K.*
(b) *For any affine function $f : K \to \mathbb{R}$ we have $f(\kappa) = \int_K f(\eta)\mu(\mathrm{d}\eta)$.*

PROOF OF THEOREM 6.6. We can apply Choquet's result to $K = \mathcal{G}^{\Phi}_{\text{tr.inv.}}$. For $a \in \mathcal{A}$ we consider the affine function $f_a(\gamma) = \gamma(a)$, $\gamma \in \mathcal{G}^{\Phi}_{\text{tr.inv.}}$. The existence of the measure in Theorem 6.6 then follows immediately from Proposition 6.7.

There remains to establish uniqueness. Let $\langle\cdot\rangle$ be a state and μ a measure on $\mathcal{G}^{\Phi}_{\text{tr.inv.}}$ such that $\langle\cdot\rangle = \int \gamma(\cdot)\mu(\mathrm{d}\gamma)$ and where μ is concentrated on extremal states. We check that the μ-expectation of any continuous function depends solely on the state. Then μ is indeed unique.

Recall the Stone–Weierstrass theorem (continuous functions can be approximated by polynomials); it is then enough to check expectations of polynomials. Let $f_{a_1}, \dots, f_{a_k}$ be functions as above, for some $a_1, \dots, a_k \in \mathcal{A}$. Then, since the measure μ is concentrated on extremal states with short-range correlations, we have

$$\begin{aligned}\mu(f_{a_1} \dots f_{a_k}) &= \int_K \gamma(a_1) \dots \gamma(a_k)\mu(\mathrm{d}\gamma) \\ &= \lim_{n\to\infty} \int_K \gamma(m_n(a_1)) \dots \gamma(m_n(a_k))\mu(\mathrm{d}\gamma) \\ &= \lim_{n\to\infty} \int_K \gamma\big(m_n(a_1) \dots m_n(a_k)\big)\mu(\mathrm{d}\gamma) \\ &= \lim_{n\to\infty} \big\langle m_n(a_1) \dots m_n(a_k)\big\rangle.\end{aligned} \tag{6.6}$$

The last term does not depend explicitly on μ, but solely on the state. □

6.3. GNS representation of states

The GNS representation is due to Gelfand, Naimark, and Segal. Loosely speaking, any state can be written as a projection onto a single vector, at the cost of enlarging the Hilbert space. In order to understand why it is useful, it is worth reflecting on the theory of classical spin systems. There it is possible to turn the set of infinite configurations into a measurable space, which allows to define macroscopic observables (those that are independent of any finite set of spins). Extremal states are characterised by the fact that macroscopic observables take deterministic values. These theory is used to prove Theorem 6.5 in the classical case.

This structure cannot be generalised in the quantum case since there is no Hilbert space for the whole of $\mathbb{Z}^d$ and the space of quasi-local observables $\mathcal{A}$ does not contain macroscopic observables. We recover much of this structure with the GNS representation.

The GNS representation can be stated for general C^* algebras but we only consider here the case of states on the algebra of quasi-local operators.

THEOREM 6.8 (**GNS representation of states**). *Let $\langle\cdot\rangle$ be a translation invariant state on $\mathcal{A}$. There exists a Hilbert space $\mathscr{H}$, a vector $v \in \mathscr{H}$, and a representation (i.e. a $*$-isomorphism) $\pi : \mathcal{A} \to \mathcal{B}(\mathscr{H})$ such that*

(i) *$\|v\| = 1$ and $\pi(\mathcal{A})v$ is dense in $\mathscr{H}$.*
(ii) *$\langle a\rangle = \langle v, \pi(a)v\rangle$ for all $a \in \mathcal{A}$.*
(iii) *There are unitary operators $(U_x)_{x\in\mathbb{Z}^d}$ such that $U_x v = v$ and $\pi(\tau_x a) = U_x \pi(a) U_x^{-1}$ for all $a \in \mathcal{A}$ and $x \in \mathbb{Z}^d$.*

The representation is unique in the sense that any two representations are related by isomorphisms.

PROOF. We first assume that the state satisfies $\langle a^* a\rangle > 0$ for all $a \in \mathcal{A} \setminus \{0\}$; the construction is then much simpler. We explain afterwards how to handle the general situation.

The idea is to introduce the following inner product on $\mathcal{A}$:

$$\langle a, b\rangle = \langle a^* b\rangle. \tag{6.7}$$

The space $(\mathcal{A}, \langle\cdot,\cdot\rangle)$ is not necessarily complete, so we define $\mathscr{H}$ to be its completion. Then we can view $\mathcal{A}$ as a dense subspace of $\mathscr{H}$. The vector v is the operator $\mathbb{1}$. The operator $\pi(a)$ in $\mathscr{H}$ acts as follows; if $b \in \mathcal{A}$, then

$$\pi(a)b = ab. \tag{6.8}$$

Then $\pi(\mathcal{A})v = \mathcal{A}$ is indeed dense and $\langle v, \pi(a)v\rangle = \langle \mathbb{1} a \mathbb{1}\rangle = \langle a\rangle$. Finally, the operator U_x acts on $a \in \mathcal{A}$ as

$$U_x a = \tau_x a. \tag{6.9}$$

Then $\pi(\tau_x a)b = (\tau_x a)b = \tau_x(a \tau_x^{-1} b) = U_x a U_x^{-1} b$.

Isomorphism between different representations is straightforward; if $(\mathscr{H}', \pi')$ is another representation, we can consider the map V such that $V\pi(a)v = \pi'(a)v'$.

Let us now consider the general case where $\langle a^* a\rangle = 0$ for nontrivial operators. We introduce the space $\mathcal{R} = \{a \in \mathcal{A} : \langle a^* a\rangle = 0\}$. One can check that $\mathcal{R} = \{a \in \mathcal{A} : \langle ab\rangle = \langle ba\rangle = 0 \ \forall\, b \in \mathcal{A}\}$ so that $\mathcal{R}$ is an ideal of $\mathcal{A}$ (that is, $ab, ba \in \mathcal{R}$ whenever $a \in \mathcal{A}$ and $b \in \mathcal{R}$). We introduce an equivalence relation on $\mathcal{A}$, namely $a \sim b$ whenever $a - b \in \mathcal{R}$. This allows to define the inner product on the quotient algebra $\mathcal{A}/\mathcal{R}$ using (6.7); the right side does not depend on the representatives of the equivalence classes. The Hilbert space $\mathscr{H}$ is then defined as the completion of $\mathcal{A}/\mathcal{R}$. The rest of the proof is exactly as above; one can check that all relations are independent of the choice of representatives. □

Among the advantages of the GNS representations is that the space average operator converges. Recall the definition (6.1) of m_n. Although the limit $n \to \infty$ of m_n does not exist, the limit of $\pi(m_n)$ does. We let P denote the projector in $\mathscr{H}$ onto the subspace of vectors that are invariant under all U_x; this subspace is not null, it contains v.

PROPOSITION 6.9. *We have*

$$\text{s-lim} \frac{1}{n^d} \sum_{x \in \{1,\dots,n\}^d} U_x = P.$$

(Recall that a sequence of operators u_n converges strongly to u if $\|(u_n - u)\varphi\| \to 0$ for each vector φ.)

PROOF. It is enough to prove it for $d = 1$ since everything factorises according to dimensions. The operator U_{e_1} is unitary and it can be written with the help of the spectral theorem as

$$U_{e_1} = \frac{1}{2\pi} \int_0^{2\pi} \mathrm{e}^{\mathrm{i}\theta} \, P(\theta) \mathrm{d}\nu(\theta). \tag{6.10}$$

Using $U_{ke_1} = U_{e_1}^k$, we get

$$\frac{1}{n} \sum_{k=1}^n U_{ke_1} = \frac{1}{2\pi} \int_0^{2\pi} \frac{1}{n} \sum_{k=1}^n \mathrm{e}^{\mathrm{i}k\theta} \, P(\theta) \mathrm{d}\nu(\theta). \tag{6.11}$$

Now we have

$$\frac{1}{n} \sum_{k=1}^n \mathrm{e}^{\mathrm{i}k\theta} = \begin{cases} 1 & \text{if } \theta = 0, \\ \frac{1}{n} \, \mathrm{e}^{\mathrm{i}\theta} \, \frac{1-\mathrm{e}^{\mathrm{i}\theta n}}{1-\mathrm{e}^{\mathrm{i}\theta}} & \text{if } \theta \in (0, 2\pi). \end{cases} \tag{6.12}$$

Applying the operators to a vector $w \in \mathscr{H}$, we can use the dominated convergence theorem to see that the limit converges to $P(0)w = Pw$. □

PROOF OF LEMMA 6.3. Using Theorem 6.8 we have

$$\langle m_n(a)^k \rangle = \Big\langle v, \Big(\frac{1}{n^d} \sum_{x \in \{1,\dots,n\}^d} U_x \pi(a) U_x^{-1} \Big)^k v \Big\rangle. \tag{6.13}$$

We now use Proposition 6.9, which involves the projector P onto translation invariant vectors in the GNS Hilbert space $\mathscr{H}$. If $w \in \mathscr{H}$ is translation invariant, then $\frac{1}{n^d} \sum_{x \in \{1,\dots,n\}^d} U_x \pi(a) U_x^{-1} w$ converges strongly to $P\pi(a)w$ (which is translation invariant). Applying this above, we get that $\langle m_n(a)^k \rangle$ converges to $\langle v, (P\pi(a))^k v \rangle$. □

We now introduce the "algebra at infinity". First, given a finite set $\Lambda \Subset \mathbb{Z}^d$, let

$$\mathcal{A}_{\Lambda^c} = \overline{\bigcup_{\substack{\Lambda' \Subset \mathbb{Z}^d \\ \Lambda' \cap \Lambda = \emptyset}} \mathcal{A}_{\Lambda'}}. \tag{6.14}$$

(The closure is with respect to the operator norm.)

DEFINITION 6.10 (**Observables at infinity**). *Let $\langle \cdot \rangle$ be a state with GNS representation $(\mathscr{H}, \pi, v)$. The algebra at infinity $\mathcal{O}$ is the von Neumann algebra of operators in $\mathscr{H}$ defined by*

$$\mathcal{O} = \overline{\bigcap_{\Lambda \Subset \mathbb{Z}^d} \pi(\mathcal{A}_{\Lambda^c})}^{\text{weak}}. \tag{6.15}$$

Here the completion is taken with the weak operator topology (the topology such that $a_n \to a$ iff $\langle u, a_n w \rangle \to \langle u, aw \rangle$ for all vectors u, w).

Notice that $\mathbb{1} \in \mathcal{O}$. In contrast to the classical case $\mathcal{O}$ depends on the state; this does not play a major rôle. We now complete the proof of Theorem 6.5.

PROOF OF THEOREM 6.5. (a) $\implies$ (e): We prove the contrapositive. There exists $T \in \mathcal{O}$ that is not a multiple of the identity. Since $\mathcal{O}$ is a linear subspace that is closed under taking the adjoint, we can suppose that $0 \leq T \leq \mathbb{1}$. We define a linear functional on $\mathcal{A}$ by

$$\lambda(a) = \langle v, T\pi(a)v \rangle. \tag{6.16}$$

Notice that $[T, \pi(a)] = 0$ since a is quasi-local; then $0 \leq \lambda \leq \langle \cdot \rangle$. We can approximate T by a sequence $\pi(c_n)$ where $c_n \in \mathcal{A}$ and convergence is in the weak operator topology. Recall the space $\tilde{\mathcal{A}}$ of observables that appeared in the definition of KMS states, we have for all $a \in \mathcal{A}$ and $b \in \tilde{\mathcal{A}}$ that

$$\begin{aligned} \lambda(ab) &= \langle v, T\pi(ab)v \rangle = \lim_n \langle c_n ab \rangle = \lim_n \langle \alpha_{-\mathrm{i}}(b) c_n a \rangle \\ &= \lim_n \langle \pi(\alpha_{-\mathrm{i}}(b))^* v, \pi(c_n)\pi(a)v \rangle = \langle v, \pi(\alpha_{-\mathrm{i}}(b)) T\pi(a)v \rangle \\ &= \langle v, T\pi(\alpha_{-\mathrm{i}}(b))\pi(a)v \rangle = \lambda(\alpha_{-\mathrm{i}}(b)a). \end{aligned} \tag{6.17}$$

We used that $T \in \mathcal{O}$ commutes with all quasi-local observables. This shows that λ satisfies the KMS condition, so we can define the two KMS states

$$\langle a \rangle^{(1)} = \frac{\lambda(a)}{\lambda(\mathbb{1})}, \qquad \langle a \rangle^{(2)} = \frac{\langle a \rangle - \lambda(a)}{1 - \lambda(\mathbb{1})}. \tag{6.18}$$

Then $\langle \cdot \rangle = \lambda(\mathbb{1}) \langle \cdot \rangle^{(1)} + (1 - \lambda(\mathbb{1})) \langle \cdot \rangle^{(2)}$, and the latter states are distinct, when T is not a multiple of the identity.

(e) $\Longrightarrow$ (c): Again we prove the contrapositive. If $\langle\cdot\rangle$ does not have short-range correlations we can find $a, b \in \mathcal{A}$, and a sequence $x_n \in \mathbb{Z}^d$ with $\|x_n\| \to \infty$, such that

$$\big|\langle a\,\tau_{x_n} b\rangle - \langle a\rangle\langle \tau_{x_n} b\rangle\big| \geq 1 \tag{6.19}$$

for all n. A limiting argument allows to take a, b in $\mathcal{A}_{\rm loc}$. Then there exists a sequence of domains $\Lambda_n \Uparrow \mathbb{Z}^d$ such that $\tau_{x_n} b \in \mathcal{A}_{\Lambda_n^{\rm c}}$.

Since the unit ball of a von Neumann algebra is compact (in the weak operator topology) we can suppose that $\pi(\tau_{x_n} b)$ converges, taking a subsequence if necessary; let B denote the limit. Then B belongs to the von Neumann algebra generated by $\pi(\mathcal{A}_{\Lambda_n^{(\rm c)}})$ for all n, and therefore $B \in \mathcal{O}$. There remains to verify that B is not a multiple of the identity. We have

$$\begin{aligned}\big|\langle v, \pi(a) B v\rangle - \langle v, \pi(a) v\rangle\langle v, Bv\rangle\big| &= \lim_n \big|\langle v, \pi(a)\pi(\tau_{x_n} b) v\rangle - \langle v, \pi(a) v\rangle\langle v, \pi(\tau_{x_n} b) v\rangle\big| \\ &= \lim_n \big|\langle a\,\tau_{x_n} b\rangle - \langle a\rangle\langle \tau_{x_n} b\rangle\big| \\ &\geq 1.\end{aligned} \tag{6.20}$$

We used (6.19). If B were equal to $\mathbb{1}$ the first term in the above equation would be 0. □

Bibliographical references

Theorem 6.6 was first proposed by Lanford and Ruelle [1969] for classical systems. A more general result in the quantum case can be found in Ruelle [1970].

The topic of C^* algebras is vast and we have only borrowed the minimum amount of material that was needed for our purpose. Interested readers are encouraged to learn more in the books of Ruelle [1969], Israel [1979], Simon [1993], and especially the first volume of Bratteli and Robinson [1987].

Exercise 6.1. *Show that* $\mathcal{R} = \{a \in \mathcal{A} : \langle a^* a\rangle = 0\}$ *is indeed an ideal (hint: Cauchy–Schwarz).*