Introduction to Quantum Lattice Systems

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Information:

- Online lectures Tuesdays 10-12 except 25 Nov.
- Lecture notes
- Assessment on demand

Topics:

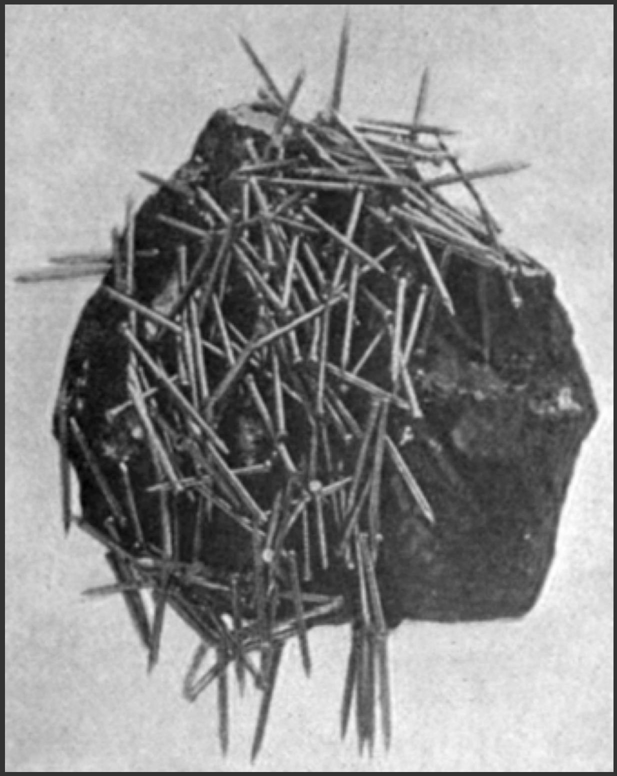
- Quantum spin systems (XY, Heisenberg)
  Phase diagrams
- Fermionic & bosonic systems
- Equilibrium states (general theory)
- Uniqueness / non uniqueness
- Mean-field systems
- 2D systems with continuous symmetry (Mermin-Wagner)

Background:

- Electronic properties of condensed matter
- More specifically: Magnetism
- Physicists' approach: simplified models and description by Statistical Mechanics
- Not an exact description of Nature!
  But important nonetheless.
  In fact, very attractive for mathematicians:
  models have most interesting features and properties, they involve many domains
  (probability theory, analysis, algebra, combinatorics, ...)

- Much interest in recent years, underlined by the Fields medals of Wendelin Werner ('06), Stanislav Smirnov ('10), Martin Hairer ('14), Hugo Duminil-Copin ('22)

Brief history of magnetism 1/2



μαγνῆτις λίθος

First mentioned by Thales of Miletus (625 - 545 B.C.)

Mentioned in a Chinese text, 4th Century B.C.

Indian medical text Sushrata Samhita (before 3rd Century A.D):
magnetite can be used to remove arrows from a body.

Brief history of magnetism 2/2

11th century AD: Shen Kuo writes about a magnetite needle used as compass.

1269: First treatise on magnets, Epistola de magnete by Pierre de Maricourt.

1282: Magnets described by the Yemeni Al-Ashraf Umar.

1600: De Magnete, Magneticisque Corporibus, et de Magno Magnete Tellure by William Gilbert. He understands that the Earth is magnetic. (It was previously suggested that Earth's magnetic field was due to Polaris, or to a magnetic island in the North Pole.)

1819: Hans Christian Ørsted understands that magnetism is related to electricity.

Then many important studies by Ampère, Gauss, Biot, Savart, Faraday, Maxwell, ...

Spin matrices

$[A, B] = AB - BA$

On $\mathbb{C}^n$, we introduce:

S^x, S^y, S^z

$J \in \frac{1}{2}\mathbb{N}, \quad n = 2J+1 \leftrightarrow J = \frac{n-1}{2}$

Definition 1.1. *Let $n \in \{2, 3, 4, \dots\}$.* Spin-matrices *are $n \times n$ hermitian matrices $S^{(1)}, S^{(2)}, S^{(3)}$ that satisfy the following:*

$$[S^{(1)}, S^{(2)}] = \mathrm{i}S^{(3)}, \quad [S^{(2)}, S^{(3)}] = \mathrm{i}S^{(1)}, \quad [S^{(3)}, S^{(1)}] = \mathrm{i}S^{(2)}, \tag{1.5}$$

$$(S^{(1)})^2 + (S^{(2)})^2 + (S^{(3)})^2 = \frac{n^2-1}{4}\mathbb{1}. \tag{1.6}$$

$J(J+1)$

Such matrices exist! (See notes)

Examples: $n=2$: $S^1 = \frac{1}{2}\sigma^1 = \frac{1}{2}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, S^2 = \frac{1}{2}\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, S^3 = \frac{1}{2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$

$[S^1, S^2] = \frac{1}{4}\begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix} - \frac{1}{4}\begin{pmatrix} -i & 0 \\ 0 & i \end{pmatrix} = \frac{1}{2}\begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix} = iS^3$

$n=3$: $S^1 = \frac{1}{\sqrt{2}}\begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}, S^2 = \frac{1}{\sqrt{2}}\begin{pmatrix} 0 & -i & 0 \\ i & 0 & -i \\ 0 & i & 0 \end{pmatrix}, S^3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix}$

Properties of spin matrices

On $\mathbb{C}^n$, $n = 2J+1$, $J \in \frac{1}{2}\mathbb{N}$

Lemma 1.3. *Assume that $S^{(1)}, S^{(2)}, S^{(3)}$ are hermitian $n \times n$ matrices that satisfy the relations (1.5) and (1.6). Then each $S^{(i)}$ has eigenvalues $\{-J, -J+1, \ldots, J\}$.*

Proof: We prove that S^3 has eigenvalues $\{-J, \ldots, J\}$

let $S^+ = S^1 + iS^2$, $S^- = S^1 - iS^2$

$$S^+S^- = J(J+1) - (S^3)^2 + S^3$$
$$S^-S^+ = J(J+1) - (S^3)^2 - S^3$$

let $|a\rangle$ an eigenvector of S^3

$$\| S^+ |a\rangle \|^2 = \langle a | S^- S^+ | a \rangle = J(J+1) - a^2 - a \geq 0$$

$$\| S^- |a\rangle \|^2 = \qquad\qquad + a \geq 0$$

$\Rightarrow |a| \leq J$

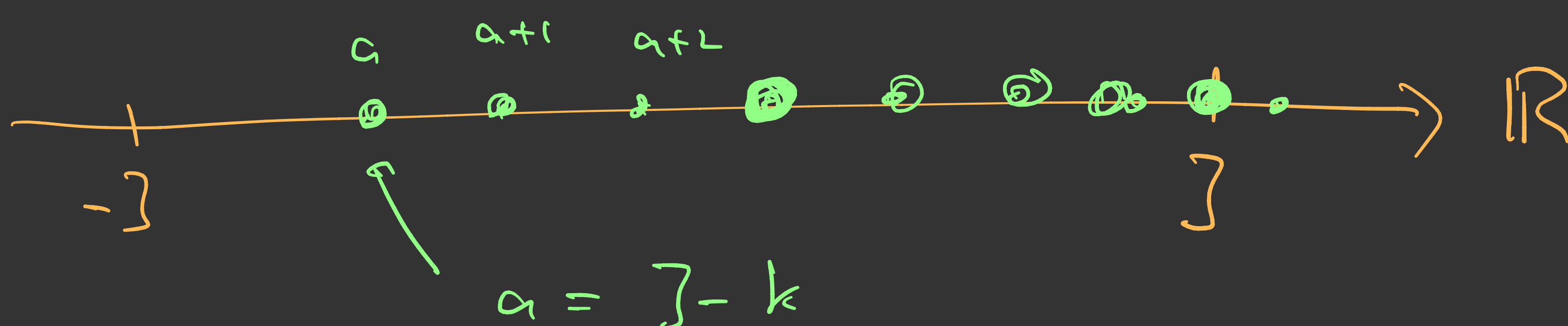
Next: $S^+|a\rangle \neq 0$ if $a \neq j$

$S^3S^+ - S^+S^3$

$[S^3, S^+] = S^+$

Then $S^3 S^+ |a\rangle = (S^+S^3 + S^+)|a\rangle$

$$= (a+1)\, S^+ |a\rangle$$



Thus: $a \in \{-j, \dots, j\}$. And given $a \neq j$, $a+1$ is e.v.

$\square$

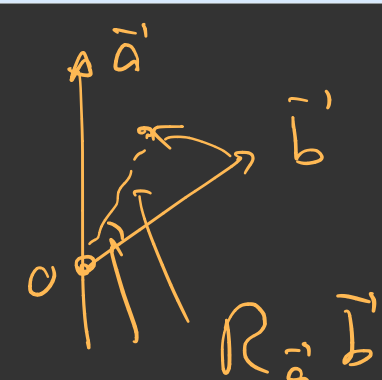
Rotation of spins

Let $\vec a \in \mathbb{R}^3$ and: $S^{\vec a} = \vec a \cdot \vec S = a_1 S^1 + a_2 S^2 + a_3 S^3$

Then $[S^{\vec a}, S^{\vec b}] = i S^{\vec a \times \vec b}$

Lemma 1.4.

$$\underbrace{e^{-iS^{\vec a}}}_{=U} S^{\vec b} \underbrace{e^{iS^{\vec a}}}_{=U^{-1}} = S^{R_{\vec a}\vec b}.$$



angle is $\|\vec a\|$

Proof:

Replace $\vec a$ with $s\vec a$.

$$\frac{d}{ds}\underbrace{e^{-is\vec a\cdot\vec S}\, S^{b}\, e^{is\vec a\cdot\vec S}} = S^{R_{s\vec a}\vec b}$$

$$= e^{-is\vec a\vec S}\left((-is\vec a\vec S)\, S^b + S^b\, (is\vec a\vec S)\right) e^{is\vec a\vec S}$$

$$= -i\left[S^{\vec a}, e^{-isS^{\vec a}} S^b e^{isS^{\vec a}}\right]$$

$$\frac{d}{ds} S^{R_{s\vec a}\vec b} = \left(\frac{d}{ds} R_{s\vec a}\vec b\right)\cdot\vec S = \vec a \times R_{s\vec a}\vec b \cdot \vec S = S^{\vec a\times R_{s\vec a}\vec b} = -i\left[S^{\vec a}, S^{R_{s\vec a}\vec b}\right]$$

$\square$

Spin systems (with several spins)

2 equivalent definitions:
- Tensor product of Hilbert spaces
- Hilbert space spanned by classical spin configurations

Tensor products:

If $\mathcal{H}_1$ and $\mathcal{H}_2$ are two Hilbert spaces, then $\mathcal{H}_1 \otimes \mathcal{H}_2$ consists of vectors of the form $v \otimes w$ with $v \in \mathcal{H}_1$, $w \in \mathcal{H}_2$, and of their linear combinations.

Inner product on $\mathcal{H}_1 \otimes \mathcal{H}_2$: If $v, v' \in \mathcal{H}_1$ and $w, w' \in \mathcal{H}_2$, then

$$\langle v \otimes w | v' \otimes w' \rangle_{\mathcal{H}_1 \otimes \mathcal{H}_2} = \langle v | v' \rangle_{\mathcal{H}_1} \langle w | w' \rangle_{\mathcal{H}_2}$$

This extends by linearity to general vectors in $\mathcal{H}_1 \otimes \mathcal{H}_2$.

Let $\Lambda \Subset \mathbb{Z}^d$.

The Hilbert space of the spin system on Λ is $\mathcal{H}_\Lambda = \bigotimes_{x\in\Lambda} \mathbb{C}^n$.

$$\mathbb{C}^n \otimes \dots \otimes \mathbb{C}^n$$

site x_1 site $x_{|\Lambda|}$

Spin operators: S^i_x where $x \in \Lambda$ and $i = 1, 2, 3$.

$$S^i_x = S^i \otimes \mathbb{1}_{\Lambda \setminus \{x\}}$$

If $v = \bigotimes_{x\in\Lambda} v_x = v_{x_1} \otimes \dots \otimes v_{x_{|\Lambda|}}$:

$$S^i_{x_j} = v_{x_1} \otimes \dots \otimes v_{x_{j-1}} \otimes S^i v_{x_j} \otimes v_{x_{j+1}} \otimes \dots \otimes v_{x_{|\Lambda|}}.$$

$$S^i_{x_j} = \mathbb{1} \otimes \mathbb{1} \otimes \dots \otimes S^i \otimes \dots \otimes \mathbb{1}$$

in $\mathbb{C}^n$

Linear span of classical spin configurations: E.g. $J=\frac{1}{2}$

A classical configuration is $\omega = (\omega_x)_{x\in\Lambda}$, $\omega_x \in \{-J,\dots,J\}$

Let $\Omega_\Lambda = \{-J,\dots,J\}^\Lambda$ and $\mathcal{H}_\Lambda = \operatorname{span}\Omega_\Lambda$.

Dirac notation: If $\omega\in\Omega_\Lambda$, let $|\omega\rangle \in \mathcal{H}_\Lambda$.

Then $\{|\omega\rangle\}_{\omega\in\Omega_\Lambda}$ is a basis of $\mathcal{H}_\Lambda$

$\mathbb{C}^n = \operatorname{span}\{1,\dots,n\}$

Remark: $\dim \mathcal{H}_\Lambda = \cancel{2J|\Lambda|}$. or $n^{|\Lambda|} = (2J+1)^{|\Lambda|}$

$|\Lambda| \simeq 10^{20}$

Spin operators $(n=2)$:

$$S^1_x |\omega\rangle = \frac{1}{2}|\omega^{(x)}\rangle$$

$$S^2_x |\omega\rangle = i\omega_x |\omega^{(x)}\rangle$$

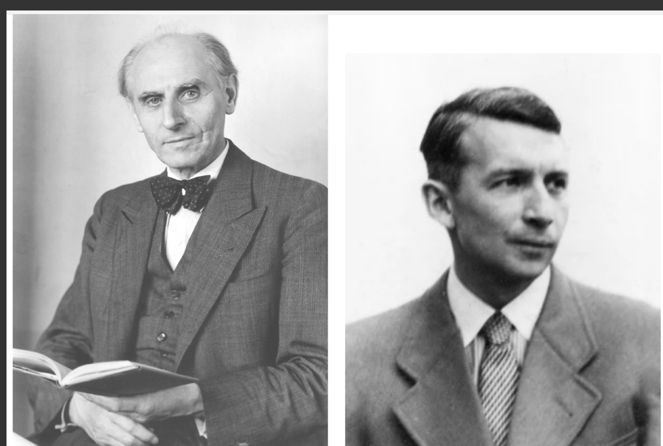
$$S^3_x |\omega\rangle = \omega_x |\omega\rangle$$

XYZ Hamiltonian

$$H_{\Lambda,h} = - \sum_{xy \in \mathcal{E}_\Lambda} \left(J^1 S^1_x S^1_y + J^2 S^2_x S^2_y + J^3 S^3_x S^3_y \right) - h \sum_{x \in \Lambda} S^3_x$$

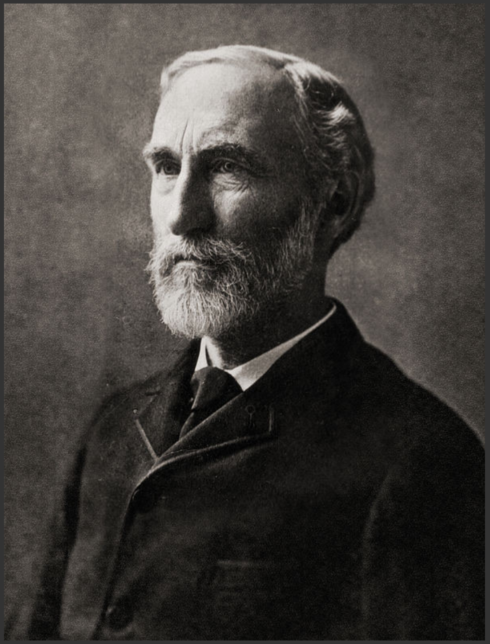
Special cases:

- $n = 2$ and $J^1 = J^2 = 0$: Classical Ising (or Lenz-Ising)



- $J^1 = J^2 = J^3$: Quantum Heisenberg model

Gibbs states & correlation functions



Let $\beta \geq 0$: Inverse temperature

Equilibrium state (Gibbs state):

the map $\mathcal{B}(\mathcal{H}_\Lambda) \longrightarrow \mathbb{C}$ given by

$$\langle A \rangle_{\Lambda,\beta,h} = \frac{1}{Z_{\Lambda,\beta,h}} \operatorname{Tr} A\, e^{-\beta H_{\Lambda,h}}$$

where the partition function is

$$Z_{\Lambda,\beta,h} = \operatorname{Tr}_{\mathcal{H}_\Lambda} e^{-\beta H_{\Lambda,h}} .$$

Total magnetisation operator: $M_\Lambda = \sum_{x \in \Lambda} S_x^3$

Average magnetisation: $m_{\Lambda,\beta,h} = \frac{1}{|\Lambda|} \langle M_\Lambda \rangle_{\Lambda,\beta,h}$

Spontaneous magnetisation: $m_\beta^* = \lim_{h \downarrow 0} \lim_{\Lambda \nearrow \mathbb{Z}^d} m_{\Lambda,\beta,h}$

$$\boxed{\langle S_0^3 S_x^3 \rangle - \langle S_0^3 \rangle \langle S_x^3 \rangle}$$

Two-point correlation: $\langle S_0^3 S_x^3 \rangle_{\Lambda,\beta,h} = \frac{1}{Z_{\Lambda,\beta,h}} \operatorname{Tr} S_0^3 S_x^3 \, e^{-\beta H_{\Lambda,h}}$

DEFINITION 1.6. *The system exhibits* **long-range order** *if there exists a sequence of domains* Λ_n*, where either* $\Lambda_n \Uparrow \mathbb{Z}^d$*, or* $\Lambda_n = \{1, \dots, m_n\}^d_{\text{per}}$ *with* $m_n \to \infty$*, such that*

$$\frac{1}{|\Lambda_n|^2} \sum_{x,y \in \Lambda_n} \langle S_x^{(3)} S_y^{(3)} \rangle_{\Lambda_n,\beta,0} \geq c > 0,$$

for all n.

Phase diagrams

$$H_{\Lambda,h} = -\sum_{xy\in\mathcal{E}_\Lambda}\left(J^1 S^1_x S^1_y + J^2 S^2_x S^2_y + J^3 S_x S_y\right) - h\sum_{x\in\Lambda} S^3_x$$

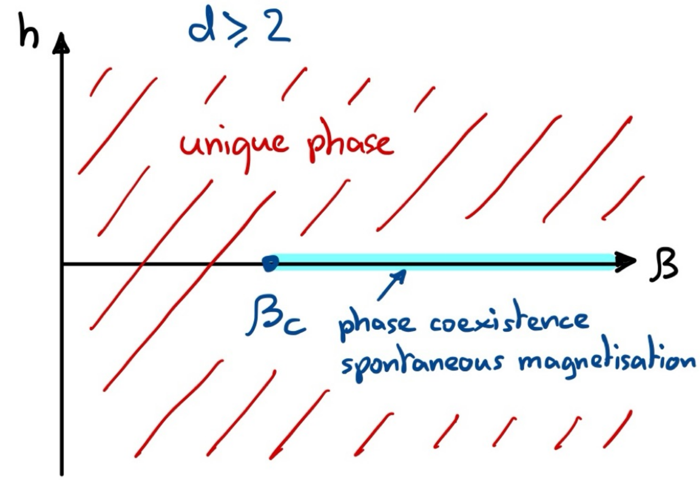


FIGURE 1.1. Phase diagram of the **Ising model**, and of the **XXZ model** for $J^{(3)} > J^{(1)} = J^{(2)} \geq 0$, for all dimensions $d \geq 2$.

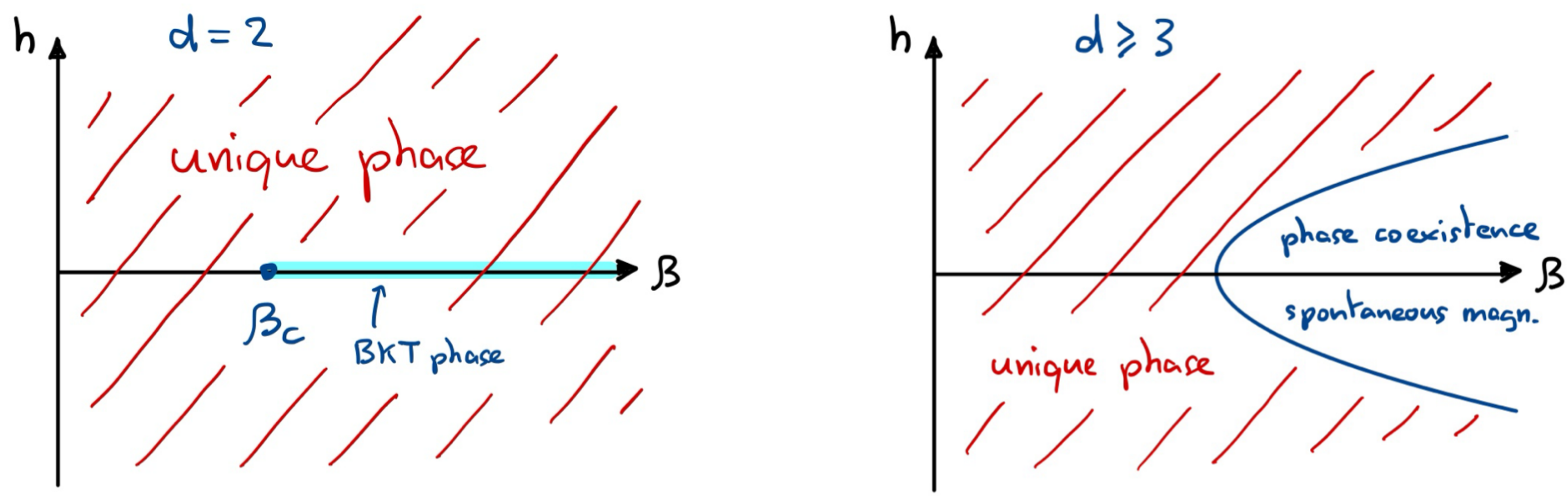


FIGURE 1.2. Phase diagrams of the **XXZ model** for $J^{(1)} = J^{(2)} > |J^{(3)}|$. The BKT phase is the Berezinsky-Kosterlitz-Thouless phase where the two point correlation function has power-law decay (in the unique phase, decay is exponential).

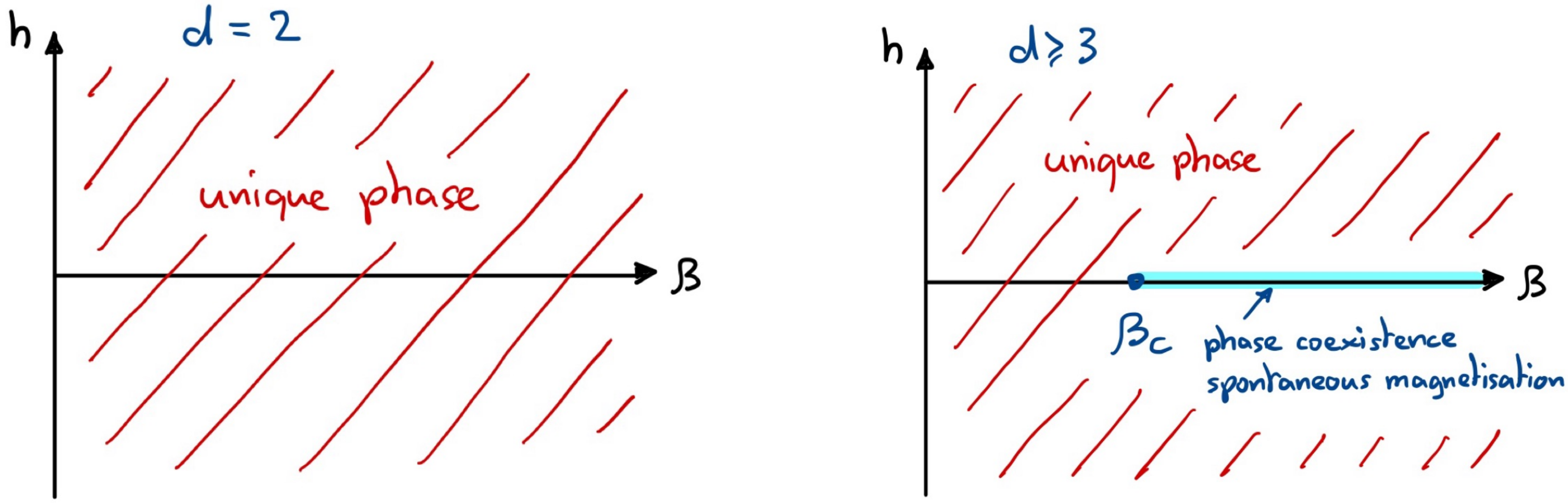


FIGURE 1.3. Phase diagrams of the **Heisenberg** or **XXX model** for which $J^{(1)} = J^{(2)} = J^{(3)} \geq 0$.