Long-range order in XXZ model

① Peierls argument for Ising

② Kennedy's extension to XXZ with any $J^3 > J^1 = J^2$.

Note: long-range order $\Rightarrow$ not mixing $\Rightarrow$ Gibbs state not extremal

$$\Rightarrow \boxed{|\xi^{\phi}_{\beta}| > 1}$$

Phase diagram of the XXZ model

$$H_{\Lambda,h} = -\sum_{xy\in\mathcal{E}_\Lambda} \left(J^1 S^1_x S^1_y + J^2 S^2_x S^2_y + J^3 S_x S_y \right) - h \sum_{x\in\Lambda} S^3_x$$

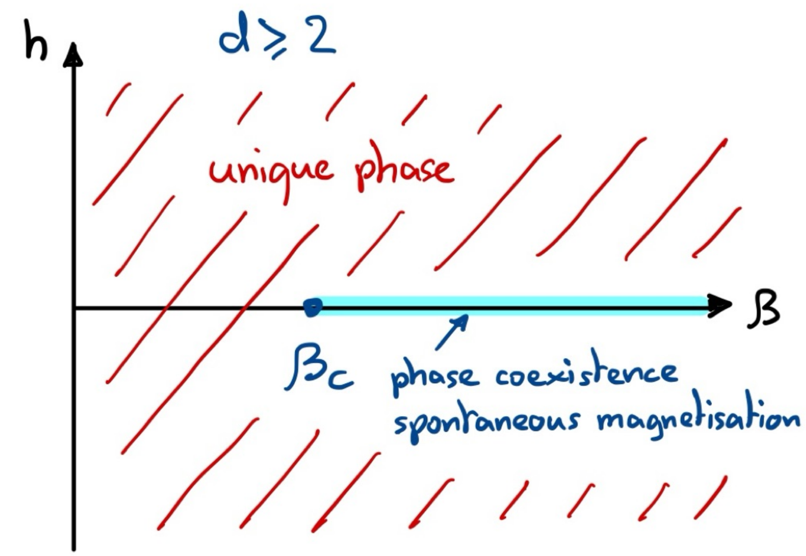


FIGURE 1.1. Phase diagram of the **Ising model**, and of the **XXZ model** for $J^{(3)} > J^{(1)} = J^{(2)} \geq 0$, for all dimensions $d \geq 2$.

$(N=2)$

Ising hamiltonian:

$$H_\Lambda^{\text{ISING}} = -2 \sum_{xy\in\mathcal{E}(\Lambda)} (S_x^{(3)} S_y^{(3)} - \tfrac{1}{4}) = -\tfrac{1}{2} \sum_{xy\in\mathcal{E}(\Lambda)} (\sigma_x^{(3)} \sigma_y^{(3)} - 1).$$

Recall that $\mathcal{H}_\Lambda = \bigotimes_{x\in\Lambda} \mathbb{C}^2 \simeq \ell^2(\{-1,1\}^\Lambda)$.

Notation: $\omega = (\omega_x)_{x\in\Lambda}$, $\omega_x = \pm 1$.

Probability measure on classical configurations:

$$\mathbb{P}_\Lambda(\omega) = \frac{1}{Z_\Lambda} \exp\left(\tfrac{1}{2}\beta \textstyle\sum_{xy\in\mathcal{E}(\Lambda)} (\omega_x\omega_y - 1)\right),$$

Then:

$$\langle \sigma_x^{(3)} \sigma_y^{(3)} \rangle_\Lambda = \mathbb{E}_\Lambda[\omega_x\omega_y].$$

Here: $\Lambda = \{-N, \dots, N\}^d$.

Occurrence of long-range order

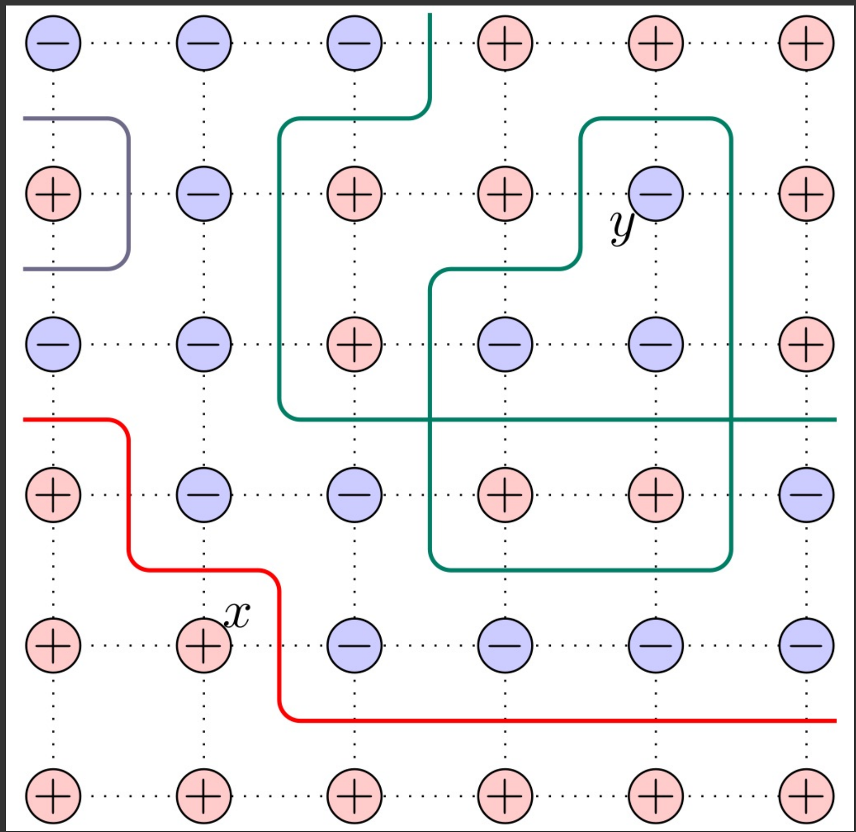
Theorem 4.2. *Consider the model (4.13) with $d \geq 2$. There exist $\beta_0 < \infty$ and $c(\beta) > 0$ (that depend on d but not on N) such that for $\beta > \beta_0$, we have*

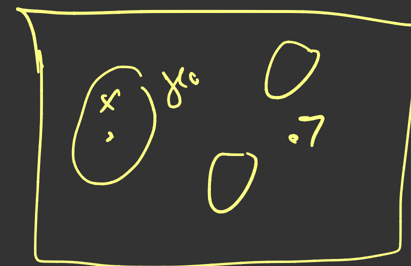
$$\langle \sigma_x^{(3)} \sigma_y^{(3)} \rangle_\Lambda \geq c \tag{4.18}$$

for all $x, y \in \Lambda_N$.

Proof.

Contour representation:





$$Z_{\Lambda,\beta} = \mathrm{Tr}\; \mathrm{e}^{-\beta H_\Lambda^{\mathrm{ISING}}} = 2 \sum_{g \in G_\Lambda} \prod_{\gamma \in g} w(\gamma).$$

where the weights of the contours are $$w(\gamma) = \mathrm{e}^{-\beta|\gamma|},$$

$\mathbb{1}_{\omega_x = \omega_y} - \mathbb{1}_{\omega_x \neq \omega_y} = 1 - 2 \cdot \mathbb{1}_{\omega_x \neq \omega_y}$

$$\langle \sigma_x^{(3)} \sigma_y^{(3)} \rangle_\Lambda = \mathbb{E}_\Lambda[\omega_x \omega_y] = 1 - 2\mathbb{P}_\Lambda(\omega_x \neq \omega_y).$$

set of sets of disjoint contours

$$\begin{aligned}
\mathbb{P}_\Lambda(\omega_x \neq \omega_y) &\leq \frac{2}{Z_{\Lambda,\beta}} \sum_{\substack{g \in G_\Lambda \\ x,y \text{ separated}}} \prod_{\gamma \in g} w(\gamma) \\
&\leq \frac{2}{Z_{\Lambda,\beta}} \left(\sum_{\gamma_0 : \mathrm{Int}\gamma_0 \ni x} w(\gamma_0) \sum_{g : g \cup \{\gamma_0\} \in G_\Lambda} \prod_{\gamma \in g} w(\gamma) + [\text{same with } y] \right).
\end{aligned}$$

interior of γ_0

We have:

$$\frac{2}{Z_{\Lambda,\beta}} \sum_{g : g \cup \{\gamma_0\} \in G_\Lambda} \prod_{\gamma \in g} w(\gamma) \leq 1.$$

Then

$$\mathbb{P}_\Lambda(\omega_x \neq \omega_y) \leq \sum_{\gamma_0 : \mathrm{Int}\,\gamma_0 \ni x} w(\gamma_0) + \sum_{\gamma_0 : \mathrm{Int}\,\gamma_0 \ni y} w(\gamma_0).$$

$D_x(\gamma_0) \leq |\gamma_0|$

$D_x(\gamma_0)$: length of shortest path between x and γ_0

$\delta^{|\gamma_0| - D_x(\gamma_0)} \geq 1$

$$\sum_{\gamma_0 : \mathrm{Int}\,\gamma_0 \ni x} w(\gamma_0) \leq \sum_{\gamma_0 \in \Gamma_\Lambda} \mathrm{e}^{-(\beta - \log\delta)|\gamma_0|}\, \delta^{-D_x(\gamma_0)}.$$ $\forall\ \delta \geq 1$

We now estimate the sum over contours:

$$\sum_{\gamma_0 \in \Gamma_\Lambda} \mathrm{e}^{-(\beta - \log\delta)|\gamma_0|}\, \delta^{-D_x(\gamma_0)} \leq \left(\tfrac{2}{\delta - 1}\right)^d \left(c_d^{-1}\, \mathrm{e}^{\beta - \log\delta} - 1\right)^{-1}.$$

We get a bound for the probability that $\omega_x \neq \omega_y$, hence for $\langle S_x^3 S_y^3 \rangle$:

$$\mathbb{P}_\Lambda(\omega_x \neq \omega_y) \leq 2\left(\tfrac{2}{\delta - 1}\right)^d \left(c_d^{-1}\, \mathrm{e}^{\beta - \log\delta} - 1\right)^{-1}.$$ $\square$

Next: XXZ model

$$H_\Lambda^{\mathrm{XXZ}} = -\frac{1}{2} \sum_{xy\in\mathcal{E}(\Lambda)} \left(t\sigma_x^{(1)}\sigma_y^{(1)} + t\sigma_x^{(2)}\sigma_y^{(2)} + \sigma_x^{(3)}\sigma_y^{(3)} - 1\right) = H_\Lambda^{\mathrm{ISING}} + tV_\Lambda.$$

THEOREM 4.4. *Let $d \geq 2$. Consider the model (4.28) with $t \in [0,1)$. There exists $c > 0$ and $\beta_0(t) < \infty$ such that for all $\beta > \beta_0(t)$, all finite boxes Λ, and all $x, y \in \Lambda$, we have*

$$\langle \sigma_x^{(3)}\sigma_y^{(3)}\rangle_\Lambda \geq c. \tag{4.29}$$

Consequence: $|\mathcal{G}_\beta^{XXZ}| > 1$ when β is large

Many open questions:
- Unique β_c ?
- $d \geq 3$: Can we get $\beta_0(t) \not\to \infty$ as $t \to 1_+$?
- $d = 2$: The situation as $t \to 1+$ is unclear.

Proof of long-range order in XXZ model

Lie-Trotter expansion:

$$\mathrm{e}^{a+b} = \lim_{N\to\infty} \left[\mathrm{e}^{\frac{1}{N}a} \overbrace{\left(1 + \tfrac{1}{N}b\right)}^{\approx e^{\frac{1}{N}b}} \right]^N .$$

(Proposition A.5 in lecture notes)

$$\mathrm{e}^{-\beta H_\Lambda^{\mathrm{XXZ}}} = \lim_{N\to\infty} \left[\mathrm{e}^{-\frac{\beta}{N} H_\Lambda^{\textsc{Ising}}} \left(1 - \tfrac{\beta t}{N} V_\Lambda\right) \right]^N .$$

Inserting $\mathbb{1} = \sum_w |w\rangle\langle w|$, N times:

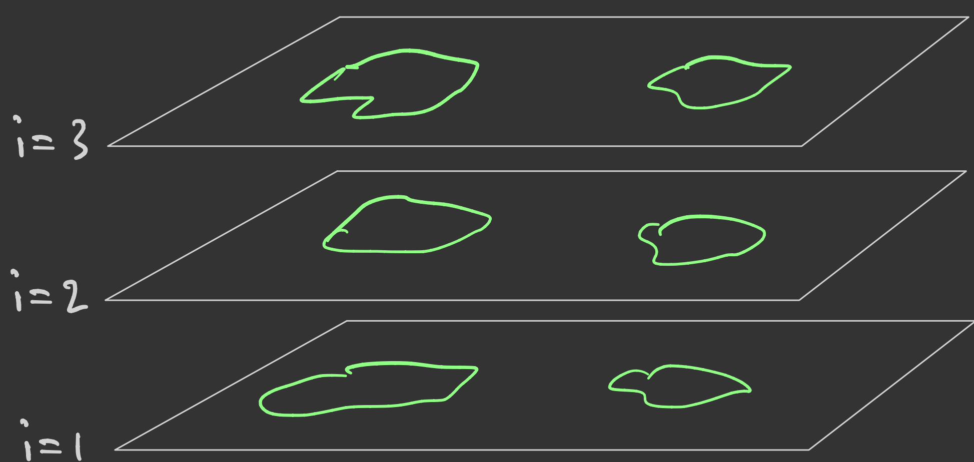
$$\begin{aligned}
\operatorname{Tr} \mathrm{e}^{-\beta H_\Lambda^{\mathrm{XXZ}}} &= \lim_{N\to\infty} \sum_{\omega_\Lambda^{(1)},\dots,\omega_\Lambda^{(N)} \in \{-1,+1\}^\Lambda} \left\langle \omega_\Lambda^{(1)} \middle| \mathrm{e}^{-\frac{\beta}{N} H_\Lambda^{\textsc{Ising}}} \left(1 - \tfrac{\beta t}{N} V_\Lambda\right) \middle| \omega_\Lambda^{(2)} \right\rangle \\
&\qquad \dots \left\langle \omega_\Lambda^{(N)} \middle| \mathrm{e}^{-\frac{\beta}{N} H_\Lambda^{\textsc{Ising}}} \left(1 - \tfrac{\beta t}{N} V_\Lambda\right) \middle| \omega_\Lambda^{(1)} \right\rangle, \\
\operatorname{Tr} \sigma_x^{(3)} \sigma_y^{(3)} \mathrm{e}^{-\beta H_\Lambda^{\mathrm{XXZ}}} &= \lim_{N\to\infty} \sum_{\omega_\Lambda^{(1)},\dots,\omega_\Lambda^{(N)} \in \{-1,+1\}^\Lambda} \left\langle \omega_\Lambda^{(1)} \middle| \omega_x^{(1)} \omega_y^{(1)} \mathrm{e}^{-\frac{\beta}{N} H_\Lambda^{\textsc{Ising}}} \left(1 - \tfrac{\beta t}{N} V_\Lambda\right) \middle| \omega_\Lambda^{(2)} \right\rangle \\
&\qquad \dots \left\langle \omega_\Lambda^{(N)} \middle| \mathrm{e}^{-\frac{\beta}{N} H_\Lambda^{\textsc{Ising}}} \left(1 - \tfrac{\beta t}{N} V_\Lambda\right) \middle| \omega_\Lambda^{(1)} \right\rangle.
\end{aligned}$$

The terms in the expansion:

$$\left\langle \omega_\Lambda \middle| \mathrm{e}^{-\frac{\beta}{N} H_\Lambda^{\mathrm{ISING}}} \middle| \omega_\Lambda \right\rangle = \mathrm{e}^{\frac{\beta}{2N} \sum_{xy}(1-\omega_x\omega_y)} = \prod_{\gamma \in g(\omega)} \mathrm{e}^{-\frac{\beta}{N}|\gamma|} .$$

$$\left\langle \omega_\Lambda^{(i)} \middle| \left(1-\tfrac{\beta t}{N} V_\Lambda\right) \middle| \omega_\Lambda^{(i+1)} \right\rangle = \begin{cases} 1 & \text{if } \omega_\Lambda^{(i)} = \omega_\Lambda^{(i+1)}, \\ \frac{\beta t}{N} & \text{if } |\omega_\Lambda^{(i)}\rangle = \sigma_x^{(+)}\sigma_y^{(-)}|\omega_\Lambda^{(i+1)}\rangle \text{ for some neighbours } x,y \in \Lambda, \\ 0 & \text{otherwise.} \end{cases}$$

"Space-time" contours:



Weight of space-time contours:

\# transitions

$$w(\gamma) = \exp\Big(-\frac{\beta}{N}\sum_{i=1}^{N}|\gamma_i|\Big)\Big(\frac{\beta t}{N}\Big)^{n(\gamma)},$$

$$\mathrm{Tr}\ \mathrm{e}^{-\beta H_\Lambda^{\mathrm{XXZ}}} = 2\sum_{\{\gamma_1,\dots,\gamma_k\}}\prod_{j=1}^{k} w(\gamma_j).$$

γ_i: space-time contour (connected in space-time)

Let $D_x(\gamma^{(i)})$: minimal distance between x and all connected elements of $\gamma^{(i)}$. We have:

$$D_x(\gamma^{(i)}) \le |\gamma^{(i)}| + n(\gamma).$$

Goal: $\langle \sigma_0^{(3)} \sigma_x^{(3)} \rangle_{\Lambda,\beta} \gtrsim c$

Use $\langle \sigma_0^{(3)} \sigma_x^{(3)} \rangle = 1 - \cancel{\langle \sigma_0^{(3)} \sigma_x^{(3)} \rangle} = 1 - \lim\limits_{N\to\infty} \mathbb{P}_{\Lambda,\beta,N} [\, \exists \text{ contour separating } 0 \text{ and } x \,]$

Let $\eta > 0$ and $\delta > 1$.

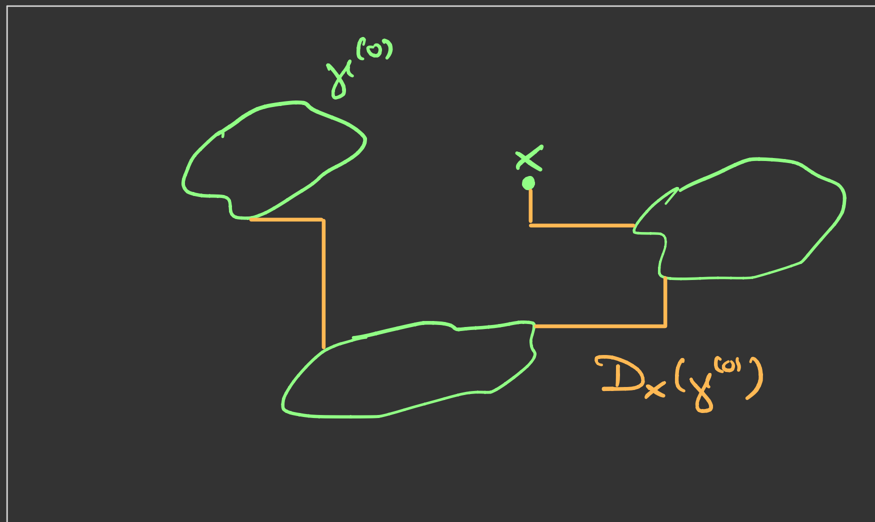
$$
\begin{aligned}
\sum_{\gamma\in\mathbf{\Gamma}_\Lambda:\gamma^{(1)} \text{ surrounds } x} w(\gamma) &\overset{=}{\leq} \sum_{\gamma\in\mathbf{\Gamma}_\Lambda:\gamma^{(1)} \text{ surrounds } x} \mathrm{e}^{-\frac{\beta\eta}{N}\sum_{j=1}^N |\gamma^{(j)}|}\, \mathrm{e}^{-\frac{\beta}{N}(1-\eta)\sum_{i=1}^N |\gamma^{(i)}|} \left(\frac{\beta t}{N}\right)^{n(\gamma)} \\
&\overset{\text{Jensen}}{\leq} \sum_{\gamma\in\mathbf{\Gamma}_\Lambda:\gamma^{(1)} \text{ surrounds } x} \frac{1}{N}\sum_{j=1}^N \mathrm{e}^{-\beta\eta|\gamma^{(j)}|}\, \mathrm{e}^{-\frac{\beta}{N}(1-\eta)\sum_{i=1}^N |\gamma^{(i)}|} \left(\frac{\beta t}{N}\right)^{n(\gamma)} \\
&\leq \sum_{\gamma\in\mathbf{\Gamma}_\Lambda} \frac{1}{N}\sum_{j=1}^N \mathrm{e}^{-(\beta\eta-\log\delta)|\gamma^{(j)}|}\, \delta^{-D_x(\gamma^{(j)})}\, \mathrm{e}^{-\frac{\beta}{N}(1-\eta)\sum_{i=1}^N |\gamma^{(i)}|} \left(\frac{\beta t\delta}{N}\right)^{n(\gamma)} \\
&= \sum_{\gamma^{(0)}\in G_\Lambda} \mathrm{e}^{-(\beta\eta-\log\delta)|\gamma^{(0)}|}\, \delta^{-D_x(\gamma^{(0)})} \sum_{\gamma\in\mathbf{\Gamma}_\Lambda:\gamma^{(1)}=\gamma^{(0)}} \mathrm{e}^{-\frac{\beta}{N}(1-\eta)\sum_{i=1}^N |\gamma^{(i)}|} \left(\frac{\beta t\delta}{N}\right)^{n(\gamma)}.
\end{aligned}
$$

Choose $1-\eta > t\delta$ $\quad (\eta > 0,\ \delta > 1,\ t < 1)$

$$
\sum_{\gamma\in\mathbf{\Gamma}_\Lambda:\gamma^{(1)}=\gamma^{(0)}} \mathrm{e}^{-\frac{\beta}{N}(1-\eta)\sum_{i=1}^N |\gamma^{(i)}|} \left(\frac{\beta t\delta}{N}\right)^{n(\gamma)} \leq \operatorname{Tr} P_{\gamma^{(0)}}\, \mathrm{e}^{-\beta t\delta H_\Lambda^{\mathrm{XXX}}} \leq 2.
$$

We have obtained

$$\sum_{\gamma : \gamma^{(1)} \text{ surrounds } x} w(\gamma) \leq 2 \sum_{\gamma^{(0)} \in G_\Lambda} e^{-(\beta\eta - \log\delta)|\gamma^{(0)}|} \delta^{-D_x(\gamma^{(0)})}$$



Proposition 4.5. *For any $\delta > 1$ and $\varepsilon > 0$ there is a constant β_0 (that depends on d, ε, δ but not on Λ) such that for any $\beta \geq \beta_0$, we have*

$$\sum_{\gamma^{(0)} \in G_\Lambda} e^{-\beta|\gamma^{(0)}|} \delta^{-D_x(\gamma^{(0)})} \leq \varepsilon.$$

See the proof in the notes. $\square$

Long-range order – proof using the "infrared bound"

Theorem 4.6 (Infrared bound). *Assume that $\ell \in 2\mathbb{N}$ and that*

$$J^{(1)}, J^{(3)} \geq 0 \geq J^{(2)}.$$

$\Lambda_\ell = \{1, \dots, \ell\}^d_{per}$

Then we have for all $k \in \Lambda^*_\ell \setminus \{0\}$ that

$\varepsilon(k) = 2 \sum_{i=1}^{d} (1 - \cos k_i)$

$$\langle \widehat{S^{(3)}_0 S^{(3)}_x} \rangle_{\Lambda_\ell,\beta}(k) \leq S \sqrt{\frac{d(J^{(1)} - J^{(2)})}{J^{(3)}}} \frac{1}{\sqrt{\varepsilon(k)}} + \frac{1}{2\beta J^{(3)} \varepsilon(k)}.$$

Theorem 4.7. *Assume that $\ell \in 2\mathbb{N}$ and that*

$$J^{(3)} \geq J^{(1)} \geq -J^{(2)} \geq 0.$$

Then

$$\frac{1}{\ell^d} \sum_{x \in \Lambda_\ell} \langle S^{(3)}_0 S^{(3)}_x \rangle_{\Lambda_\ell,\beta} \geq \tfrac{1}{3} S(S+1) - \frac{1}{\ell^d} \sum_{k \in \Lambda^*_\ell \setminus \{0\}} \left(S \sqrt{\frac{d(J^{(1)} - J^{(2)})}{J^{(3)}}} \frac{1}{\sqrt{\varepsilon(k)}} + \frac{1}{2\beta J^{(3)} \varepsilon(k)} \right). \tag{4.46}$$

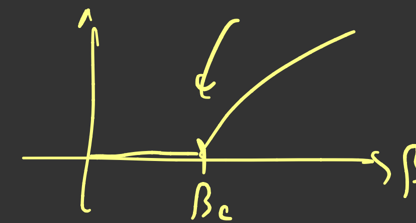
Proof:

$$\tfrac{1}{3} S(S+1) \leq \langle S^{(3)}_0 S^{(3)}_0 \rangle_{\Lambda_\ell,\beta} = \underbrace{\frac{1}{\ell^d} \langle \widehat{S^{(3)}_0 S^{(3)}_x} \rangle_{\Lambda_\ell,\beta}(0)}_{= \frac{1}{\ell^d} \sum_{x \in \Lambda_\ell} \langle S^{(3)}_0 S^{(3)}_x \rangle_{\Lambda_\ell,\beta}} + \underbrace{\frac{1}{\ell^d} \sum_{k \in \Lambda^*_\ell \setminus \{0\}} \langle \widehat{S^{(3)}_0 S^{(3)}_x} \rangle_{\Lambda_\ell,\beta}(k)}_{\text{small because of IRB}}.$$

$\square$

Mean-field systems (i.e. on the complete graph)

$\sim \left(\frac{\beta-\beta_c}{\beta_c}\right)^{\gamma}$



- Simpler to study
- Describe the limit $d \to \infty$. However:
- Expected to have the same properties as in $\mathbb{Z}^d$ when the dimension is greater than the upper critical dimension (often $d=4$).

Here: spin-$\frac{1}{2}$ $(N=2)$ Heisenberg model

$$H_N = -\frac{2}{N-1} \sum_{\substack{x,y=1 \\ x \neq y}}^{N} \vec{S}_x \cdot \vec{S}_y \quad \text{on } \mathbb{C}^{2N}$$

Setting

$$\mathcal{A}_N = \mathcal{A}_{\{1,\dots,N\}} = B(\mathbb{C}^{2N})$$

$$\mathcal{A}_{loc} = \bigvee_{N\geq 1} \mathcal{A}_N \,, \qquad \mathcal{A} = \overline{\mathcal{A}_{loc}}^{\,\text{norm}}$$

States are positive normalised linear functionals $\mathcal{A} \to \mathbb{C}$.

translation-invariance on $\mathbb{Z}^d$ $\longleftrightarrow$ permutation invariance

We restrict to permutation-invariant states.

Gibbs states: those that are tangent to the free energy, or satisfy the Gibbs variational principle, or the KMS condition. The set of Gibbs states is convex.

Properties of Gibbs states: mixing, extremal, product.

Definition 5.3 (Product state). *A state* $\boldsymbol{\rho}$ *on* $\mathcal{A}$ *is called a* product state *if there is a density matrix* $\rho \in \mathcal{B}(\mathcal{H}_n)$ *such that for all* $N \geq 1$ *and all* $A \in \mathcal{A}_N$ *we have that* $\boldsymbol{\rho}(A) = \mathrm{tr}\,(\rho^{\otimes N} A)$. *(Equivalently,* $\rho_N = \rho^{\otimes N}$ *for all* N.*)*

Theorem 5.5. *Assume that* $\boldsymbol{\rho} \in \mathfrak{E}_{\text{p.i.}}$. *The following are equivalent:*

(a) $\boldsymbol{\rho}$ *is extremal in* $\mathfrak{E}_{\text{p.i.}}$

(b) $\boldsymbol{\rho}$ *is mixing*

(c) $\boldsymbol{\rho}$ *is a product state.*

Furthermore, if $\boldsymbol{\rho} \in \mathcal{G}^{\beta\Phi}_{\text{p.i.}}$, *then the above are all equivalent to:*

(d) $\boldsymbol{\rho}$ *is extremal in* $\mathcal{G}^{\beta\Phi}_{\text{p.i.}}$

Then any permutation-invariant state can be written as a convex combination of product states.

(This is analogue to the quantum de Finetti theorem.)

Determining the Gibbs states

Case of pair interactions, $\Phi_X \neq 0$ only if $|X| = 2$.

Let

$$H_\rho = \operatorname{Tr}_2\big((\mathbb{1} \otimes \rho)\Phi_{1,2}\big).$$

THEOREM 5.6 (Fannes–Spohn–Verbeure). *Let Φ be a permutation-invariant two-body interaction. Then the extremal elements of $\mathcal{G}^{\Phi}_{\text{p.i.}}(\beta)$ are those product states whose density matrix ρ (with $\operatorname{Tr}\rho = 1$) minimizes the function*

$$\mathcal{F}(\rho) = \operatorname{Tr}\rho H_\rho + \tfrac{1}{\beta}\operatorname{Tr}\rho\log\rho.$$

Necessary condition:

PROPOSITION 5.7 (Mean-field equation). *Let ρ be a density matrix ($\operatorname{Tr}\rho = 1$) which minimizes the function $\mathcal{F}(\rho)$ in Theorem 5.6. Then ρ satisfies*

$$\rho = \frac{\mathrm{e}^{-2\beta H_\rho}}{\operatorname{Tr}\,\mathrm{e}^{-2\beta H_\rho}}.$$

5.3.1. The spin-$\frac{1}{2}$ Heisenberg model. Consider the fully isotropic model for spin $\frac{1}{2}$, that is we take $n = 2$ and $J_1 = J_2 = J_3 = 1$. Thus we have the interaction

$$\Phi_L = \begin{cases} -\vec{S}_x \cdot \vec{S}_y, & \text{if } L = \{x, y\}, x \neq y, \\ 0, & \text{otherwise.} \end{cases} \tag{5.23}$$

We identify the density matrices ρ corresponding to extremal states. From Exercise 1.7 we know that $\Phi_{1,2} = -(\frac{1}{2}T_{1,2} - \frac{1}{4})$ where $T_{1,2}$ is the transposition operator. Thus

$$(\mathbb{1} \otimes \rho)\Phi_{1,2} = -(\mathbb{1} \otimes \rho)(\tfrac{1}{2}T_{1,2} - \tfrac{1}{4}) = -\tfrac{1}{2}\rho \otimes \mathbb{1} + \tfrac{1}{4}\mathbb{1} \otimes \rho. \tag{5.24}$$

Thaking the partial trace we get

$$\boxed{H_\rho = -\rho + \tfrac{1}{4}\mathbb{1}.} \tag{5.25}$$

Any 2×2 density matrix can be written (compare Exercise 3.4) in the form

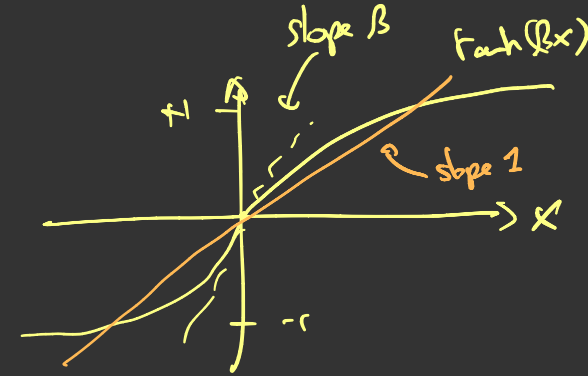
$$\boxed{\rho = \tfrac{1}{2}\mathbb{1} + \vec{a} \cdot \vec{S},} \qquad \vec{a} = (a_1, a_2, a_3) \in \mathbb{R}^3. \tag{5.26}$$

With this parameterization,

$$H_\rho = -\vec{a} \cdot \vec{S} - \tfrac{1}{4}\mathbb{1}. \tag{5.27}$$

We need to compute $\frac{\mathrm{e}^{-2\beta H_\rho}}{\mathrm{Tr}\ \mathrm{e}^{-2\beta H_\rho}}$, and we see that the scalar term $-\frac{1}{4}\mathbb{1}$ in H_ρ cancels. To compute $\mathrm{e}^{2\beta\vec{a}\cdot\vec{S}}$, note that $(\vec{a} \cdot \vec{S})^2 = \frac{1}{4}\|\vec{a}\|^2$, so from the power series expansion of the exponential we get

$$\mathrm{e}^{2\beta\vec{a}\cdot\vec{S}} = \cosh\left(\beta\|\vec{a}\|\right)\mathbb{1} + \frac{\sinh\left(\beta\|\vec{a}\|\right)}{\frac{1}{2}\|\vec{a}\|}(\vec{a} \cdot \vec{S}). \tag{5.28}$$



non-trivial sols when $\beta > 1$

Then $\mathrm{Tr}\ \mathrm{e}^{2\beta\vec{a}\cdot\vec{S}} = 2\cosh\left(\frac{\beta}{2}\|\vec{a}\|\right)$, so

$$\frac{\mathrm{e}^{-2\beta H_\rho}}{\mathrm{Tr}\ \mathrm{e}^{-2\beta H_\rho}} = \tfrac{1}{2}\mathbb{1} + (\vec{a} \cdot \vec{S})\frac{\tanh\left(\beta\|\vec{a}\|\right)}{\|\vec{a}\|}. \tag{5.29}$$

The mean-field equation reduces to

$$\boxed{(\vec{a} \cdot \vec{S})\left(\frac{\tanh\left(\beta\|\vec{a}\|\right)}{\|\vec{a}\|} - 1\right) = 0.} \tag{5.30}$$

One solution is always $\vec{a} = 0$. If $\beta > 1 =: \beta_c$ then the equation $\tanh(\beta x) = x$ has a (unique) positive solution $x = x^\star(\beta)$. Any $\vec{a}$ with $\|\vec{a}\| = x^\star$ thus solves the mean-field equation. One may check that all such $\vec{a}$ give the same value of the free energy, and that it is smaller than for $\vec{a} = 0$ (see Exercise 5.1). It follows that the extremal permutation-invariant Gibbs states for the spin-$\frac{1}{2}$ Heisenberg XXX-model are indexed by the points on a sphere, that is by SO(3).