No continuous symmetry breaking in 2D

"Mermin-Wagner theorem": confusing name, since it groups several related, by distinct, properties:

① The original proof of Mermin and Wagner (1966) of the absence of spontaneous magnetisation:

$$\lim_{h \to 0+} \langle S_0^{(3)} \rangle_{\beta,h} = 0 \quad \forall \beta$$

② No long-range order: $\langle S_0^{(3)} S_x^{(3)} \rangle_{\beta,0} \leq C \, \|x\|^{-\frac{c}{\beta}}$.

(Fisher, Jasnow '71; McBryan, Spencer '77; Koma, Tasaki '92)

③ All Gibbs states retain the continuous symmetry.

(Dobrushin, Shlosman '75; Fröhlich, Pfister '81)

Main claim

$d = 2$.

Assume $\exists\, S_x \in \mathfrak{A}_{\{x\}}$ such that $[\Phi_X, \sum_{x \in X} S_x] = 0 \;\forall X$.

Let $U_\Lambda = \exp(i\theta \sum_{x \in \Lambda} S_x)$, $\theta \in \mathbb{R}$.

Then:

THEOREM 6.1. *Under the assumptions above, assume that* $\langle\cdot\rangle \in \mathcal{G}^{\Phi}_{\text{t.i.}}$ *is a translation-invariant Gibbs state. Then for all* $\Lambda \Subset \mathbb{Z}^2$ *and all* $A \in \mathcal{A}_\Lambda$ *we have that*

$$\langle U^*_\Lambda(\theta) A U_\Lambda(\theta)\rangle = \langle A\rangle. \tag{6.1}$$

Main ingredients in the proof

LEMMA 6.2. *Let* $\langle\cdot\rangle \in \mathcal{G}^{\Phi}_{\mathrm{t.i.}}$ *be a translation-invariant Gibbs state for an interaction* $\Phi \in \mathcal{I}$. *Let* $\Lambda \Subset \mathbb{Z}^d$ *and let* $\Lambda_n \uparrow \mathbb{Z}^d$. *Then there is a sequence of interactions* Ψ_n *such that* $\|\Psi_n\| \to 0$, *such that for all* $A \in \mathcal{A}_\Lambda$ *we have*

$$\langle A\rangle = \lim_{n\to\infty} \langle A\rangle^{\Phi+\Psi_n}_{\Lambda_n}.$$

DEFINITION A.16. *The* **relative entropy** $S(\cdot\|\cdot)$ *is the following function of two positive-definite matrices* $a, b \in \mathcal{M}_n$:

$$S(a\|b) = \mathrm{Tr}\, a\big(\log a - \log b\big).$$

LEMMA 6.3 (Quantum Pinsker's inequality). *For two density-matrices* $\rho, \sigma \in \mathcal{B}(\mathcal{H})$ *on a finite-dimensional Hilbert space* $\mathcal{H}$, *define the* relative entropy

$$S(\rho\|\sigma) := \mathrm{Tr}\,\rho(\log\rho - \log\sigma). \tag{6.4}$$

Then $S(\rho\|\sigma) \geq \frac{1}{2}\|\rho-\sigma\|_1^2$. *In particular* $S(\rho\|\sigma) \geq 0$.

Proof of the quantum Pinsker inequality

Lemma A.13. *Let $a, b, h \in \mathcal{M}_n$ such that $a, b \geq 0$, $[a,b] = 0$, and $h = h^*$. Assume that $\left(\begin{smallmatrix} a & h \\ h & b \end{smallmatrix}\right) \geq 0$. Then $h \leq a^{1/2}b^{1/2}$.*

Proof. Let $u, v \in \mathbb{C}^n$ such that $\|u\| = \|v\| = 1$ and define $x = a^{-1/2}u$, $y = b^{-1/2}v$. Then, with $\mathbf{0}$ the zero vector in $\mathbb{C}^n$, we have

$$0 \leq \underbrace{\begin{pmatrix} x^* & \mathbf{0}^* \\ \mathbf{0}^* & y^* \end{pmatrix}}_{2\times 2n} \underbrace{\begin{pmatrix} a & h \\ h & b \end{pmatrix}}_{2n\times 2n} \underbrace{\begin{pmatrix} x & \mathbf{0} \\ \mathbf{0} & y \end{pmatrix}}_{2n\times 2} = \begin{pmatrix} x^*ax & x^*hy \\ y^*hx & y^*hy \end{pmatrix} = \begin{pmatrix} 1 & u^*a^{-1/2}hb^{-1/2}v \\ v^*b^{-1/2}ha^{-1/2}u & 1 \end{pmatrix}. \tag{A.36}$$

The latter is a 2×2 matrix; its determinant is nonnegative so that $|u^*a^{-1/2}hb^{-1/2}v| \leq 1$. This implies that $\|a^{-1/2}hb^{-1/2}\| \leq 1$. Next one can check that $a^{-1/4}b^{-1/4}ha^{-1/4}b^{-1/4}$ has the same eigenvalues as $a^{-1/2}hb^{-1/2}$ (if v is eigenvector of the first matrix, then $a^{-1/4}b^{1/4}v$ is eigenvector of the second matrix with the same eigenvalue). Then

$$\|a^{-1/4}b^{-1/4}ha^{-1/4}b^{-1/4}\| \leq 1, \tag{A.37}$$

and, since this matrix is hermitian, we have

$$a^{-1/4}b^{-1/4}ha^{-1/4}b^{-1/4} \leq \mathbb{1} \qquad \Longleftrightarrow \qquad h \leq a^{-1/2}b^{-1/2}. \tag{A.38}$$

$\square$

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Theorem A.14 (Lieb's concavity). *Let $a_1, a_2, b_1, b_2 \geq 0$ be $n \times n$ complex matrices and let $\alpha \in [0, 1]$. Then*

$$(a_1 + a_2)^\alpha \otimes (b_1 + b_2)^{1-\alpha} \geq a_1^\alpha \otimes b_1^{1-\alpha} + a_2^\alpha \otimes b_2^{1-\alpha}.$$

Proof. Let $x(\alpha) = a_1^\alpha \otimes b_1^{1-\alpha}$, $y(\alpha) = a_2^\alpha \otimes b_2^{1-\alpha}$, $z(\alpha) = (a_1 + a_2)^\alpha \otimes (b_1 + b_2)^{1-\alpha}$. We need to show that $z(\alpha) \geq x(\alpha) + y(\alpha)$ for all $\alpha \in [0, 1]$. It is actually enough to show this in a dense subset since all expressions are continuous in α. This clearly holds for $\alpha \in \{0, 1\}$. We now show that if it holds for α and β, then it also holds for $\frac{\alpha+\beta}{2}$.

We have $x(\frac{\alpha+\beta}{2}) = x(\alpha)^{1/2} x(\beta)^{1/2}$, and the same relations for y and z. Then

$$\begin{pmatrix} x(\alpha) & x(\frac{\alpha+\beta}{2}) \\ x(\frac{\alpha+\beta}{2}) & x(\beta) \end{pmatrix} = \underbrace{\begin{pmatrix} x(\alpha)^{1/2} \\ x(\beta)^{1/2} \end{pmatrix}}_{2n \times n} \underbrace{\begin{pmatrix} x(\alpha)^{1/2} & x(\beta)^{1/2} \end{pmatrix}}_{n \times 2n} \geq 0. \tag{A.39}$$

The latter inequality holds quite generally, only using that $x(\alpha)$ is hermitian. We have a similar inequality for y. Then

$$0 \leq \begin{pmatrix} x(\alpha) & x(\frac{\alpha+\beta}{2}) \\ x(\frac{\alpha+\beta}{2}) & x(\beta) \end{pmatrix} + \begin{pmatrix} y(\alpha) & y(\frac{\alpha+\beta}{2}) \\ y(\frac{\alpha+\beta}{2}) & y(\beta) \end{pmatrix} \leq \begin{pmatrix} z(\alpha) & x(\frac{\alpha+\beta}{2}) + y(\frac{\alpha+\beta}{2}) \\ x(\frac{\alpha+\beta}{2}) + y(\frac{\alpha+\beta}{2}) & z(\beta) \end{pmatrix}. \tag{A.40}$$

The second holds because the difference is equal to $\left(\begin{smallmatrix} z(\alpha)-x(\alpha)-y(\alpha) & 0 \\ 0 & z(\beta)-x(\beta)-y(\beta) \end{smallmatrix}\right)$, which is nonnegative by assumption. We now use Lemma A.13 and we get

$$x\left(\tfrac{\alpha+\beta}{2}\right) + y\left(\tfrac{\alpha+\beta}{2}\right) \leq z(\alpha)^{1/2} z(\beta)^{1/2} = z\left(\tfrac{\alpha+\beta}{2}\right). \tag{A.41}$$

We can start with $\alpha = 0$ and $\beta = 1$ and iterate the inequality, so it applies to all multiples of 2^{-k} for arbitrary k; this set is dense in $[0, 1]$. □

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Corollary A.15. *Let $a_1, a_2, b_1, b_2 \geq 0$ be $n \times n$ complex matrices and let $\alpha \in [0,1]$. Then*

$$\operatorname{Tr}\left((a_1 + a_2)^\alpha (b_1 + b_2)^{1-\alpha}\right) \geq \operatorname{Tr}\left(a_1^\alpha b_1^{1-\alpha}\right) + \operatorname{Tr}\left(a_2^\alpha b_2^{1-\alpha}\right).$$

Proof. We use the following correspondence between $\mathcal{M}_n$ and $\mathbb{C}^n \otimes \mathbb{C}^n$:

$$\operatorname{Tr} a^{\mathrm{T}} b = \sum_{i,j=1}^{n} a_{i,j} b_{i,j} = \sum_{i,j=1}^{n} \langle i| \otimes \langle \underset{i}{\cancel{j}}| \,(a \otimes b)\, |\underset{j}{\cancel{i}}\rangle \otimes |j\rangle. \tag{A.42}$$

This allows to use Theorem A.14. $\square$

Lieb's concavity

Recall: $S(a\|b) = \operatorname{Tr} a(\log a - \log b)$.

Lemma A.17. *We have for all $a, b \geq 0$ that*

$$S(a\|b) = \lim_{\varepsilon \to 0+} \tfrac{1}{\varepsilon}\left(\operatorname{Tr} a - \operatorname{Tr} a^{1-\varepsilon} b^\varepsilon\right).$$

Proof. Let $f(\varepsilon) = \operatorname{Tr} a^{1-\varepsilon} b^\varepsilon$. The right side is the derivative $f'(0)$ which is equal to $S(a\|b)$. $\square$

Theorem A.18 (Joint convexity of the relative entropy). *If $a_1, a_2, b_1, b_2 \geq 0$ are complex matrices in $\mathcal{M}_n$, then*

$$S\big(a_1 + a_2 \| b_1 + b_2\big) \leq S(a_1 \| b_1) + S(a_2 \| b_2).$$

Since $S(\lambda a \| \lambda b) = \lambda S(a \| b)$, the joint convexity of the relative entropy follows immediately. And since the entropy is equal to $S(a) = S(a \| \mathbb{1})$, it is convex too.

Proof. Starting with Lemma A.17, and using Corollary A.15, we have

$$\begin{aligned} S(a_1 + a_2 \| b_1 + b_2) &= \lim_{\varepsilon \to 0+} \tfrac{1}{\varepsilon}\Big(\operatorname{Tr}(a_1 + a_2) - \operatorname{Tr}(a_1 + a_2)^{1-\varepsilon}(b_1 + b_2)^{\varepsilon}\Big) \\ &\leq \lim_{\varepsilon \to 0+} \tfrac{1}{\varepsilon}\Big(\operatorname{Tr} a_1 - \operatorname{Tr} a_1^{1-\varepsilon} b_1^{\varepsilon} + \operatorname{Tr} a_2 - \operatorname{Tr} a_2^{1-\varepsilon} b_2^{\varepsilon}\Big) \\ &= S(a_1 \| b_1) + S(a_2 \| b_2). \end{aligned} \tag{A.43}$$

$\square$

Lemma A.19. *Let $U = \operatorname{diag}\big(1, \mathrm{e}^{\frac{2\pi\mathrm{i}}{n}}, \ldots, \mathrm{e}^{\frac{2\pi\mathrm{i}}{n}(n-1)}\big)$. Then for any matrix $a \in \mathcal{M}_n$ we have*

$$\operatorname{diag} a = \frac{1}{n} \sum_{k=0}^{n-1} U^k a\, U^{-k}.$$

Proof. The element (ℓ, m) of the left side is equal to

$$\frac{1}{n} \sum_{k=0}^{n-1} \mathrm{e}^{\frac{2\pi\mathrm{i}}{n} k\ell}\, a_{\ell,m}\, \mathrm{e}^{-\frac{2\pi\mathrm{i}}{n} km} = \frac{1}{n} a_{\ell,m} \sum_{k=0}^{n-1} \mathrm{e}^{\frac{2\pi\mathrm{i}}{n} k(\ell - m)} = \begin{cases} a_{\ell,\ell} & \text{if } \ell = m, \\ 0 & \text{if } \ell \neq m. \end{cases} \tag{A.44}$$

$\square$

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Let $P = \operatorname{diag}(1, \ldots, 1, 0, \ldots, 0)$ with k elements equal to 1, where $k \in \{1, \ldots, n-1\}$. We consider the map $\Phi : \mathcal{M}_n \to \mathcal{M}_n$ defined by

$$\Phi(a) = \operatorname{diag}\big(\underbrace{\tfrac{1}{k}\operatorname{Tr} Pa \ldots \tfrac{1}{k}\operatorname{Tr} Pa}_{k \text{ elements}} \; \underbrace{\tfrac{1}{n-k}\operatorname{Tr}(1-P)a \ldots \tfrac{1}{n-1}\operatorname{Tr}(1-P)a}_{n-k \text{ elements}}\big). \tag{A.45}$$

$$\begin{pmatrix} c & & & & \\ & c & & & \\ & & c & & \\ & & & d & \\ & & & & d \\ & & & & & \ddots \end{pmatrix}$$

The image $\Phi(a)$ is a simple matrix with just two values.

LEMMA A.20. *There exists a finite number L and unitary matrices $U_1, \ldots, U_L$ such that*

$$\Phi(a) = \frac{1}{L} \sum_{\ell=1}^{L} U_\ell a U_\ell^{-1}.$$

PROOF. If a is diagonal we can consider the permutation $(1, \ldots, k)(k+1, \ldots, n)$ and its permutation matrix V. Then $\Phi(a) = \frac{1}{k(n-k)} \sum_{\ell=1}^{k(n-k)} V^\ell a V^{-\ell}$. Indeed, this amounts to average over diagonal matrices where the first k elements of a have been rotated, as well as the last $n-k$ elements.

For the general case we combine this with Lemma A.19 to get

$$\Phi(a) = \frac{1}{nk(n-k)} \sum_{\ell=1}^{k(n-k)} \sum_{m=0}^{n-1} V^\ell U^m a U^{-m} V^{-\ell}. \tag{A.46}$$

□

Finally: proof of Pinsker's inequality

Proof of Lemma 6.3. Let P be the projector onto the subspace of the eigenvectors of $\rho - \sigma$ with nonnegative eigenvalues and let Φ be the map defined in Eq. (A.45). By Lemma A.20 and the convexity of the relative entropy (Theorem A.18), we obtain that

$$S(\rho\|\sigma) \geq S\big(\Phi(\rho)\|\Phi(\sigma)\big). \tag{A.47}$$

The latter is equal to the classical relative entropy of two Bernoulli random variables with parameters $\operatorname{Tr} P\rho$ and $\operatorname{Tr} P\sigma$. Using the classical Pinsker inequality (see Exercise A.1), we get that it is greater than

$$\tfrac{1}{2}\big(|\operatorname{Tr} P\rho - \operatorname{Tr} P\sigma| + |\operatorname{Tr}(1-P)\rho - \operatorname{Tr}(1-P)\sigma|\big)^2 = \tfrac{1}{2}\|\rho - \sigma\|_1^2. \tag{A.48}$$

The last identity uses the fact that P is the projector onto the suitable eigensubspace of $\rho - \sigma$. $\square$

Back to Theorem 6.1

THEOREM 6.1. *Under the assumptions above, assume that* $\langle\cdot\rangle \in \mathcal{G}^{\Phi}_{\text{t.i.}}$ *is a translation-invariant Gibbs state. Then for all* $\Lambda \Subset \mathbb{Z}^2$ *and all* $A \in \mathcal{A}_\Lambda$ *we have that*

$$\langle U^*_\Lambda(\theta) A U_\Lambda(\theta)\rangle = \langle A\rangle. \tag{6.1}$$

(1) Introduce angles $\boldsymbol{\theta}^{(m)} = (\theta^{(m)}_x)$ such that $\theta^{(m)}_x = \theta$ for all $x \in \Lambda$ and all m. Let $U_m = \sum_x \theta^{(m)}_x S_x$.

(2) Using Lemma 6.2, it is enough to show that for all $A \in \mathcal{A}_\Lambda$, we have

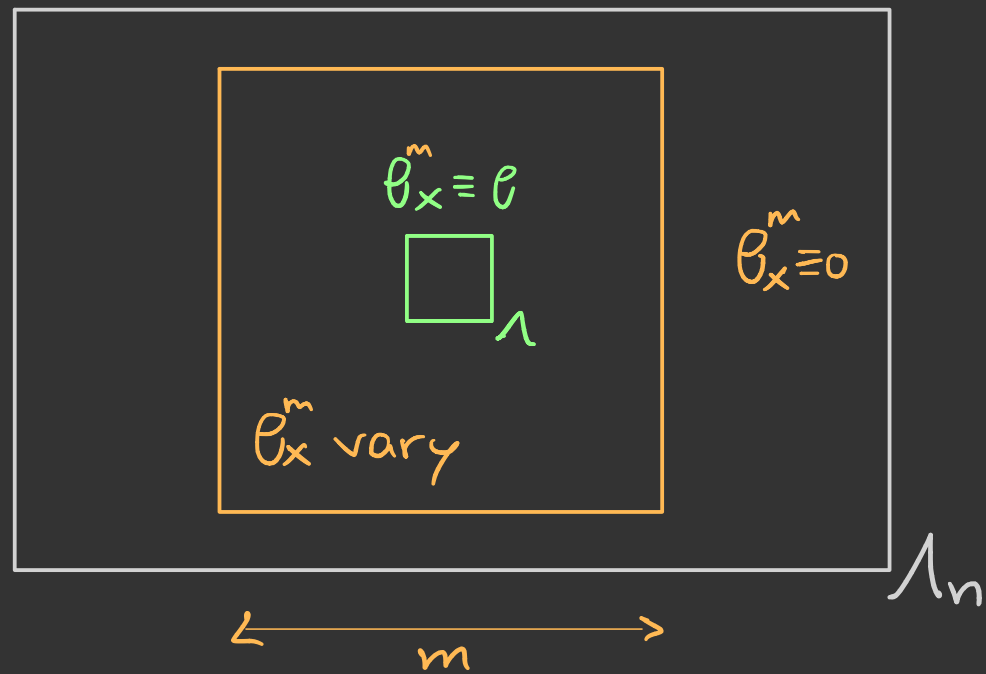
$$\lim_{m\to\infty}\lim_{n\to\infty}\Big(\langle A\rangle^{\Phi+\Psi_n}_{\Lambda_n} - \langle U^*_m A U_m\rangle^{\Phi+\Psi_n}_{\Lambda_n}\Big) = 0. \tag{6.5}$$

(3) We estimate the difference above in terms of the relative entropy. This in turns gives an estimate involving the difference of hamiltonians $H^{\Phi+\Psi_n}_{\Lambda_n}$ and $H^{U_m(\Phi+\Psi_n)U^*_m}_{\Lambda}$.

(4) The difference above can be bounded by $C\sum_{\|x-y\|=1}(\theta^{(m)}_x - \theta^{(m)}_y)^2$.

(5) In two dimensions, we can find $\boldsymbol{\theta}^{(m)}$ such that $\theta^{(m)}_x = \theta$ for all $x \in \Lambda$ and all m, such that the gradient above goes to 0 as $m \to \infty$.

Step (1):



Step (3):

LEMMA 6.4. *We have the bound*

$$\left|\langle A\rangle^{\Phi+\Psi_n}_{\Lambda_n} - \langle U_m^* A U_m\rangle^{\Phi+\Psi_n}_{\Lambda_n}\right|^2 \le 2\left\|2H^{\Phi}_{\Lambda_n} - H^{U_m\Phi U_m^*}_{\Lambda_n} - H^{U_m^*\Phi U_m}_{\Lambda_n}\right\| + 2|\!|\!|\Psi_n|\!|\!| \|\boldsymbol{\theta}^{(m)}\|_1 \|A\|.$$

PROOF. This uses Pinsker inequality and the trick of rotating the angles in both directions. □

Step (4):

Now we get Step (4). For $X \Subset \mathbb{Z}^2$

$$\overline{U}_{m,X} = \sum_{x \in X} (\theta_x^{(m)} - \bar{\theta}^{(m)}) S_x, \tag{6.6}$$

where $\bar{\theta}^{(m)}$ is the average of $\theta_x^{(m)}$ on X (it depends on X, although we do not indicate it). We have $U_m \Phi_X U_m^* = \overline{U}_{m,X} \Phi_X \overline{U}_{m,X}^*$.

LEMMA 6.5. *We have*

$$\left\| 2H_{\Lambda_n}^{\Phi} - H_{\Lambda_n}^{U_m \Phi U_m^*} - H_{\Lambda_n}^{U_m^* \Phi U_m} \right\| \leq 2 \sum_{X \subset \Lambda_n} \|\Phi_X\| \, \|\overline{U}_{m,X}\|^2 \, \mathrm{e}^{2\|\overline{U}_{m,X}\|} .$$

PROOF. Use the Lie-Schwinger expansion, and observe that odd powers of commutators vanish. The inequality $\cosh u - 1 \leq \frac{1}{2} u^2 \, \mathrm{e}^u$ is used. □

To get $\sum_{\|x-y\|=1} |\theta_x - \theta_y|^2$: use a discrete Poincaré inequality.

Step (5):

Lemma 6.6. *The following choice for $\boldsymbol{\theta}^{(m)}$ gives the desired rotation on Λ and its gradient vanishes when $m \to \infty$: With m_0 large enough so that all sites in Λ are at distance at most m_0 from the origin, let*

$$\theta_x = \begin{cases} \theta & \text{if } \|x\|_1 \le m_0, \\ \theta\left(1 - \frac{\log(\|x\|_1 - m_0)}{\log m}\right) & \text{if } m_0 < \|x\|_1 < m_0 + m, \\ 0 & \text{if } \|x\|_1 \ge m. \end{cases}$$

Then

$$\sum_{\substack{\{x,y\} \subseteq \Lambda_n \\ \|x-y\|=1}} |\theta_x - \theta_y|^2 \le \frac{\text{const}}{\log m}.$$

Proof. We can bound

$$\sum_{\substack{\{x,y\} \subseteq \Lambda_n \\ \|x-y\|=1}} |\theta_x - \theta_y|^2 = \sum_{r=0}^{\substack{m+m_0 \\ \cancel{n}}} \sum_{x:\|x\|_1 = r} \sum_{y:\|y\|_1 = r+1} |\theta_x - \theta_y|^2$$

$$\le \sum_{r=m_0}^{m} 4 \cdot 8 r \theta^2 \left(\frac{\log(r+1) - \log r}{\log m}\right)^2 \tag{6.7}$$

$$\le \frac{\text{const}}{(\log m)^2} \sum_{r=1}^{m} r\Big(\underbrace{\log(1 + \tfrac{1}{r})}_{\sim \frac{1}{r}}\Big)^2 \le \frac{\text{const}}{\log m}.$$

□

Conclusion

- Family of quantum lattice systems with interesting phase diagrams.
- General theory for infinite-volume Gibbs states.
- Unique Gibbs state at high temperature (or when the magnetic field is large).
- Existence of long-range at low temperatures, proved sometimes.
- Mean-field systems.
- Absence of continuous symmetry breaking in 2D.