KMS condition. Extremal state decomposition

Recall: Finite volume Gibbs state: $\langle A \rangle_{H,\beta} = \frac{1}{Z} \operatorname{Tr} e^{-\beta H} A$

PROPOSITION 3.3. *The Gibbs state* $\langle\cdot\rangle_{H,\beta}$ *is the unique state* $\langle\cdot\rangle = \operatorname{Tr} \cdot \rho$ *satisfying any of the following four conditions:*

(a) Tangent condition: *Let* $F_\beta(H) := -\frac{1}{\beta} \log \operatorname{Tr} \mathrm{e}^{-\beta H}$. *Then*

$$F_\beta(H + A) \leq F_\beta(H) + \langle A \rangle$$

for all $A = A^* \in \mathcal{B}(\mathcal{H})$.

(b) Gibbs variational principle: *The density matrix* ρ *minimizes the function*

$$\mathcal{F}_\beta(\rho) := \operatorname{Tr} H\rho + \frac{1}{\beta} \operatorname{Tr} \rho \log \rho.$$

(c) KMS condition: *Define the time evolution*

$$\alpha_t(A) = \mathrm{e}^{\mathrm{i}tH} A \, \mathrm{e}^{-\mathrm{i}tH}, \qquad t \in \mathbb{C}. \tag{3.11}$$

Then

$$\langle AB \rangle = \langle B \, \alpha_{\mathrm{i}\beta}(A) \rangle$$

for all $A, B \in \mathcal{B}(\mathcal{H})$.

(d) RAS condition: *For all* $A \in \mathcal{B}(\mathcal{H})$ *such that* $\langle AA^* \rangle > 0$,

$$\langle A^*[H, A] \rangle \geq \frac{1}{\beta} \langle A^* A \rangle \log \frac{\langle A^* A \rangle}{\langle AA^* \rangle} \tag{3.12}$$

Infinite-volume setting

$X \Subset \mathbb{Z}^d$: $\mathcal{A}_X$ is space of observables with support in X

$$\mathcal{A}_{loc} = \bigvee_{X \Subset \mathbb{Z}^d} \mathcal{A}_X \quad : \text{ local observables}$$

$$\mathcal{A} = \overline{\mathcal{A}_{loc}}^{\,norm} \quad : \text{ quasi-local observables}$$

We consider states on $\mathcal{A}$.

Interaction: $(\Phi_X)_{X \Subset \mathbb{Z}^d}$ (here: always translation-invariant)

Norms: $$|||\Phi||| = \sum_{X \ni 0} \frac{1}{|X|} \|\Phi_X\| \qquad (\text{if } |||\Phi||| < \infty, \ \Phi \in \mathcal{I})$$

$$\|\Phi\|_r = \sum_{X \ni 0} e^{r|X|} \|\Phi_X\| \qquad (\text{if } \|\Phi\|_r < \infty : \ \Phi \in \mathcal{I}_r)$$

Back to finite-volume KMS condition

For $\Lambda \Subset \mathbb{Z}^d$: hamiltonian $H_\Lambda^\Phi = \sum_{X \subset \Lambda} \Phi_X$

Evolution operator $\alpha_{\Lambda,t}^\Phi(A) = e^{itH_\Lambda^\Phi} A \, e^{-itH_\Lambda^\Phi}$

The (finite-vol.) KMS condition:

$$\langle AB \rangle = \langle B \, \alpha_{\Lambda, i\beta}^\Phi(A) \rangle \quad \forall A, B \in \mathfrak{A}_\Lambda$$

Method: Extend $\alpha_{\Lambda,t}^\Phi$ to $\mathbb{C}$.

Infinite-volume of evolution operator

Lemma 3.15. *Assume that $\Phi \in \mathcal{I}_r$ for some $r > 0$ and that $t \in \mathbb{C}$ with $|t| < \frac{r}{2\|\Phi\|_r}$. Then $(\alpha^{\Phi}_{\Lambda,t}(A))_{\Lambda \in \mathbb{Z}^d}$ is a Cauchy sequence for each fixed $A \in \mathcal{A}_{\mathrm{loc}}$.*

Further, we found that

$$\| \alpha^{\Phi}_{\Lambda,t}(A) \| \leq \|A\| \, e^{r|Y|} \left(1 - |t| \frac{2\|\Phi\|_r}{r} \right)^{-1}$$

$\forall A \in \mathcal{A}_Y$. Here, $t \in \mathbb{C}$.

We can define α^{Φ}_t on $\mathcal{A}_{\mathrm{loc}}$ by

$$\alpha^{\Phi}_t(A) = \lim_{\Lambda \uparrow \mathbb{Z}^d} \alpha^{\Phi}_{\Lambda,t}(A) \qquad |t| < \frac{r}{2\|\Phi\|_r}.$$

We need $\alpha^{\Phi}_{i\beta}$, with β arbitrary large.

Infinite-volume of evolution operator

Theorem 3.17 (Infinite-volume limit of the evolution operator).
Let $\Phi \in \mathcal{I}_r$ for some $r > 0$. There exists a family of $*$-automorphisms α_t^Φ on $\mathcal{A}$ such that

(a) $\lim_{n\to\infty} \alpha^\Phi_{\Lambda_n,t}(A) = \alpha^\Phi_t(A)$ *along any sequence* $\Lambda_n \uparrow \mathbb{Z}^d$, *for any* $t \in \mathbb{R}$, *and for any* $A \in \mathcal{A}_{\mathrm{loc}}$.

(b) $\|\alpha^\Phi_t\| = 1$ *for any* $t \in \mathbb{R}$.

(c) *It satisfies the group property*

$$\alpha^\Phi_{s+t}(A) = \alpha^\Phi_s\left(\alpha^\Phi_t(A)\right) \qquad \text{for all } A \in \mathcal{A},\ s,t \in \mathbb{R}.$$

Proof: Extend $\alpha^\Phi_{\Lambda,t}$ to $\mathcal{A}$:

$$A = \underset{\mathcal{A}_\Lambda}{\underset{\cap}{B}} \otimes \underset{\mathcal{A}_{\Lambda^c}}{\underset{\cap}{C}} : \qquad \boxed{\alpha^\Phi_{\Lambda,t}(A) = \mathrm{e}^{\mathrm{i}tH^\Phi_\Lambda}\, B\, \mathrm{e}^{-\mathrm{i}tH^\Phi_\Lambda} \otimes C.}$$

Note that $\|\alpha^\Phi_{\Lambda,t}\| = 1$, so the definition extends to $\mathcal{A}$.

For small t: $\quad \alpha^\Phi_t(A) \doteq \lim_{\Lambda\uparrow\mathbb{Z}^d} \alpha^\Phi_{\Lambda,t}(A), \quad A \in \mathcal{A}_{\mathrm{loc}}$

Let $t \in \mathbb{R}$ and $t_1, \dots, t_k \in \mathbb{R}$ such that

$$\sum_{i=1}^{k} t_i = t, \qquad |t_i| < \frac{r}{2\|\Phi\|_r}, \quad i=1,\dots,k.$$

We have

$$\alpha^{\Phi}_{\Lambda,t_k} \circ \dots \alpha^{\Phi}_{\Lambda,t_1}(A) - \alpha^{\Phi}_{t_k} \circ \dots \alpha^{\Phi}_{t_1}(A) = \sum_{j=1}^{k} \alpha^{\Phi}_{\Lambda,t_k} \circ \dots (\alpha^{\Phi}_{\Lambda,t_j} - \alpha^{\Phi}_{t_j}) \circ \dots \alpha^{\Phi}_{t_1}(A).$$

Then

$$\Big\|\underbrace{\alpha^{\Phi}_{\Lambda,t_k} \circ \dots \alpha^{\Phi}_{\Lambda,t_1}(A)}_{= \alpha^{\Phi}_{\Lambda,t}(A)} - \alpha^{\Phi}_{t_k} \circ \dots \alpha^{\Phi}_{t_1}(A)\Big\| \leq \sum_{j=1}^{k} \big\|\alpha^{\Phi}_{\Lambda,t_j} - \alpha^{\Phi}_{t_j}\big\| \, \|A\|.$$

Then

- $\alpha^{\Phi}_{\Lambda,t} = \alpha^{\Phi}_{\Lambda,t_k} \circ \dots \circ \alpha^{\Phi}_{\Lambda,t_1}$ converges as $\Lambda \uparrow \mathbb{Z}^d$.
- Group property: $\alpha^{\Phi}_{s+t} = \alpha^{\Phi}_{s} \circ \alpha^{\Phi}_{t}$.

$\square$

The space $\tilde{\mathfrak{A}}$ of quasi-local operators

We have α_t^ϕ for $t \in \mathbb{R}$. We need $\alpha_{i\beta}^\phi$, $\beta \in \mathbb{R}$.

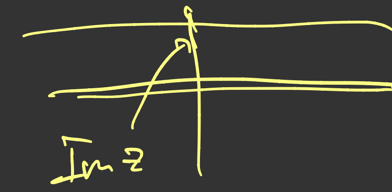
Given $f \in C_c^\infty(\mathbb{R})$, define the complex function

$$\hat{f}(z) = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} e^{i\xi z} f(\xi)\, d\xi \,, \qquad z \in \mathbb{C}.$$

If $A \in \mathfrak{A}$, let

$$A_f = \int_{\mathbb{R}} \hat{f}(t)\, \alpha_t^\phi(A)\, dt$$

A_f is a "Paley-Wiener" operator. Then $A_f \in \tilde{\mathfrak{A}}$.



Note that $\tilde{\mathfrak{A}}$ is dense in $\mathfrak{A}$.

For $A_f \in \tilde{\mathfrak{A}}$ we define

$$\simeq \int_{\mathbb{R}+z} \hat{f}(t')\, \underbrace{\alpha^{\Phi}_{t'+z}}_{\alpha_z(\alpha_{t'}(A))}(A)\, dt'$$

$$\alpha_z^{\Phi}(A_f) = \int_{\mathbb{R}} \hat{f}(t-z)\, \alpha_t^{\Phi}(A)\, dt$$

Note: $\hat{f}(t-z) = \hat{f}_z(t)$ where $f_z(t) = e^{-izt} f(t) \in C_c^\infty$.

Then $$\alpha_z^{\Phi}(A_f) = A_{f_z}$$

and α_z^{Φ} is a linear map $\tilde{\mathfrak{A}} \to \tilde{\mathfrak{A}}$.

Proposition 3.18. *With the definition (3.56), α_z^{Φ} maps $\tilde{A}$ to $\tilde{A}$ and satisfies the group property $\alpha_z^{\Phi} \circ \alpha_w^{\Phi} = \alpha_{z+w}^{\Phi}$ for all $z, w \in \mathbb{C}$.*

The KMS condition in infinite volumes

Definition 3.19. *We say that the state $\langle\cdot\rangle$ on $\mathcal{A}$ satisfies* **the KMS condition** *for the interaction $\Phi \in \mathcal{I}_r$ at inverse temperature β if for all $A \in \tilde{\mathcal{A}}$ and all $B \in \mathcal{A}$, we have*

$$\langle AB\rangle = \langle B\alpha^{\Phi}_{\mathrm{i}\beta}(A)\rangle. \tag{3.55}$$

Equivalently, we have for all $A, B \in \mathcal{A}$, and all functions f such that $\widehat{f} \in C^{\infty}_{\mathrm{c}}$, that

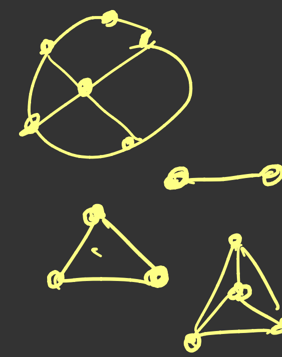
$$\int_{\mathbb{R}} f(t)\langle\alpha^{\Phi}_{t}(A)B\rangle\mathrm{d}t = \int_{\mathbb{R}} f(t-\mathrm{i}\beta)\langle B\,\alpha^{\Phi}_{t}(A)\rangle\mathrm{d}t. \tag{3.56}$$

Let $\mathcal{G}^{\Phi}_{\beta}$: set of KMS states for the interaction Φ.

Proposition 3.20. *Let $\langle\cdot\rangle$ be a translation-invariant state on $\mathcal{A}$. Then $\langle\cdot\rangle$ satisfies the KMS-condition if and only if it satisfies the tangent- and variational definitions.*

Next: beautiful theory, that holds (and can be proved) very generally!

Extremal Gibbs states



Context: $\mathcal{G}_\beta^\Phi$: set of Gibbs states for Φ

↖ KMS or tangent

Clear: $\mathcal{G}_\beta^\Phi$ is a convex set.

4 properties: extremal; short-range correlations; mixing; ergodic.

Definition 3.25. *An element x of a convex set $\mathcal{X}$ is called* **extremal** *if it cannot be written as a convex combination $tx_1 + (1-t)x_2$, $t \in (0,1)$, of distinct elements $x_1, x_2 \in \mathcal{X}$. In particular, this definition applies to elements of $\mathfrak{E}$, $\mathcal{G}_\beta^\Phi$ and $\mathcal{G}_{\mathrm{t.i.}}^\Phi$.*

Definition 3.26. *A state $\langle\cdot\rangle$ is said to have* **short-range correlations** *if*

$$\lim_{\Lambda \Uparrow \mathbb{Z}^d} \sup_{\substack{B \in \mathcal{A}_{\Lambda^c} \\ \|B\|=1}} \big|\langle AB\rangle - \langle A\rangle\langle B\rangle\big| = 0.$$

It is said to be **mixing** *if for all $A, B \in \mathcal{A}$ we have*

$$\lim_{\|x\|\to\infty} \big(\langle A\tau_x B\rangle - \langle A\rangle\langle \tau_x B\rangle\big) = 0.$$

$$m_n(A) = \frac{1}{n^d} \sum_{x \in \{1,\dots,n\}^d} \tau_x A$$

translation

DEFINITION 3.27. *A translation-invariant state* $\langle\cdot\rangle$ *is* **ergodic** *if*

$$\limsup_{n\to\infty} \left\langle \left(m_n(A) - \langle A\rangle\right)^2 \right\rangle = 0$$

for all $A \in \mathcal{A}$.

THEOREM 3.28. *Let* $\langle\cdot\rangle \in \mathcal{G}^{\Phi}_{\mathrm{t.i.}}$. *The following properties are equivalent.*

(a) $\langle\cdot\rangle$ *is extremal in* $\mathcal{G}^{\Phi}$.
(b) $\langle\cdot\rangle$ *is extremal in* $\mathfrak{E}$.
(c) $\langle\cdot\rangle$ *has short-range correlations.*
(d) $\langle\cdot\rangle$ *is mixing.*
(e) $\langle\cdot\rangle$ *is ergodic.*

Theorem 3.28. *Let $\langle\cdot\rangle \in \mathcal{G}^{\Phi}_{\text{t.i.}}$. The following properties are equivalent.*

(a) *$\langle\cdot\rangle$ is extremal in $\mathcal{G}^{\Phi}$.*
(b) *$\langle\cdot\rangle$ is extremal in $\mathfrak{E}$.*
(c) *$\langle\cdot\rangle$ has short-range correlations.*
(d) *$\langle\cdot\rangle$ is mixing.*
(e) *$\langle\cdot\rangle$ is ergodic.*

Proof: (b) $\Rightarrow$ (a) $\Rightarrow$ (c) $\Rightarrow$ (d) $\Rightarrow$ (e) $\Rightarrow$ (b).

(b) $\Rightarrow$ (a) : clear.

(a) $\Rightarrow$ (c) : We prove contrapositive :

no short-range correlations $\Rightarrow$ not extremal.

$\exists\, A \in \mathcal{A}_{loc}$, $B_n \in \mathcal{A}_{\Lambda_n^c}$ where $\Lambda_n \uparrow \mathbb{Z}^d$, $\|B_n\| = 1$

s.t. $$\liminf_{n\to\infty} |\langle A B_n\rangle - \langle A\rangle\langle B_n\rangle| > 0$$

Take A, B_n hermitian, in $\mathcal{A}_{loc}$

Also: $\frac{1}{4}\mathbb{1} \le B_n \le \frac{3}{4}\mathbb{1}$.

$$\langle \cdot \rangle^{(1)} = \lim_n \frac{\langle \cdot \, B_n \rangle}{\langle B_n \rangle} = \lim_n \frac{\langle \cdot \, B_n \rangle}{\theta}$$

$$\langle \cdot \rangle^{(2)} = \lim_n \frac{\langle \cdot \, (1-B_n) \rangle}{1-\theta}$$

$$\langle \cdot \rangle = \theta \langle \cdot \rangle^{(1)} + (1-\theta) \langle \cdot \rangle^{(2)}$$

$$\langle A \rangle^{(1)} = \lim_{n\to\infty} \frac{\langle A B_n \rangle}{\langle B_n \rangle} \neq \langle A \rangle$$

THEOREM 3.28. *Let* $\langle\cdot\rangle \in \mathcal{G}^{\Phi}_{\text{t.i.}}$. *The following properties are equivalent.*

(a) $\langle\cdot\rangle$ *is extremal in* $\mathcal{G}^{\Phi}$.

(b) $\langle\cdot\rangle$ *is extremal in* $\mathfrak{E}$.

(c) $\langle\cdot\rangle$ *has short-range correlations.*

(d) $\langle\cdot\rangle$ *is mixing.*

(e) $\langle\cdot\rangle$ *is ergodic.*

(c) $\Rightarrow$ (d): Obvious from the definition:

DEFINITION 3.26. *A state* $\langle\cdot\rangle$ *is said to have* **short-range correlations** *if*

$$\lim_{\Lambda \Uparrow \mathbb{Z}^d} \sup_{\substack{B \in \mathcal{A}_{\Lambda^c} \\ \|B\|=1}} \left|\langle AB\rangle - \langle A\rangle\langle B\rangle\right| = 0.$$

It is said to be **mixing** *if for all* $A, B \in \mathcal{A}$ *we have*

$$\lim_{\|x\|\to\infty} \big(\langle A\tau_x B\rangle - \langle A\rangle\langle\tau_x B\rangle\big) = 0.$$

Theorem 3.28. *Let $\langle\cdot\rangle \in \mathcal{G}^{\Phi}_{\text{t.i.}}$. The following properties are equivalent.*

(a) $\langle\cdot\rangle$ is extremal in $\mathcal{G}^{\Phi}$.
(b) $\langle\cdot\rangle$ is extremal in $\mathfrak{E}$.
(c) $\langle\cdot\rangle$ has short-range correlations.
(d) $\langle\cdot\rangle$ is mixing.
(e) $\langle\cdot\rangle$ is ergodic.

(d) $\Rightarrow$ (e): $\forall A \in \mathfrak{A}$: $\forall \varepsilon > 0$, $\exists R$ s.t.

$$\left| \langle A \tau_x A \rangle - \langle A \rangle^2 \right| < \varepsilon \quad \forall |x| > R$$

Then $$\left| \langle m_n(A)^2 \rangle - \langle A \rangle^2 \right| = \frac{1}{n^{2d}} \left| \sum_{x,y \in \{1,\dots n\}^d} \left(\langle \tau_x A \tau_y A \rangle - \langle A \rangle^2 \right) \right|$$

$$= \Bigg| \underbrace{\frac{1}{n^{2d}} \sum_{\substack{x,y \in \{\ \}^d \\ |x-y| \le R}} \left(\langle \tau_x A \tau_y B \rangle - \langle A \rangle^2 \right)}_{\le\, n^d (2R)^d \, 2 \|A\|^2} + \underbrace{\frac{1}{n^{2d}} \sum_{\substack{x,y \\ |x-y| > R}} \left(\dots \right)}_{<\, \varepsilon} \Bigg|$$

$$\overset{n \to \infty}{\le} \varepsilon .$$

THEOREM 3.28. *Let $\langle\cdot\rangle \in \mathcal{G}^{\Phi}_{\text{t.i.}}$. The following properties are equivalent.*

(a) $\langle\cdot\rangle$ is extremal in $\mathcal{G}^{\Phi}$.
(b) $\langle\cdot\rangle$ is extremal in $\mathfrak{E}$.
(c) $\langle\cdot\rangle$ has short-range correlations.
(d) $\langle\cdot\rangle$ is mixing.
(e) $\langle\cdot\rangle$ is ergodic.

(e) $\Rightarrow$ (b): We prove the contrapositive:
not extremal $\Rightarrow$ not ergodic.

Not extremal: $\langle\cdot\rangle = \frac{1}{2}\langle\cdot\rangle^{(1)} + \frac{1}{2}\langle\cdot\rangle^{(2)}$, $\quad \langle A\rangle^{(1)} \neq \langle A\rangle^{(2)}$

Use: $(s+t)^2 < 2s^2 + 2t^2$ if $s \neq t$.

$$\langle A\rangle^2 = \left(\tfrac{1}{2}\langle A\rangle^{(1)} + \tfrac{1}{2}\langle A\rangle^{(2)}\right)^2 < \tfrac{1}{2}\left(\langle A\rangle^{(1)}\right)^2 + \tfrac{1}{2}\left(\langle A\rangle^{(2)}\right)^2$$

$$\leq \tfrac{1}{2}\langle m_n(A)^2\rangle^{(1)} + \tfrac{1}{2}\langle m_n(A)^2\rangle^{(2)}$$

$$= \langle m_n(A)^2\rangle$$

$\square$

Ising: $\langle\cdot\rangle = \theta \langle\cdot\rangle^+ + (1-\theta)\langle\cdot\rangle^-$

Decomposition of states

THEOREM 3.29. *Let $\Phi \in \mathcal{I}$ and $\langle\cdot\rangle \in \mathcal{G}^{\Phi}_{\text{tr.inv}}$. There exists a measure μ on $\mathcal{G}^{\Phi}$, that is concentrated on the extremal states of $\mathcal{G}^{\Phi}$, such that for all $A \in \mathcal{A}$, we have*

$$\langle A\rangle = \int_{\mathcal{G}^{\Phi}} \gamma(A)\,\mu(\mathrm{d}\gamma).$$

If the measure is concentrated on $\mathcal{G}^{\Phi}_{\text{tr.inv.}}$, it is unique.

This is based on Choquet's theorem:

PROPOSITION 3.30 (Choquet). *Let K be a metrisable compact convex set and $\kappa \in K$. Then there exists a probability measure μ on K such that*

(a) μ is concentrated on the extremal points of K.

(b) For any affine function $f : K \to \mathbb{R}$ we have $f(\kappa) = \int_K f(\eta)\mu(\mathrm{d}\eta)$.

Here: $K = \mathcal{G}^{\Phi}$. The affine functions are given by $A \in \mathcal{A}$, namely: $f(\gamma) = \gamma(A)$.

Proof of Thm 3.29 (Decomposition of states):

Existence of the measure follows from Choquet's thm.

We still need to prove that the measure is unique (in the translation-invariant case).

Let $f_{A_1} \ldots f_{A_k}$ be fcts as above.

We check that $\int f_{A_1}(\gamma) \cdots f_{A_k}(\gamma)\, d\mu(\gamma)$ depends on $\langle \cdot \rangle$ only.

$$= \int \gamma(A_1) \ldots \gamma(A_k)\, d\mu(\gamma)$$

$$= \lim_{n\to\infty} \int \gamma(m_n(A_1)) \ldots \gamma(m_n(A_k))\, d\mu(\gamma)$$

$$= \lim_{n\to\infty} \int \gamma(m_n(A_1) \ldots m_n(A_k))\, d\mu(\gamma)$$

$$= \lim_{n\to\infty} \langle m_n(A_1) \ldots m_n(A_k) \rangle \qquad \square$$