Infinite-volume setting: observables, interactions, states

Context: For $\Lambda \Subset \mathbb{Z}^d$, we have

- Hilbert space $\mathcal{H}_\Lambda = \bigotimes_{x \in \Lambda} \mathbb{C}^{2J+1} \simeq \ell^2(\Omega_\Lambda)$

- Hamiltonian: $H_\Lambda = \sum_{X \subset \Lambda} \Phi_X$

- Gibbs state: $\langle A \rangle = \frac{1}{Z} \operatorname{Tr} A e^{-\beta H_\Lambda}$

As $\Lambda \nearrow \mathbb{Z}^d$: $\mathcal{H}_{\mathbb{Z}^d}$ not useful (not separable!)

$H_{\mathbb{Z}^d}$ not defined.

Classical spin systems: $\Omega_{\mathbb{Z}^d}$, DLR measures.

Does not work here.

Local observables

Given $X \Subset \mathbb{Z}^d$, let $\mathcal{H}_X = \bigotimes_{x \in X} \mathbb{C}^{2J+1}$

$$\mathcal{A}_X = \mathcal{B}(\mathcal{H}_X)$$

If $X \subset Y$: injection $\iota : \mathcal{A}_X \longrightarrow \mathcal{A}_Y$

$$\iota A = A \otimes \mathbb{1}_{Y \setminus X}$$

Exercise: Show that $\| \iota A \| = \| A \|$.

Algebra of local observables: inductive union $\mathcal{A}_{loc} = \bigvee_{X \Subset \mathbb{Z}^d} \mathcal{A}_X$.

Algebra of quasi-local observables: $\mathcal{A} = \overline{\mathcal{A}_{loc}}^{\,norm}$, the norm completion.

Then if $A \in \mathcal{A}$, $\exists$ Cauchy sequence $A_n \in \mathcal{A}_{X_n}$ s.t. $\| A - A_n \| \to 0$ as $n \to \infty$.

Remark: $\mathcal{A}$ has the structure of a Banach C^* algebra.

Translations: $\tau_y : \mathcal{H}_X \longrightarrow \mathcal{H}_{X+y}$ $\quad y \in \mathbb{Z}^d$

$$\tau_y \bigotimes_{x \in X} \varphi_x = \bigotimes_{x \in X} \varphi_{x+y}$$

This extends to operators:

$$\tilde{\tau}_y : \mathcal{B}(\mathcal{H}_X) \longrightarrow \mathcal{B}(\mathcal{H}_{X+y})$$

$\tilde{\tau}_y A$ is the operator such that

$$(\tilde{\tau}_y A)\, \underset{\mathcal{H}_{X+y}}{\underset{\cap}{\varphi}} = \tau_y \left(A\, \tau_y^{-1} \varphi \right)$$

The translation is a $*$-automorphism:

$$\tau_y(AB) = \tau_y A\ \tau_y B \quad \text{and} \quad (\tau_y A)^* = \tau_y A^*.$$

Interactions

An interaction Φ is collection $(\Phi_X)_{X \in \mathbb{Z}^d}$ where $\Phi_X \in \mathcal{A}_X$.

An interaction is translation-invariant if $\tau_y \Phi_X = \Phi_{X+y}$

All our interactions are translation-invariant.

Two norms of interactions:

$$|||\Phi||| = \sum_{X \ni 0} \frac{\|\Phi_X\|}{|X|} \qquad (\text{then } \Phi \in \mathcal{I})$$

$$\|\Phi\|_r = \sum_{X \ni 0} \|\Phi_X\| e^{r|X|}, \quad r \geq 0 \qquad (\text{then } \Phi \in \mathcal{I}_r)$$

Hamiltonians & finite-volume Gibbs states

Given $\Phi \in \mathcal{I}$ and $\Lambda \Subset \mathbb{Z}^d$, let

$$H_\Lambda^\Phi = \sum_{X \subset \Lambda} \Phi_X$$

$$\langle A \rangle_{\Lambda,\beta}^{\Phi} = \frac{1}{Z_{\Lambda,\beta}(\Phi)} \operatorname{Tr}_{\mathcal{H}_\Lambda} A \, e^{-\beta H_\Lambda^\Phi}$$

Infinite-volume states

These are states on $\mathfrak{A}$ (the algebra of quasilocal obs.)

Definition 3.4. *A* **state** $\langle\cdot\rangle$ *is a normalised, positive linear functional on* $\mathcal{A}$. *That is,* $\langle\cdot\rangle$ *satisfies*

(i) $\langle sA + tB\rangle = s\langle A\rangle + t\langle B\rangle$ *for all* $A, B \in \mathcal{A}$ *and* $s, t \in \mathbb{C}$.

(ii) $\langle \mathbb{1}\rangle = 1$.

(iii) $\langle A^* A\rangle \geq 0$ *for all* $A \in \mathcal{A}$.

We write $\mathfrak{E}$ *for the set of states. A state is called* **translation-invariant** *if* $\langle A\rangle = \langle \tau_x A\rangle$ *for all* $x \in \mathbb{Z}^d$.

Exercise: States have norm 1.

Gibbs state:

Definition 3.5 (State as cluster point). *Let* $\Phi \in \mathcal{I}$, *and let* Ψ_n *be a sequence of interactions in* $\mathcal{I}$ *such that* $\|\Psi_n\| \to 0$ *as* $n \to \infty$. *Let* $\Lambda_n = \{-n, \dots, n\}^d$ *and for* $A \in \mathcal{A}_{\mathrm{loc}}$, *let*

$$\langle A\rangle^{\Phi}_{\Lambda_n,\beta} = \frac{1}{Z_{\Lambda_n,\beta}(\Phi + \Psi_n)} \mathrm{Tr}\, A\, \mathrm{e}^{-\beta H^{\Phi+\Psi_n}_{\Lambda_n}} .$$

The cluster points of the sequence $(\langle\cdot\rangle^{\Phi}_{\Lambda_n,\beta})_{n\geq 1}$ *are infinite-volume Gibbs states for the interaction* Φ.

Examples

Ising model:

$$\Phi_X = \begin{cases} -S_x^{(3)} S_y^{(3)} & \text{if } X = \{x,y\} \in \mathcal{E}_{\mathbb{Z}^d} \\ -h\, S_x^{(3)} & \text{if } X = \{x\} \\ 0 & \text{oth.} \end{cases}$$

Then H_Λ^Φ is the Ising hamiltonian with free b.c.

Heisenberg model:

$$\Phi_X = \begin{cases} -\vec{S}_x \cdot \vec{S}_y & \text{if } X = \{x,y\} \in \mathcal{E}_{\mathbb{Z}^d} \\ 0 & \text{oth.} \end{cases}$$

Main goals & relevant questions

A **model** is given by an interaction $\Phi = (\Phi_X)$ or a family of interactions (Φ_X^h).

- For given $\beta > 0$, how many infinite-volume Gibbs states?
- Properties of the Gibbs states, and of the correlation functions?

Partial answers:

- A common general theory (extremal states, ...).
- More detailed results for specific models.

First step: Various characterisations of ∞-vol. Gibbs states.

Recall: finite fixed domain:

Gibbs state: $\langle A \rangle_{H,\beta} = \frac{1}{Z_\beta} \operatorname{Tr} A\, e^{-\beta H}$.

Proposition 3.3. *The Gibbs state $\langle \cdot \rangle_{H,\beta}$ is the unique state $\langle \cdot \rangle = \operatorname{Tr} \cdot \rho$ satisfying any of the following four conditions:*

(a) Tangent condition: *Let $F_\beta(H) := -\frac{1}{\beta} \log \operatorname{Tr} \mathrm{e}^{-\beta H}$. Then*

$$F_\beta(H + A) \le F_\beta(H) + \langle A \rangle$$

for all $A = A^* \in \mathcal{B}(\mathcal{H})$.

(b) Gibbs variational principle: *The density matrix ρ minimizes the function*

$$\mathcal{F}_\beta(\rho) := \operatorname{Tr} H\rho + \tfrac{1}{\beta} \operatorname{Tr} \rho \log \rho.$$

(c) KMS condition: *Define the time evolution*

$$\alpha_t(A) = \mathrm{e}^{\mathrm{i}tH} A\, \mathrm{e}^{-\mathrm{i}tH}, \qquad t \in \mathbb{C}. \tag{3.11}$$

Then

$$\langle AB \rangle = \langle B\, \alpha_{\mathrm{i}\beta}(A) \rangle$$

for all $A, B \in \mathcal{B}(\mathcal{H})$.

(d) RAS condition: *For all $A \in \mathcal{B}(\mathcal{H})$ such that $\langle AA^* \rangle > 0$,*

$$\langle A^*[H, A] \rangle \ge \frac{1}{\beta} \langle A^* A \rangle \log \frac{\langle A^* A \rangle}{\langle AA^* \rangle} \tag{3.12}$$

All characterisations can be generalised to infinite volumes!

Here: (a) and (c).

Finite-volume free energy

$\mathrm{tr}_Y(A \otimes \mathbb{1}_{Y \setminus X})$
$= \mathrm{tr}_X A$

Free energy as function of the interactions:

$$f_\Lambda(\Phi,\beta) = -\frac{1}{\beta|\Lambda|}\log \mathrm{tr}\ \mathrm{e}^{-\beta H_\Lambda^\Phi}.$$

$\frac{1}{\dim \mathcal{H}_\Lambda} \mathrm{Tr}_{\mathcal{H}_\Lambda}$

Proposition 3.7.

(a) *The free energy f_Λ is a concave function of the interactions.*

(b) *$(f_\Lambda)_{\Lambda\in\mathbb{Z}^d}$ are equicontinuous: for any $\Phi,\Phi'\in\mathcal{I}$ we have*

$$\left|f_\Lambda(\Phi,\beta) - f_\Lambda(\Phi',\beta)\right| \le \|\Phi-\Phi'\|.$$

Proof: (a) $f_\Lambda(a\Phi + b\Psi, \beta) = -\frac{1}{\beta|\Lambda|}\log \mathrm{Tr}\, e^{-\beta(aH_\Lambda^\Phi + bH_\Lambda^\Psi)}$

$$\mathrm{Tr}\, e^{-a\beta H_\Lambda^\Phi - b\beta H_\Lambda^\Psi} \underset{\text{Golden-Thompson}}{\le} \mathrm{Tr}\, e^{-a\beta H_\Lambda^\Phi} e^{-b\beta H_\Lambda^\Psi} = \| e^{-a\beta H_\Lambda^\Phi} e^{-b\beta H_\Lambda^\Psi}\|_1$$

$$\underset{\text{Hölder}}{\le} \|e^{-a\beta H_\Lambda^\Phi}\|_{\frac{1}{a}} \|e^{-b\beta H_\Lambda^\Psi}\|_{\frac{1}{b}} = \left(\mathrm{Tr}\, e^{-\beta H_\Lambda^\Phi}\right)^a \left(\mathrm{Tr}\, e^{-\beta H_\Lambda^\Psi}\right)^b$$

$$f_\Lambda(a\phi + b\psi, \beta) \geq -\frac{1}{\beta|\Lambda|} a \log \operatorname{Tr} e^{-\beta H_\Lambda^\phi} - \frac{1}{\beta|\Lambda|} b \log \ldots$$

$$= a f_\Lambda(\phi, \beta) + b f_\Lambda(\psi, \beta)$$

(b) We use $\operatorname{Tr} e^{A - \|B\|} \leq \operatorname{Tr} e^{A+B} \leq \operatorname{Tr} e^{A + \|B\|}$

$$A = -\beta H_\Lambda^\phi, \quad B = \beta H_\Lambda^\phi - \beta H_\Lambda^{\phi'}$$

$$\left| f_\Lambda(\phi, \beta) - f_\Lambda(\phi', \beta) \right| \leq \frac{1}{|\Lambda|} \| H_\Lambda^\phi - H_\Lambda^{\phi'} \|$$

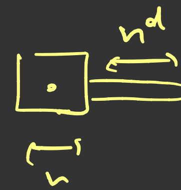
$$\| H_\Lambda^\phi - H_\Lambda^{\phi'} \| \leq \sum_{X \subset \Lambda} \| \phi_X - \phi'_X \| \underbrace{\sum_{y \in \Lambda} \frac{1}{|X|} 1_{y \in X}}_{=1}$$

$$= \sum_{y \in \Lambda} \sum_{X \ni y} \frac{1}{|X|} \| \phi_X - \phi'_X \| \underbrace{1_{X \subset \Lambda}}_{\leq 1}$$

$$= |\Lambda| \, ||| \phi - \phi' ||| .$$

$\square$

Infinite-volume limit of $f_\Lambda(\Phi, \beta)$



DEFINITION 3.8. *A sequence of finite domains* $(\Lambda_n)_{n\geq 1}$ *converges to* $\mathbb{Z}^d$ **in the sense of van Hove** *if*

(i) *it is increasing:* $\Lambda_{n+1} \supset \Lambda_n$ *for all* n*;*

(ii) *it invades* $\mathbb{Z}^d$*:* $\cup_{n\geq 1}\Lambda_n = \mathbb{Z}^d$*;*

(iii) *the ratio boundary/bulk vanishes:* $\frac{|\partial_r \Lambda_n|}{|\Lambda_n|} \to 0$ *as* $n \to \infty$, $\forall r$.

Here, the r*-boundary is* $\partial_r\Lambda = \{x \in \Lambda^c : \mathrm{dist}(x, \Lambda) \leq r\}$.

THEOREM 3.9. *Assume that* $\Phi \in \mathcal{I}$*, i.e.* $\|\Phi\| < \infty$*. Then the limit*

$$f(\Phi, \beta) := \lim_{n\to\infty} f_{\Lambda_n}(\Phi, \beta)$$

exists and is the same along all van Hove sequences $\Lambda_n \Uparrow \mathbb{Z}^d$*. It is a concave function of the interactions.*

Fundamental result in Statistical Mechanics!

This is a gate from the microscopic to the macroscopic World!

For the proof, we use:

Lemma A.12. *Assume that* $(a_\Lambda)_{\Lambda \Subset \mathbb{Z}^d}$ *is a set-indexed sequence of real numbers that satisfies*

- *translation invariance:* $a_{\Lambda+x} = a_\Lambda$ *for all* $\Lambda \Subset \mathbb{Z}^d$ *and* $x \in \mathbb{Z}^d$;
- *bulk property: there exists a constant* c *such that for any* k *and any mutually disjoint* $\Lambda_1, \dots, \Lambda_k \Subset \mathbb{Z}^d$, *we have*

$$\left| a_{\cup_{i=1}^k \Lambda_i} - \sum_{i=1}^k a_{\Lambda_i} \right| \leq c \sum_{i=1}^k |\partial_1 \Lambda_i|.$$

Then there exists $\lambda \in \mathbb{R}$ *such that*

$$\lambda = \lim_{n \to \infty} \frac{a_{\Lambda_n}}{|\Lambda_n|}$$

along any van Hove sequence (Λ_n) *of domains in* $\mathbb{Z}^d$.

Proof of Theorem 3.9:

$$H^\Phi_\Lambda = \sum_{i=1}^k H^\Phi_{\Lambda_i} + B$$

$$\|B\| \leq \sum_{i=1}^k \sum_{\substack{X \cap \Lambda_i \neq \emptyset \\ X \not\subset \Lambda_i}} \|\Phi_X\| \leq \sum_{i=1}^k \|\Phi\|_0 \, |\partial_1 \Lambda_i|$$

$$\left| |\Lambda| f_\Lambda(\Phi,\beta) - \sum_{i=1}^k |\Lambda_i| f_{\Lambda_i}(\Phi,\beta) \right| \leq \|\Phi\|_0 \sum_{i=1}^k |\partial_1 \Lambda_i| .$$

$\square$

Gibbs states as tangent functional

Given $\Psi \in \mathcal{I}$, let $\quad A_\Psi = \sum_{X \ni 0} \frac{1}{|X|} \Psi_X$.

Note: $\|A_\Psi\| \leq |||\Psi|||$.

Definition 3.10 (States as tangent functionals). *A translation-invariant state $\langle\cdot\rangle$ on $\mathcal{A}$ is an equilibrium state for the interaction Φ, in the sense of tangent functionals to the free energy, if*

$$f(\Phi+\Psi) \leq f(\Phi) + \langle A_\Psi \rangle$$

for all $\Psi \in \mathcal{I}$.

Remark: We saw that $\langle \cdot \rangle^{\Phi}_{\Lambda,\beta} = \frac{1}{Z^{\Phi}_{\Lambda,\beta}} \operatorname{Tr} \cdot e^{-\beta H^{\Phi}_{\Lambda}}$ satisfies

$$f_\Lambda(\Phi+\Psi,\beta) \leq f_\Lambda(\Phi,\beta) + \underbrace{\langle \frac{1}{|\Lambda|} H^{\Psi}_{\Lambda} \rangle^{\Phi}_{\Lambda,\beta}}_{\longrightarrow \langle A_\Psi \rangle \text{ as } \Lambda \Uparrow \mathbb{Z}^d}.$$

The "cluster point Gibbs states" are then tangent states!

KMS states

Recall:

Proposition 3.3. *The Gibbs state $\langle\cdot\rangle_{H,\beta}$ is the unique state $\langle\cdot\rangle = \mathrm{Tr}\cdot\rho$ satisfying any of the following four conditions:*

(a) Tangent condition: *Let $F_\beta(H) := -\frac{1}{\beta}\log\mathrm{Tr}\,\mathrm{e}^{-\beta H}$. Then*

$$F_\beta(H+A) \le F_\beta(H) + \langle A\rangle$$

for all $A = A^* \in \mathcal{B}(\mathcal{H})$.

(b) Gibbs variational principle: *The density matrix ρ minimizes the function*

$$\mathcal{F}_\beta(\rho) := \mathrm{Tr}\,H\rho + \tfrac{1}{\beta}\mathrm{Tr}\,\rho\log\rho.$$

(c) KMS condition: *Define the time evolution*

$$\alpha_t(A) = \mathrm{e}^{\mathrm{i}tH}A\,\mathrm{e}^{-\mathrm{i}tH}, \qquad t \in \mathbb{C}. \tag{3.11}$$

Then

$$\langle AB\rangle = \big\langle B\,\alpha_{\mathrm{i}\beta}(A)\big\rangle$$

for all $A, B \in \mathcal{B}(\mathcal{H})$.

(d) RAS condition: *For all $A \in \mathcal{B}(\mathcal{H})$ such that $\langle AA^*\rangle > 0$,*

$$\langle A^*[H,A]\rangle \ge \frac{1}{\beta}\langle A^*A\rangle\log\frac{\langle A^*A\rangle}{\langle AA^*\rangle} \tag{3.12}$$

Given an interaction $\Phi \in \mathcal{I}$, let $\boxed{\alpha^{\Phi}_{\Lambda,t}(A) = \mathrm{e}^{\mathrm{i}tH^\Phi_\Lambda}A\,\mathrm{e}^{-\mathrm{i}tH^\Phi_\Lambda}.}$

(for $A \in \mathcal{A}_\Lambda$.)

We first show the existence of the limit $\Lambda \Uparrow \mathbb{Z}^d$.

Infinite-volume limit of the evolution operator

Proposition 3.15 (Infinite-volume limit of the evolution operator). *Assume that $\Phi \in \mathcal{I}_r$ for some $r > 0$ and that $t \in \mathbb{R}$. There exists a $*$-automorphism $\alpha_t^\Phi : \mathcal{A} \to \mathcal{A}$ such that*

$$\lim_{\Lambda \Uparrow \mathbb{Z}^d} \|\alpha_{\Lambda,t}^\Phi(A) - \alpha_t^\Phi(A)\| = 0 \quad \textit{for all } A \in \mathcal{A}_{\text{loc}}. \tag{3.39}$$

Further $\|\alpha_t^\Phi\| = 1$ and α_t^Φ satisfies the group property

$$\alpha_{s+t}^\Phi(A) = \alpha_s^\Phi\big(\alpha_t^\Phi(A)\big) \qquad \textit{for all } A \in \mathcal{A}, s, t \in \mathbb{R}.$$

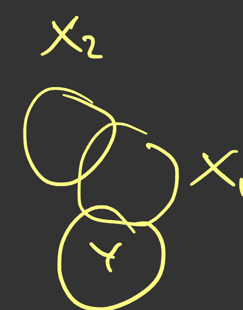
Proof:

(1) For $|t| < \frac{r}{2\|\Phi\|_r}$, we show that $(\alpha_{\Lambda,t}^\Phi)_{\Lambda \Subset \mathbb{Z}^d}$ is Cauchy for each fixed $A \in \mathcal{A}_{\text{loc}}$. We denote the limit $\alpha_t^\Phi(A)$.
(2) For $t \in \mathbb{R}$, we have $\|\alpha_{\Lambda,t}^\Phi(A)\| = \|A\|$ for all Λ, so $\|\alpha_t^\Phi\| = 1$.
(3) We use the group property to extend α_t^Φ to the whole real line.

Let $A \in \mathcal{A}_Y$. We show that $\left(\alpha^{\Phi}_{\Lambda,t}(A) \right)_{\Lambda \in \mathbb{Z}^d}$ is Cauchy.

$$\alpha^{\Phi}_{\Lambda,t}(A) = e^{itH^{\Phi}_{\Lambda}} A \, e^{-itH^{\Phi}_{\Lambda}} = \sum_{n \geq 0} \frac{(it)^n}{n!} \, \mathrm{ad}^n_{H^{\Phi}_{\Lambda}}(A)$$

$$= \sum_{n \geq 0} \frac{(it)^n}{n!} \sum_{X_1, \dots, X_n \subset \Lambda} [\Phi_{X_n}, [\Phi_{X_{n-1}}, \dots [\Phi_{X_1}, A] \dots]$$



For the commutators to be $\neq 0$:

$$X_1 \cap Y \neq \emptyset$$

$$X_2 \cap (X_1 \cup Y) \neq \emptyset$$

$$\vdots$$

$$X_n \cap (X_1 \cup \dots \cup X_{n-1} \cup Y) \neq \emptyset$$

Then

$$\alpha^{\Phi}_{\Lambda,t}(A) = \sum_{n\geq 0} \frac{(it)^n}{n!} \sum_{X_1,\dots,X_n \ni 0} \left(\prod_{i=1}^{n} \frac{1}{|X_i|} \right)$$

$$\sum_{\substack{x_1,\dots x_n \in \mathbb{Z}^d \\ X_i + x_i \subset \Lambda}} [\Phi_{X_n+x_n}, [\Phi_{X_{n-1}+x_{n-1}}, \dots [\Phi_{X_1+x_1}, A]\dots]$$

There are no more than

$|X_1|\,|Y|$ choices for x_1

$|X_2|\,(|X_1|+|Y|)$ choices for x_2

$\vdots$

$|X_n|\,(|X_1|+\dots+|X_{n-1}|+|Y|)$ choices for x_n

We get

$$\Big\| \sum_{X_1 \dots X_n \ni 0} \Big(\prod^{k} \frac{1}{|X_i|} \Big) \sum_{\substack{x_1 \dots x_n \\ X_i + x_i \subset \Lambda}} [\Phi_{X_n + x_n}, \dots [\Phi_{X_1 + x_1}, A] \dots] \Big\|$$

$$\leq \|A\| \, 2^n \sum_{X_1 \dots X_n \ni 0} \underbrace{(|X_1| + \dots + |X_n| + |Y|)^n}_{\leq\, n!\, r^{-n} e^{r(|X_1| + \dots)}} \prod_{i=1}^{k} \|\Phi_{X_i}\|$$

$$\leq \|A\| \, e^{r|Y|} \, n! \left(\frac{2 \|\Phi\|_r}{r} \right)^n$$

$$\sum_{n \geq 0} \frac{(it)^n}{n!} \dots \quad \text{converges when } |t| \leq \frac{r}{2 \|\Phi\|_r},$$

$$\alpha^{\Phi}_{\Lambda', t}(A) - \alpha^{\Phi}_{\Lambda, t}(A) :$$