CHAPTER 11

Long-range order via reflection positivity

We show that in the case of some quantum models with continuous symmetry, it is possible to prove long-range order, and hence non-differentiability of the infinite-volume pressure. The latter implies that the set of Gibbs states contains more than one element.

11.1. Non-differentiability of the pressure from long-range order

In what follows we need to consider periodic boundary conditions. Given $\ell \in \mathbb{N}$, let $\Lambda_\ell = \{0, 1, \dots, \ell - 1\}^d$, and let the hamiltonian $H^{\rm per}_{\Lambda_\ell,h}$ given by (8.18) but with coupling parameters $J^{(i)}_{xy} = J^{(i)}_{x-y}$ replaced by the following periodised ones:

$$J^{(i)}_{x,\rm per} = \sum_{z \in \mathbb{Z}^d} J^{(i)}_{x+\ell z}. \tag{11.1}$$

We can define the periodised partition function $Z^{\rm per}(\Lambda_\ell, \beta, h)$ accordingly, and the pressure

$$p^{\rm per}_{\Lambda_\ell}(\beta, h) = \frac{1}{\ell^d} \log Z^{\rm per}(\Lambda_\ell, \beta, h). \tag{11.2}$$

As $\ell \to \infty$, these pressures converge by Theorem 3.6. Recall the definition of finite-volume equilibrium states:

$$\langle \cdot \rangle_{\Lambda,\beta,h} = \frac{\operatorname{Tr}\big[\cdot\, \mathrm{e}^{-\beta H_{\Lambda,h}}\,\big]}{Z_{\Lambda,\beta,h}}. \tag{11.3}$$

We also consider the states $\langle \cdot \rangle^{\rm per}_{\Lambda_\ell,\beta,h}$ with periodic boundary conditions, where we use $H^{\rm per}_{\Lambda_\ell,h}$ instead of $H_{\Lambda_\ell,h}$.

Definition 11.1 (Long-range order). *There exists a sequence of domains Λ_n, where either $\Lambda_n \Uparrow \mathbb{Z}^d$, or $\Lambda_n = \{1, \dots, m_n\}^d_{\rm per}$ with $m_n \to \infty$, such that*

$$\frac{1}{|\Lambda_n|^2} \sum_{x,y \in \Lambda_n} \langle S^{(3)}_x S^{(3)}_y \rangle_{\Lambda_n,\beta,0} \geq c > 0,$$

for all n.

In the next section we prove the existence of long-range order in some models. We see now that it implies non-differentiability of the pressure as function of h.

THEOREM 11.2. *We assume that the system displays long-range order in the form of Definition 11.1. Then*

$$\frac{\partial}{\partial h} p(\beta,h)\Big|_{h=0-} \;<\; 0 \;<\; \frac{\partial}{\partial h} p(\beta,h)\Big|_{h=0+}.$$

PROOF. Let us introduce the magnetisation operator

$$M_\Lambda = \sum_{x\in\Lambda} S_x^{(3)}. \tag{11.4}$$

We first give a simplified proof in the case where $[H_{\Lambda,h}, M_\Lambda] = 0$. For the general case, we will use a result of Koma and Tasaki [1993].

Let $|M_\Lambda|$ be the unique positive semi-definite square root of M_Λ^2. We have $|M_\Lambda| \leq |\Lambda| S \,\mathrm{Id}$, so that $M_\Lambda^2 \leq |\Lambda| S |M_\Lambda|$. Since Gibbs states are positive linear functionals, we get

$$\Big\langle \frac{M_\Lambda^2}{|\Lambda|^2} \Big\rangle_{\Lambda,\beta,0} \leq S \Big\langle \frac{|M_\Lambda|}{|\Lambda|} \Big\rangle_{\Lambda,\beta,0}. \tag{11.5}$$

Long-range order implies that $\frac{1}{|\Lambda|^2} \langle M_\Lambda^2 \rangle_{\Lambda,\beta,0} \geq c$, so the right side above is positive.

In order to get an inequality for the derivative of the pressure, let us introduce $\tilde p_\Lambda(\beta,h)$ to be the pressure of the model with hamiltonian

$$\tilde H_{\Lambda,h} = -\sum_{i=1}^3 \sum_{x,y\in\Lambda} J_{x-y}^{(i)} S_x^{(i)} S_y^{(i)} - h|M_\Lambda|. \tag{11.6}$$

We now check that $\tilde p_\Lambda(\beta,h)$ converges as $\Lambda \Uparrow \mathbb{Z}^d$ to the pressure $p(\beta,h)$ for $h \geq 0$. For this, notice that M_Λ (and $|M_\Lambda|$) commute with $H_{\Lambda,0} = \tilde H_{\Lambda,0}$. For $h \geq 0$ we have the inequalities (for the second one, observe that the spectrum of M_Λ is symmetric around 0)

$$\mathrm{Tr}\, \mathrm{e}^{-\beta H_{\Lambda,\beta,0} + \beta h M_\Lambda} \leq \mathrm{Tr}\, \mathrm{e}^{-\beta H_{\Lambda,\beta,0} + \beta h |M_\Lambda|} \leq 2\, \mathrm{Tr}\, \mathrm{e}^{-\beta H_{\Lambda,\beta,0} + \beta h M_\Lambda}. \tag{11.7}$$

Taking the logarithm and dividing by $\Lambda|$, and taking the relevant limits, we get that p and $\tilde p$ are equal.

We now use the convexity in h of $\tilde p_\Lambda$ and the fact that $\inf_n(\limsup_m a_{m,n}) \geq \limsup_m(\inf_n a_{m,n})$ and we get

$$\begin{aligned}
\frac{\partial}{\partial h} p(\beta,h)\Big|_{h=0+} &= \inf_{h>0} \frac{f(\beta,h) - p(\beta,0)}{h} = \inf_{h>0} \limsup_{\Lambda\Uparrow\mathbb{Z}^d} \frac{\tilde p_\Lambda(\beta,h) - \tilde p_\Lambda(\beta,0)}{h} \\
&\geq \limsup_{\Lambda\Uparrow\mathbb{Z}^d} \inf_{h>0} \frac{\tilde p_\Lambda(\beta,h) - \tilde p_\Lambda(\beta,0)}{h} = \limsup_{\Lambda\Uparrow\mathbb{Z}^d} \frac{\partial}{\partial h} \tilde p_\Lambda(\beta,h)\Big|_{h=0} \\
&= \limsup_{\Lambda\Uparrow\mathbb{Z}^d} \beta \Big\langle \frac{|M_\Lambda|}{|\Lambda|} \Big\rangle_{\Lambda,\beta,0}.
\end{aligned} \tag{11.8}$$

The last expectation is with respect to the Gibbs state with Hamiltonian $\tilde{H}_{\Lambda,0} = H_{\Lambda,0}$. The right side and thus $\frac{\partial}{\partial h} p(\beta, h)\big|_{h=0+}$ are indeed positive. Since p is even in h we get the other inequality as well.

When M_Λ does not commute with the Hamiltonian, it does not seem possible to show that p and $\tilde{p}$ are equal. But since right-derivatives of convex functions are right-continuous, we can proceed as above and get

$$\frac{\partial}{\partial h} p(\beta, h)\Big|_{h=0+} = \lim_{h'\to 0+} \frac{\partial}{\partial h} p(\beta, h)\Big|_{h=h'+} \geq \beta \lim_{h'\to 0+} \limsup_{\Lambda \Uparrow \mathbb{Z}^d} \Big\langle \frac{M_\Lambda}{|\Lambda|} \Big\rangle_{\Lambda,\beta,h'}. \tag{11.9}$$

Koma and Tasaki [1993] have proved that long-range order (in the sense of (11.1)) implies that the right side is strictly positive. □

11.2. Long-range order

We state two results about long-range order. The first theorem holds for a larger class of coupling constants and for S large enough. The second theorem is restricted to nearest-neighbour interactions, but it has the advantage of applying to more values of S and more dimensions. To briefly summarise the consequences of those results, we will see that long-range order (in the form of Definition 11.1) holds under the following conditions:

- for certain long-range interactions (specified below) if $\beta \geq \beta_0$ for some $\beta_0 < \infty$ provided $d \geq 3$ and S is large enough, or
- for nearest-neighbour interactions if $\beta \geq \beta_0$ for some $\beta_0 < \infty$ provided $d \geq 3$ and $S \geq \frac{1}{2}$, or
- for nearest-neighbour interactions in the ground-state $\beta = \infty$ provided $d \geq 2$ and either $S \geq 1$, or $S \geq \frac{1}{2}$ and $-J^{(2)}/J^{(1)} \leq 0.13$.

We consider the case of nearest-neighbour interactions, $J_x^{(i)} = 0$ unless $\|x\|_1 = 1$ (in which case it equals some constant $J^{(i)}$); and longer-range interactions that are given by a Fourier transform, $J_x^{(i)} = \int_{\mathbb{R}^d} \mathrm{d}\nu^{(i)}(k)\, \mathrm{e}^{ik\cdot x}$ where $\nu^{(i)}$ is a positive, finite measure on $\mathbb{R}^d$. The latter case allows us to include the following examples:

- $J_x^{(i)} = a^{(i)} \mathrm{e}^{-b^{(i)}\|x\|_p^p}$ for $p \in (0, 2]$ and constants $a^{(i)} \in \mathbb{R}$, $b^{(i)} > 0$. Indeed, this follows from the fact that the characteristic function of a *stable distribution* in probability theory is of the form $\mathrm{e}^{-c|t|^p}$. (For $p > 2$ this is not possible as the positivity of ν would be violated.) See e.g. Durrett [2019].
- $J_x^{(i)} = a^{(i)} \|x\|_p^{-c^{(i)}}$ with $p \in (0, 2]$, $a^{(i)} \in \mathbb{R}$ and $c^{(i)} > d$. Indeed, we can take linear combinations of the interactions above with non-negative coefficients, and we have

$$\int_0^\infty s^{(c-1)/p}\, \mathrm{e}^{-s\|x\|_p^p}\, \mathrm{d}s = C \|x\|_p^{-c}. \tag{11.10}$$

Here $c > d$ is required in order for the sum defining $J^{(i)}_{x,\mathrm{per}}$ to be convergent.

- Convex combinations of the above.

Let Λ^*_ℓ denote the dual of Λ_ℓ in Fourier theory, namely

$$\Lambda^*_\ell = \tfrac{2\pi}{\ell}\big\{-\tfrac{\ell}{2}+1,\dots,\tfrac{\ell}{2}\big\}^d \subset [-\pi,\pi]^d. \tag{11.11}$$

THEOREM 11.3. *Assume that $J^{(i)}_x$ is one of the interactions above; we assume in addition that ℓ is even and that*

$$J^{(3)}_x \geq J^{(1)}_x \geq -J^{(2)}_x \geq 0, \qquad \text{for all } x \in \mathbb{Z}^d.$$

Then

$$\frac{1}{\ell^d}\sum_{x\in\Lambda_\ell}\langle S^{(3)}_0 S^{(3)}_x\rangle^{per}_{\Lambda_\ell,\beta,0} \geq \tfrac13 S(S+1) - \frac{1}{\ell^d}\sum_{k\in\Lambda^*_\ell\setminus\{0\}}\sqrt{\frac{e(k)}{2\varepsilon(k)}} - \frac{1}{2\beta\ell^d}\sum_{k\in\Lambda^*_\ell\setminus\{0\}}\frac{1}{\varepsilon(k)}. \tag{11.12}$$

Here we defined

$$\varepsilon(k) = \sum_{x\in\mathbb{Z}^d} J^{(3)}_{x,\mathrm{per}}(1-\cos kx) \tag{11.13}$$

while the function $e(k)$ is defined in (11.45). Notice that $\varepsilon(k)$ is bounded and that $\varepsilon(k) \sim k^2$ around $k = 0$; it is positive for $k \neq 0$. It is worth pointing out that $e(k) \leq \mathrm{const}\, S^2$ around $k = 0$. Therefore the right-hand-side of (11.12) is necessarily positive when $d \geq 3$ and S, β are large enough.

We now assume that $J^{(i)}$ are nearest-neighbour couplings, that is,

$$J^{(i)}_x = \begin{cases} J^{(i)} & \text{if } \|x\| = 1, \\ 0 & \text{otherwise.}\end{cases} \tag{11.14}$$

We further normalise them so that $J^{(3)} = 1$. In this case we derive sharper lower bounds for long-range order. Let us introduce the following two sums:

$$\begin{aligned} I^{(d)}_\ell &= \frac{1}{\ell^d}\sum_{k\in\Lambda^*_\ell\setminus\{0\}}\sqrt{\frac{\varepsilon(k+\pi)}{\varepsilon(k)}}, \\ \tilde I^{(d)}_\ell &= \frac{1}{\ell^d}\sum_{k\in\Lambda^*_\ell\setminus\{0\}}\sqrt{\frac{\varepsilon(k+\pi)}{\varepsilon(k)}}\Big(\frac1d\sum_{i=1}^d \cos k_i\Big)_+. \end{aligned} \tag{11.15}$$

Here, $\varepsilon(k) = 2\sum_{i=1}^d (1-\cos k_i)$ and $\varepsilon(k+\pi) = 2\sum_{i=1}^d (1+\cos k_i)$, and $(\cdot)_+$ denotes the positive part. Their infinite volume limits converge to the integrals

$$\begin{aligned} I^{(d)} &= \lim_{\ell\to\infty} I_\ell^{(d)} = \frac{1}{(2\pi)^d} \int_{[-\pi,\pi]^d} \sqrt{\frac{\varepsilon(k+\pi)}{\varepsilon(k)}} \mathrm{d}k, \\ \tilde{I}^{(d)} &= \lim_{\ell\to\infty} \tilde{I}_\ell^{(d)} = \frac{1}{(2\pi)^d} \int_{[-\pi,\pi]^d} \sqrt{\frac{\varepsilon(k+\pi)}{\varepsilon(k)}} \Big(\frac{1}{d}\sum_{i=1}^d \cos k_i\Big)_+ \mathrm{d}k. \end{aligned} \tag{11.16}$$

One can check that, as $d \to \infty$, these integrals satisfy $I^{(d)} \to 1$ (Dyson, Lieb, Simon [1978]) and $\tilde{I}^{(d)} \to 1$ (Kennedy, Lieb, Shastry [1988b]). We also introduce the expression

$$\alpha_\ell(\beta) = J^{(1)} \langle S_0^{(1)} S_{e_1}^{(1)} \rangle_{\Lambda_\ell,\beta,0} + J^{(2)} \langle S_0^{(2)} S_{e_1}^{(2)} \rangle_{\Lambda_\ell,\beta,0} \tag{11.17}$$

and $\alpha(\beta) = \liminf_{\ell\to\infty} \alpha_\ell(\beta)$. We also denote by $\alpha_\ell(\infty)$ the $\beta \to \infty$ limit.

Theorem 11.4. *Assume that ℓ is even and that the nearest-neighbour coupling constants satisfy*

$$J^{(3)} = 1 \geq J^{(1)} \geq -J^{(2)} \geq 0.$$

Then we have the two lower bounds:

$$\frac{1}{\ell^d} \sum_{x\in\Lambda_\ell} \langle S_0^{(3)} S_x^{(3)} \rangle^{per}_{\Lambda_\ell,\beta,0} \geq$$

$$\begin{cases} \frac{1}{3}S(S+1) - \frac{1}{2}(I_\ell^{(d)} + \frac{\sqrt{2}}{\ell^d})\sqrt{\alpha_\ell(\beta)} - \frac{1}{2\beta\ell^d} \sum_{k\in\Lambda_\ell^*\setminus\{0\}} \frac{1}{\varepsilon(k)}, \\ \sqrt{\alpha_\ell(\beta)}\big[\frac{\sqrt{\alpha_\ell(\beta)}}{1-J^{(2)}/J^{(1)}} - \frac{1}{2}\tilde{I}_\ell^{(d)}\big] - \frac{1}{2\beta\ell^d} \sum_{k\in\Lambda_\ell^*\setminus\{0\}} \frac{1}{\varepsilon(k)} \big(\frac{1}{d}\sum_{i=1}^d \cos k_i\big)_+. \end{cases}$$

The theorem is proved at the end of Section 11.3.

We want to formulate sufficient conditions under which at least one of the lower bounds is positive, uniformly in ℓ. The terms involving $1/\beta$ converge as $\ell \to \infty$ if $d \geq 3$ and they can be made arbitrarily small by taking β sufficiently large. For $d = 2$ the bounds are useful in the ground state, i.e. when the limit $\beta \to \infty$ is taken before $\ell \to \infty$.

We get a uniform lower bound if either

$$\tfrac{1}{3}S(S+1) > \tfrac{1}{2}I^{(d)}\sqrt{\alpha(\beta)} \qquad \text{or} \qquad \frac{\sqrt{\alpha(\beta)}}{1-J^{(2)}/J^{(1)}} > \tfrac{1}{2}\tilde{I}^{(d)}.$$

Irrespective of the value of $\alpha(\beta)$, at least one of the lower bound is positive if

$$\frac{\frac{1}{3}S(S+1)}{\frac{1}{2}I^{(d)}} > \tfrac{1}{2}\tilde{I}^{(d)}(1 - J^{(2)}/J^{(1)}) \quad \Longleftrightarrow \quad 1 - J^{(2)}/J^{(1)} < \frac{\frac{4}{3}S(S+1)}{I^{(d)}\tilde{I}^{(d)}}. \tag{11.18}$$

Values of $I^{(d)}$ and $\tilde{I}^{(d)}$ can be found numerically; they are listed in Table 1 for $d = 2, 3, 4$. This allows us to verify that the condition (11.18) holds for all values of $J^{(1)}, J^{(2)}$ such that $J^{(1)} \geq -J^{(2)} \geq 0$, all dimensions $d \geq 2$, and all spin values $S \in \frac{1}{2}\mathbb{N}$, with the *one exception* of the case $d = 2$ and $S = \frac{1}{2}$. In this case, (11.18) holds when $-J^{(2)}/J^{(1)} \in [0, 0.109]$.

d	$I^{(d)}$	$\tilde{I}^{(d)}$
2	1.393	0.6468
3	1.157	0.3499
4	1.094	0.2540

TABLE 1. Numerical values of the integrals $I^{(d)}$ and $\tilde{I}^{(d)}$ defined in (11.16).

Kubo and Kishi [1988] improved the interval to $[0, 0.13]$ and this is the current best result. To do this, they use the variational principle with the constant state $\otimes_{x\in\Lambda_\ell}|\frac{1}{2}\rangle$ to get a bound on the ground state energy. Combined with the correlation inequalities stated in Theorem 8.5, they get a lower bound for $\alpha(\infty) = \lim_{\beta\to\infty} \alpha(\beta)$, namely

$$\alpha(\infty) \geq \frac{1/4}{2 - J^{(2)}/J^{(1)}}. \tag{11.19}$$

(Kubo and Kishi considered the case $J^{(1)} = J^{(3)} = 1$ but it is easily extended.) This implies that the second bound of Theorem 11.4 is positive in the interval $[0, 0.13]$.

11.3. Infrared bounds

This section explores estimates of the Fourier transform of correlations and their consequences. Such estimates are particularly relevant at small Fourier parameters; this corresponds to large wavelengths, i.e. the infrared spectrum for light, hence the name given by physicists.

We need to introduce the conventions about the Fourier transform used in this survey. Recall that $\Lambda_\ell^* = \frac{2\pi}{\ell}\big\{-\frac{\ell}{2}+1, \dots, \frac{\ell}{2}\big\}^d$. The Fourier transform of a function $f : \Lambda_\ell \to \mathbb{C}$ is

$$\widehat{f}(k) = \sum_{x\in\Lambda_\ell} \mathrm{e}^{-\mathrm{i}kx} f(x), \qquad k \in \Lambda_\ell^*, \tag{11.20}$$

where we write kx for the usual inner product $\sum_{i=1}^d k_i x_i$. One can check that the inverse relation is then

$$f(x) = \frac{1}{\ell^d} \sum_{k\in\Lambda_\ell^*} \mathrm{e}^{\mathrm{i}kx} \widehat{f}(k). \tag{11.21}$$

Note that $\varepsilon(k) = \widehat{J}^{(3)}(0) - \widehat{J}^{(3)}(k)$.

The first infrared bound involves the Duhamel correlation function $\eta(x)$, defined by

$$\eta(x) = \frac{1}{\beta} \frac{1}{Z_{\text{per}}(\Lambda_\ell, \beta, h)} \int_0^\beta \mathrm{d}s \; \mathrm{Tr}\, S_0^{(3)} \, \mathrm{e}^{-sH_{\Lambda,h}^{\text{per}}} \, S_x^{(3)} \, \mathrm{e}^{-(\beta-s)H_{\Lambda,h}^{\text{per}}} . \tag{11.22}$$

The method of reflection positivity allows us to establish the following infrared bound.

Lemma 11.5. *Let $h = 0$ and ℓ be even. Assume that the coupling constants $J^{(i)}$ satisfy the assumptions of Theorem 11.3. Then*

$$\widehat{\eta}(k) \leq \frac{1}{2\beta\varepsilon(k)}, \qquad \textit{for all } k \in \Lambda_\ell^* \setminus \{0\}.$$

The proof of this lemma can be found at the end of Section 11.4.

11.3.1. Falk–Bruch inequality. We cannot use the infrared bound directly on the Duhamel function because of a lack of suitable lower bound for $\eta(0)$. The way out is to derive another bound on the ordinary correlation function. This can be done using the Falk–Bruch inequality, which was proposed independently in Falk, Bruch [1969] and Dyson, Lieb Simon [1978].

Let $\mathcal{H}$ be a separable Hilbert space, H a bounded hermitian operator such that $\mathrm{Tr}\, \mathrm{e}^{-H} < \infty$, and let $\mathcal{B}$ denote the space of bounded operators on $\mathcal{H}$. We define the *Duhamel inner product* in $\mathcal{B}$ by

$$(A, B) = \frac{1}{Z} \int_0^1 \mathrm{d}s \; \mathrm{Tr}\; \mathrm{e}^{-(1-s)H} \, A^* \, \mathrm{e}^{-sH} \, B, \qquad A, B \in \mathcal{B}, \tag{11.23}$$

with $Z = \mathrm{Tr}\; \mathrm{e}^{-H}$. We have

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}s} \mathrm{Tr}\; \mathrm{e}^{-(1-s)H} \, A^* \, \mathrm{e}^{-sH} \, B &= \mathrm{Tr}\; \mathrm{e}^{-(1-s)H} \, [H, A^*] \, \mathrm{e}^{-sH} \, B \\ &= \mathrm{Tr}\; \mathrm{e}^{-(1-s)H} \, A^* \, \mathrm{e}^{-sH} \, [B, H], \end{aligned} \tag{11.24}$$

and we obtain the useful identity

$$\big([A, H], B\big) = \big(A, [B, H]\big). \tag{11.25}$$

Further,

$$\big(A, [B, H]\big) = \frac{1}{Z} \int_0^1 \mathrm{d}s \, \frac{\mathrm{d}}{\mathrm{d}s} \mathrm{Tr}\; \mathrm{e}^{-(1-s)H} \, A^* \, \mathrm{e}^{-sH} \, B = \big\langle [B, A^*] \big\rangle \tag{11.26}$$

where

$$\langle \cdot \rangle = \frac{1}{Z} \mathrm{Tr}\; \cdot \; \mathrm{e}^{-H} . \tag{11.27}$$

For a given $A \in \mathcal{B}$, let us introduce the function $F(s) = \mathrm{Tr}\; \mathrm{e}^{-(1-s)H} \, A^* \, \mathrm{e}^{-sH} \, A$. We have

$$\frac{\mathrm{d}^2}{\mathrm{d}s^2} F(s) = \mathrm{Tr}\; \mathrm{e}^{-(1-s)H} \, [A, H]^* \, \mathrm{e}^{-sH} \, [A, H] \geq 0 \tag{11.28}$$

(positivity can be shown by casting the right side in the form $\operatorname{Tr} B^*B$). The function $F(s)$ is therefore convex. Then

$$\tfrac{1}{2}\langle A^*A + AA^*\rangle = \frac{1}{2Z}\big(F(0)+F(1)\big) \geq \frac{1}{Z}\int_0^1 F(s)\mathrm{d}s = (A,A) \tag{11.29}$$

with equality if and only if $[A,H]=0$. The Cauchy–Schwarz inequality of the Duhamel inner product (11.23) gives

$$\big|\big(A,[B,H]\big)\big|^2 \leq (A,A)\ \big([B,H],[B,H]\big). \tag{11.30}$$

Using Eq. (11.26) to write the Duhamel inner product of commutators as expectations in the state $\langle\cdot\rangle$, and the inequalities (11.29) and (11.30) as well as cyclicity of the trace, we get *Bogolubov's inequality*

$$\big|\big\langle[B,A^*]\big\rangle\big|^2 \leq \tfrac{1}{2}\langle A^*A+AA^*\rangle\ \big\langle\big[[B,H],B^*\big]\big\rangle. \tag{11.31}$$

Inequality (11.29) gives an upper bound for the Duhamel inner product, but we actually need a lower bound. For this, we consider the function

$$\Phi(s) = \sqrt{s}\coth\tfrac{1}{\sqrt{s}}. \tag{11.32}$$

This function is increasing, concave, and is depicted in Fig. 11.1. One can check that

$$\sqrt{s} \leq \Phi(s) \leq \sqrt{s}+s. \tag{11.33}$$

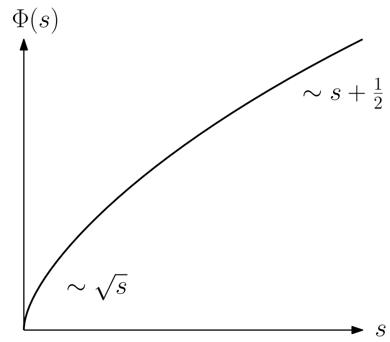

FIGURE 11.1. The function Φ of the Falk–Bruch inequality.

LEMMA 11.6 (Falk–Bruch inequality). *For all $A\in\mathcal{B}$ such that the denominators differ from zero, we have*

$$\frac{2\langle A^*A+AA^*\rangle}{\big\langle\big[A^*,[H,A]\big]\big\rangle} \leq \Phi\bigg(\frac{4(A,A)}{\big\langle\big[A^*,[H,A]\big]\big\rangle}\bigg).$$

It is worth noting that the double commutator is nonnegative, as can be seen from Eq. (11.26). Indeed, taking $A \mapsto [A^*, H]$ and $B \mapsto A^*$, we can express it using the Duhamel inner product as

$$\big\langle \big[A^*, [H, A]\big]\big\rangle = \big([A^*, H], [A^*, H]\big) \geq 0. \tag{11.34}$$

Proof of Lemma 11.6. Recall the function $F(s)$ defined before (11.28). The Falk–Bruch inequality can be written as

$$2\frac{F(0) + F(1)}{F'(1) - F'(0)} \leq \Phi\bigg(\frac{4\int_0^1 F(s)\mathrm{d}s}{F'(1) - F'(0)}\bigg). \tag{11.35}$$

If $\{\varphi_j\}$ is an orthonormal set of eigenvectors of H with eigenvalues λ_j, we can write

$$F(s) = \sum_{i,j} \big|(\varphi_i, A\varphi_j)\big|^2 \,\mathrm{e}^{-\lambda_j}\, \mathrm{e}^{(\lambda_j - \lambda_i)s} = \int_{-\infty}^{\infty} \mathrm{e}^{st}\, \mathrm{d}\mu(t), \tag{11.36}$$

where μ is a positive measure. We have

$$\begin{aligned} F(0) + F(1) &= \int (\,\mathrm{e}^t + 1)\mathrm{d}\mu(t), \\ F'(1) - F'(0) &= \int t(\,\mathrm{e}^t - 1)\mathrm{d}\mu(t), \\ \int_0^1 F(s)\mathrm{d}s &= \int \frac{\mathrm{e}^t - 1}{t}\mathrm{d}\mu(t). \end{aligned} \tag{11.37}$$

Let us consider the probability measure $\mathrm{d}\nu(t) = (\int t(\,\mathrm{e}^t - 1)\mathrm{d}\mu(t))^{-1} t(\,\mathrm{e}^t - 1)\mathrm{d}\mu(t)$. We have

$$\begin{aligned} \frac{F(0) + F(1)}{F'(1) - F'(0)} &= \int \tfrac{1}{t} \coth \tfrac{t}{2}\, \mathrm{d}\nu(t), \\ \frac{\int F(s)\mathrm{d}s}{F'(1) - F'(0)} &= \int \tfrac{1}{t^2}\, \mathrm{d}\nu(t). \end{aligned} \tag{11.38}$$

Since Φ is concave we can use Jensen's inequality and we get (11.35):

$$\begin{aligned} \Phi\bigg(\frac{4\int_0^1 F(s)\mathrm{d}s}{F'(1) - F'(0)}\bigg) &= \Phi\Big(\int \tfrac{4}{t^2}\, \mathrm{d}\nu(t)\Big) \geq \int \Phi\big(\tfrac{4}{t^2}\big)\mathrm{d}\nu(t) \\ &= \int \tfrac{2}{t} \coth \tfrac{t}{2}\, \mathrm{d}\nu(t) = 2\frac{F(0) + F(1)}{F'(1) - F'(0)}. \end{aligned} \tag{11.39}$$

□

The Falk–Bruch inequality is saturated when the measure $\mathrm{d}\mu$ is a Dirac on a single value. This is the case if H is the hamiltonian of the harmonic oscillator, and A is the creation or annihilation operator.

The following inequality follows from Lemma 11.6 and the upper bound in Eq. (11.33).

COROLLARY 11.7. *We have*

$$\tfrac{1}{2}\langle A^*A + AA^*\rangle \leq \tfrac{1}{2}\sqrt{(A,A)\,\big\langle\big[A^*,[H,A]\big]\big\rangle} + (A,A).$$

For our purpose we have $H \sim \beta$ and $(A,A) \sim \frac{1}{\beta}$ with β large, so that this inequality is quite optimal. We use it below since it is simpler.

11.3.2. Infrared bound for the usual correlation function. In the rest of this section ℓ and β will be fixed, and we drop the subscripts on $\langle\cdot\rangle^{\text{per}}_{\Lambda_\ell,\beta,0}$, writing simply $\langle\cdot\rangle$.

We introduce Fourier transforms of spin operators. This allows us to write the correlation functions in the form of Corollary 11.7. Accordingly, let

$$\widehat{S}^{(3)}_k = \sum_{x\in\Lambda_\ell} \mathrm{e}^{-\mathrm{i}kx}\, S^{(3)}_x, \qquad k \in \Lambda^*_\ell. \tag{11.40}$$

One easily checks the inverse identity

$$S^{(3)}_x = \frac{1}{\ell^d}\sum_{k\in\Lambda^*_\ell} \mathrm{e}^{\mathrm{i}kx}\, \widehat{S}^{(3)}_k, \qquad x \in \Lambda_\ell. \tag{11.41}$$

The Fourier transform of the usual correlation function is then equal to

$$\begin{aligned}\langle \widehat{S^{(3)}_0 S^{(3)}_x}\rangle(k) &= \sum_{x\in\Lambda_\ell} \mathrm{e}^{-\mathrm{i}kx}\,\langle S^{(3)}_0 S^{(3)}_x\rangle = \frac{1}{\ell^d}\sum_{x,y\in\Lambda_\ell} \mathrm{e}^{-\mathrm{i}k(x-y)}\,\langle S^{(3)}_x S^{(3)}_y\rangle \\ &= \frac{1}{\ell^d}\langle \widehat{S}^{(3)}_{-k}\widehat{S}^{(3)}_k\rangle.\end{aligned} \tag{11.42}$$

Notice that $(\widehat{S}^{(3)}_k)^* = \widehat{S}^{(3)}_{-k}$, thus

$$\langle \widehat{S^{(3)}_0 S^{(3)}_x}\rangle(k) \geq 0. \tag{11.43}$$

For the Duhamel correlation function we obtain

$$\widehat{\eta}(k) = (\widehat{S^{(3)}_0, S^{(3)}_x})(k) = \frac{1}{\ell^d}(\widehat{S}^{(3)}_k, \widehat{S}^{(3)}_k). \tag{11.44}$$

(There is no $-k$ because the Duhamel inner product involves taking the adjoint.) Let

$$e(k) = \tfrac{1}{2}\sum_{x\in\mathbb{Z}^d}\Big((J^{(1)}_{x,\text{per}} - J^{(2)}_{x,\text{per}}\cos kx)\langle S^{(1)}_0 S^{(1)}_x\rangle + (J^{(2)}_{x,\text{per}} - J^{(1)}_{x,\text{per}}\cos kx)\langle S^{(2)}_0 S^{(2)}_x\rangle\Big). \tag{11.45}$$

We will see in the proof of the next lemma that $e(k) \geq 0$, as it can be written as the expectation of a double commutator in the form of Eq. (11.34).

LEMMA 11.8 (Infrared bound for the usual correlation function). *We have for all $k \in \Lambda_\ell^* \setminus \{0\}$ that*

$$\widehat{\langle S_0^{(3)} S_x^{(3)} \rangle}(k) \leq \sqrt{\frac{e(k)}{2\varepsilon(k)}} + \frac{1}{2\beta\varepsilon(k)}.$$

PROOF. We take $A = \widehat{S}_k^{(3)}$ and $H = \beta H_{\Lambda,0}^{\rm per}$ in Corollary 11.7. We need to calculate the double commutator. First, we have

$$\begin{aligned}
[H_{\Lambda,0}^{\rm per}, \widehat{S}_k^{(3)}] &= \sum_{x \in \Lambda_\ell} [H_{\Lambda,0}^{\rm per}, S_x^{(3)}]\, \mathrm{e}^{-\mathrm{i}kx} \\
&= -\sum_{i=1}^{3} \sum_{x,y,z \in \Lambda_\ell} \mathrm{e}^{-\mathrm{i}kx}\, J_{y-z,\rm per}^{(i)} [S_y^{(i)} S_z^{(i)}, S_x^{(3)}] \\
&= -2\mathrm{i} \sum_{x,y \in \Lambda_\ell} \mathrm{e}^{-\mathrm{i}kx} \Big(-J_{x-y,\rm per}^{(1)} S_x^{(2)} S_y^{(1)} + J_{x-y,\rm per}^{(2)} S_x^{(1)} S_y^{(2)} \Big).
\end{aligned} \tag{11.46}$$

We used the fact that operators at different sites commute, and also that $J_x^{(i)} = J_{-x}^{(i)}$. Next,

$$\begin{aligned}
\big[\widehat{S}_{-k}^{(3)}, [H_{\Lambda,0}^{\rm per}, \widehat{S}_k^{(3)}]\big] &= -2\mathrm{i} \sum_{x,y \in \Lambda_\ell} \mathrm{e}^{-\mathrm{i}kx} \Big[\mathrm{e}^{\mathrm{i}kx}\, S_x^{(3)} + \mathrm{e}^{\mathrm{i}ky}\, S_y^{(3)}, -J_{x-y,\rm per}^{(1)} S_x^{(2)} S_y^{(1)} \\
&\qquad\qquad + J_{x-y,\rm per}^{(2)} S_x^{(1)} S_y^{(2)} \Big] \\
&= 2 \sum_{x,y \in \Lambda_\ell} \Big(\big(J_{x-y,\rm per}^{(1)} - \cos(k(x-y)) J_{x-y,\rm per}^{(2)} \big) S_x^{(1)} S_y^{(1)} \\
&\qquad + \big(J_{x-y,\rm per}^{(2)} - \cos(k(x-y)) J_{x-y,\rm per}^{(1)} \big) S_x^{(2)} S_y^{(2)} \Big).
\end{aligned} \tag{11.47}$$

Taking the expectation in the Gibbs state, we obtain

$$\big\langle \big[A^*, [H, A]\big] \big\rangle = \big\langle \big[\hat{S}_{-k}^{(3)}, [\beta H_{\Lambda,0}^{\rm per}, \hat{S}_k^{(3)}]\big] \big\rangle = 4\beta\ell^d e(k). \tag{11.48}$$

We also see that $e(k) \geq 0$ from Eq. (11.34). Lemma 11.8 follows from Corollary 11.7 and from the infrared bound on the Duhamel correlation function, Lemma 11.5. □

We can now prove the occurrence of long-range order.

PROOF OF THEOREM 11.3. We have the inequality (see Theorem 8.5)

$$\langle S_0^{(3)} S_0^{(3)} \rangle \geq \tfrac{1}{3} \textstyle\sum_{i=1}^{3} \langle S_0^{(i)} S_0^{(i)} \rangle = \tfrac{1}{3} S(S+1). \tag{11.49}$$

This is where we use that $J_x^{(3)} \geq J_x^{(1)} \geq -J_x^{(2)} \geq 0$.

We now use the inverse Fourier transform on the two-point correlation function, namely

$$\tfrac{1}{3}S(S+1) \leq \langle S_0^{(3)} S_0^{(3)} \rangle = \frac{1}{\ell^d} \widehat{\langle S_0^{(3)} S_x^{(3)} \rangle}(0) + \frac{1}{\ell^d} \sum_{k\in\Lambda_\ell^* \setminus \{0\}} \widehat{\langle S_0^{(3)} S_x^{(3)} \rangle}(k). \tag{11.50}$$

Notice that the first term of the right side is equal to the long-range order parameter. Then

$$\frac{1}{\ell^d} \sum_{x\in\Lambda_\ell} \langle S_0^{(3)} S_x^{(3)} \rangle = \frac{1}{\ell^d} \widehat{\langle S_0^{(3)} S_x^{(3)} \rangle}(0) \geq \tfrac{1}{3}S(S+1) - \frac{1}{\ell^d} \sum_{k\in\Lambda_\ell^* \setminus \{0\}} \widehat{\langle S_0^{(3)} S_x^{(3)} \rangle}(k). \tag{11.51}$$

We can bound the last term with the help of Lemma 11.8, which gives Theorem 11.3. □

Proof of Theorem 11.4. With nearest-neighbour interactions the function $e(k)$ can be written as

$$e(k) = \alpha_\ell(\beta) \sum_{i=1}^d (1 + r\cos k_i), \tag{11.52}$$

where

$$r = \frac{-J^{(2)} \langle S_0^{(1)} S_{e_1}^{(1)} \rangle - J^{(1)} \langle S_0^{(2)} S_{e_1}^{(2)} \rangle}{J^{(1)} \langle S_0^{(1)} S_{e_1}^{(1)} \rangle + J^{(2)} \langle S_0^{(2)} S_{e_1}^{(2)} \rangle}. \tag{11.53}$$

Here $e_1 = (1,0,\dots,0) \in \mathbb{Z}^d$ is the unit vector in the first direction. It follows from the fact that $e(k) \geq 0$ for all k, that $r \in [-1,1]$. Let

$$I_\ell^{(d)}(r) = \frac{1}{\ell^d} \sum_{k\in\Lambda_\ell^* \setminus \{0,\pi\}} \sqrt{\frac{\sum_{i=1}^d (1 + r\cos k_i)}{\sum_{i=1}^d (1 - \cos k_i)}} \tag{11.54}$$

where we have omitted the term $\frac{1}{\ell^d}\sqrt{1-r}$ for $k = \pi = (\pi,\pi,\dots,\pi)$. Adding it back and bounding it by $\sqrt{2}/\ell^d$, the lower bound is

$$\frac{1}{\ell^d} \sum_{x\in\Lambda_\ell} \langle S_0^{(3)} S_x^{(3)} \rangle \geq \tfrac{1}{3}S(S+1) - \tfrac{1}{2}\sqrt{\alpha_\ell(\beta)}\,\big(I_\ell^{(d)}(r) + \tfrac{\sqrt{2}}{\ell^d}\big) - \frac{1}{2\beta\ell^d} \sum_{k\in\Lambda_\ell^* \setminus \{0\}} \frac{1}{\varepsilon(k)}. \tag{11.55}$$

Observe that $I_\ell^{(d)}(r)$ is concave with respect to r and that its derivative at $r = 1$ is equal to

$$\frac{\mathrm{d}}{\mathrm{d}r} I_\ell^{(d)}(r)\Big|_{r=1} = \frac{1}{\ell^d} \sum_{k\in\Lambda_\ell^* \setminus \{0,\pi\}} \frac{\sum_{i=1}^d \cos k_i}{\sqrt{\sum_{i=1}^d (1-\cos k_i)\, \sum_{i=1}^d (1+\cos k_i)}}. \tag{11.56}$$

This is equal to zero, as can be seen with the change of variables $k \mapsto k + (\pi,\dots,\pi)$. Then $I_\ell^{(d)}(r) \leq I_\ell^{(d)}(1) = I_\ell^{(d)}$. Using this with the lower bound of Theorem 11.3, we obtain the first bound of Theorem 11.4.

For the second bound, we follow Kennedy, Lieb, Shastry [1988a] and use the inverse Fourier transform. In what follows, x is the dummy variable summed over inside the Fourier transform. We have

$$\begin{aligned}\langle S_0^{(3)} S_{e_1}^{(3)} \rangle &= \frac{1}{d\ell^d} \sum_{k\in\Lambda_\ell^*} \sum_{i=1}^d \mathrm{e}^{\mathrm{i}k_i} \, \langle \widehat{S_0^{(3)} S_x^{(3)}} \rangle(k) \\ &= \frac{1}{\ell^d} \langle \widehat{S_0^{(3)} S_x^{(3)}} \rangle(0) + \frac{1}{\ell^d} \sum_{k\in\Lambda_\ell^*\setminus\{0\}} \langle \widehat{S_0^{(3)} S_x^{(3)}} \rangle(k) \Bigl(\frac{1}{d} \sum_{i=1}^d \cos k_i \Bigr).\end{aligned} \tag{11.57}$$

We used lattice symmetries and the fact that $\langle \widehat{S_0^{(3)} S_x^{(3)}} \rangle \geq 0$, see Eq. (11.43). We have

$$\begin{aligned}\frac{1}{\ell^d} \langle \widehat{S_0^{(3)} S_x^{(3)}} \rangle(0) &\geq \langle S_0^{(3)} S_{e_1}^{(3)} \rangle - \frac{1}{\ell^d} \sum_{k\in\Lambda_\ell^*\setminus\{0\}} \langle \widehat{S_0^{(3)} S_x^{(3)}} \rangle(k) \Bigl(\frac{1}{d} \sum_{i=1}^d \cos k_i \Bigr)_+ \\ &\geq \langle S_0^{(3)} S_{e_1}^{(3)} \rangle - \frac{1}{\ell^d} \sum_{k\in\Lambda_\ell^*\setminus\{0\}} \Bigl(\frac{1}{d} \sum_{i=1}^d \cos k_i \Bigr)_+ \biggl[\sqrt{\frac{e(k)}{2\varepsilon(k)}} + \frac{1}{2\beta\varepsilon(k)} \biggr].\end{aligned} \tag{11.58}$$

Proceeding with $e(k)$ as we did with the first lower bound, we get

$$\frac{1}{\ell^d} \langle \widehat{S_0^{(3)} S_x^{(3)}} \rangle(0) \geq \langle S_0^{(3)} S_{e_1}^{(3)} \rangle - \tfrac{1}{2} \sqrt{\alpha_\ell(\beta)} \tilde I_\ell^{(d)}(r) - \frac{1}{2\beta\ell^d} \sum_{k\in\Lambda_\ell^*\setminus\{0\}} \frac{1}{\varepsilon(k)}, \tag{11.59}$$

where

$$\tilde I_\ell^{(d)}(r) = \frac{1}{\ell^d} \sum_{k\in\Lambda_\ell^*\setminus\{0\}} \sqrt{\frac{\sum_{i=1}^d (1 + r\cos k_i)}{\sum_{i=1}^d (1-\cos k_i)}} \Bigl(\frac{1}{d} \sum_{i=1}^d \cos k_i \Bigr)_+. \tag{11.60}$$

One easily checks that the derivative of $\tilde I_\ell^{(d)}(r)$ is positive, so it is smaller than $\tilde I_\ell^{(d)}(1) = \tilde I_\ell^{(d)}$. Finally, using Theorem 8.5, we have

$$\langle S_0^{(3)} S_{e_1}^{(3)} \rangle \geq \frac{\alpha_\ell(\beta)}{1 - J^{(2)}/J^{(1)}}. \tag{11.61}$$

The second lower bound of Theorem 11.4 follows. □

11.4. Reflection positivity

Let $\mathcal{H}$ be a separable Hilbert space, and let $\mathcal{B}_{\rm left}$, resp. $\mathcal{B}_{\rm right}$, denote the space of bounded operators on $\mathcal{H}\otimes\mathcal{H}$ that are of the form $a\otimes 1\!\!1$, resp. $1\!\!1\otimes a$, for some $a\in\mathcal{B}(\mathcal{H})$. Let $\mathcal{R}$ denote the automorphism of $\mathcal{B}(\mathcal{H}\otimes\mathcal{H})$ such that

$$\begin{aligned}\mathcal{R}(a\otimes 1\!\!1) &= 1\!\!1\otimes a, \\ \mathcal{R}(1\!\!1\otimes a) &= a\otimes 1\!\!1.\end{aligned} \tag{11.62}$$

Let us fix an orthonormal basis $\{e_i\}$ on $\mathcal{H}$, and define the complex conjugate $\overline{a}$ of a bounded operator a by

$$\langle e_i, \overline{a} e_j \rangle = \overline{\langle e_i, a e_j \rangle}. \tag{11.63}$$

In matrix notation, that means taking the complex conjugate of its elements, without transposing as for hermitian adjoints. The reason to use the complex conjugate is that for all $a, b \in \mathcal{B}(\mathcal{H})$, we have

$$\overline{a\, b} = \overline{a}\, \overline{b}. \tag{11.64}$$

Here is the key inequality that is closely related to reflection positivity. Let I be an index set and μ a positive, finite measure on I. We assume that $A, C_i \in \mathcal{B}_{\text{left}}$ and $B, D_i \in \mathcal{B}_{\text{right}}$ for all $i \in I$.

LEMMA 11.9. *We have*

$$\Big| \mathrm{Tr}\; \mathrm{e}^{A+B+\int C_i D_i \mathrm{d}\mu(i)} \Big|^2 \leq \mathrm{Tr}\; \mathrm{e}^{A+\mathcal{R}\overline{A}+\int C_i \mathcal{R}\overline{C}_i \mathrm{d}\mu(i)} \cdot \mathrm{Tr}\; \mathrm{e}^{\mathcal{R}\overline{B}+B+\int \mathcal{R}\overline{D}_i\, D_i \mathrm{d}\mu(i)}.$$

PROOF. We use the Duhamel formula in the following form. If A, B are bounded operators, then

$$\mathrm{e}^{A+B} = \sum_{n\geq 0} \int_{0<t_1<\dots<t_n<1} \mathrm{d}t_1 \dots \mathrm{d}t_n\, \mathrm{e}^{t_1 A}\, B\, \mathrm{e}^{(t_2-t_1)A}\, B \dots B\, \mathrm{e}^{(1-t_n)A}. \tag{11.65}$$

In what follows, we use the shorthands

$$\int \mathrm{d}\boldsymbol{i} \equiv \int \mathrm{d}\mu(i_1) \cdots \int \mathrm{d}\mu(i_n) \qquad \text{and} \qquad \int \mathrm{d}\boldsymbol{t} \equiv \int_{0<t_1<\dots<t_n<1} \mathrm{d}t_1 \dots \mathrm{d}t_n. \tag{11.66}$$

We also write $A = a \otimes \mathbb{1}$, $B = \mathbb{1} \otimes b$, $C_i = c_i \otimes \mathbb{1}$, and $D_i = \mathbb{1} \otimes d_i$. Then

$$\begin{aligned}
&\Big| \mathrm{Tr}_{\mathcal{H}\otimes\mathcal{H}}\, \mathrm{e}^{A+B+\int C_i D_i \mathrm{d}\mu(i)} \Big|^2 \\
&= \Big| \sum_{n\geq 0} \int \mathrm{d}\boldsymbol{i} \int \mathrm{d}\boldsymbol{t}\; \mathrm{Tr}_{\mathcal{H}\otimes\mathcal{H}}\, \mathrm{e}^{t_1(A+B)}\, C_{i_1} D_{i_1} \dots C_{i_n} D_{i_n}\, \mathrm{e}^{(1-t_n)(A+B)} \Big|^2 \\
&= \Big| \sum_{n\geq 0} \int \mathrm{d}\boldsymbol{i} \int \mathrm{d}\boldsymbol{t}\; \mathrm{Tr}_{\mathcal{H}}\, \mathrm{e}^{t_1 a}\, c_{i_1} \dots c_{i_n}\, \mathrm{e}^{(1-t_n)a}\, \mathrm{Tr}_{\mathcal{H}}\, \mathrm{e}^{t_1 b}\, d_{i_1} \dots d_{i_n}\, \mathrm{e}^{(1-t_n)b} \Big|^2 \\
&\leq \sum_{n\geq 0} \int \mathrm{d}\boldsymbol{i} \int \mathrm{d}\boldsymbol{t}\; \mathrm{Tr}_{\mathcal{H}}\, \mathrm{e}^{t_1 a}\, c_{i_1} \dots c_{i_n}\, \mathrm{e}^{(1-t_n)a}\, \mathrm{Tr}_{\mathcal{H}}\, \mathrm{e}^{t_1 \overline{a}}\, \overline{c}_{i_1} \dots \overline{c}_{i_n}\, \mathrm{e}^{(1-t_n)\overline{a}} \\
&\qquad \cdot \sum_{n\geq 0} \int \mathrm{d}\boldsymbol{i} \int \mathrm{d}\boldsymbol{t}\; \mathrm{Tr}_{\mathcal{H}}\, \mathrm{e}^{t_1 \overline{b}}\, \overline{d}_{i_1} \dots \overline{d}_{i_n}\, \mathrm{e}^{(1-t_n)\overline{b}}\, \mathrm{Tr}_{\mathcal{H}}\, \mathrm{e}^{t_1 b}\, d_{i_1} \dots d_{i_n}\, \mathrm{e}^{(1-t_n)b} \\
&= \mathrm{Tr}_{\mathcal{H}\otimes\mathcal{H}}\, \mathrm{e}^{A+\mathcal{R}\overline{A}+\int C_i \mathcal{R}\overline{C}_i \mathrm{d}\mu(i)} \cdot \mathrm{Tr}_{\mathcal{H}\otimes\mathcal{H}}\, \mathrm{e}^{\mathcal{R}\overline{B}+B+\int \mathcal{R}\overline{D}_i\, D_i \mathrm{d}\mu(i)}.
\end{aligned} \tag{11.67}$$

We used the ordinary Cauchy–Schwarz inequality for functions, here with argument $(n, \boldsymbol{i}, \boldsymbol{t})$. The complex conjugate was written with the help of (11.64). $\square$

We now derive the infrared bound for the Duhamel correlation function, Lemma 11.5. In the rest of this Section, we fix an even integer ℓ and consider periodic couplings (11.1). Recall that $\Lambda_\ell = \{0, 1, \dots, \ell - 1\}^d$. Let Δ denote the discrete Laplacian from the coupling constant $J^{(3)}_{\rm per}$, which acts on a field $v = (v_x) \in \mathbb{R}^\Lambda$ as

$$(\Delta v)_x = \sum_{y \in \Lambda_\ell} J^{(3)}_{x-y,{\rm per}} (v_y - v_x). \tag{11.68}$$

Notice the following identity, which is a discrete version of $\int f(-\Delta g) = \int \nabla f \nabla g$ for functions:

$$(u, -\Delta v) = \tfrac12 \sum_{x,y \in \Lambda_\ell} J^{(3)}_{x-y,{\rm per}} (u_x - u_y)(v_x - v_y). \tag{11.69}$$

In the left side, $(\cdot, \cdot)$ stands for the usual inner product on $\mathbb{R}^{\Lambda_\ell}$, i.e. $(u, v) = \sum_{x \in \Lambda_\ell} u_x v_x$. We introduce the following partition function that depends on a field v:

$$Z(v) = {\rm Tr}\, {\rm e}^{-\beta H(v)}, \tag{11.70}$$

with hamiltonian given by

$$H(v) = H^{\rm per}_{\Lambda_\ell, 0} - \sum_{x \in \Lambda_\ell} h_x S^{(3)}_x, \tag{11.71}$$

where the local magnetic field is obtained from v by

$$h_x = (\Delta v)_x. \tag{11.72}$$

Let

$$\tilde Z(v) = {\rm e}^{\frac14 \beta (v, \Delta v)}\, Z(v). \tag{11.73}$$

We show that $\tilde Z(v)$ is maximised by the field $v \equiv 0$, which is the key to proving Lemma 11.5.

Let $\mathcal{R}$ denote a reflection across a plane cutting through edges. Namely, given a direction $i = 1, \dots, d$ and a half integer $\epsilon \in \{\frac12, \frac32, \dots, \frac{\ell-1}{2}\}$, let $\mathcal{R}$ be the bijection $\Lambda_\ell \to \Lambda_\ell$ such that

$$\mathcal{R} x = x + 2(\epsilon - x_i) e_i. \tag{11.74}$$

Let

$$\Lambda_{\rm left} = \{x \in \Lambda_\ell : \epsilon - \frac{\ell}{2} < x_i < \epsilon\}, \qquad \Lambda_{\rm right} = \{x \in \Lambda_\ell : \epsilon < x_i < \epsilon + \frac{\ell}{2}\}. \tag{11.75}$$

Given a field $v_1 \in \mathbb{R}^{\Lambda_{\rm left}}$, let $(\mathcal{R} v_1)_x = (v_1)_{\mathcal{R} x} \in \mathbb{R}^{\Lambda_{\rm right}}$.

LEMMA 11.10. *Let the couplings $J^{(i)}$ satisfy the assumptions of Theorem 11.3. Then, for any $v_1 \in \mathbb{R}^{\Lambda_{left}}$ and $v_2 \in \mathbb{R}^{\Lambda_{right}}$, we have*

$$Z(v_1, v_2)^2 \leq Z(v_1, \mathcal{R}v_1)\, Z(\mathcal{R}v_2, v_2).$$

We first prove the lemma in the case of nearest-neighbour couplings; we then consider long-range interactions.

PROOF OF LEMMA 11.10 FOR NEAREST-NEIGHBOUR COUPLINGS. We cast $Z(v_1, v_2)$ in the form of Lemma 11.9. Using (11.69), we get

$$\begin{aligned}\tilde{Z}(v) &= \operatorname{Tr} \exp \beta \Big\{ \tfrac{1}{8} \sum_{x,y} J^{(3)}_{x-y} (v_y - v_x)^2 + \sum_{i=1}^{3} \sum_{x,y} J^{(i)}_{x-y} S^{(i)}_x S^{(i)}_y \\ &\qquad\qquad + \sum_{x,y} J^{(3)}_{x-y} S^{(3)}_x (v_y - v_x) \Big\} \\ &= \operatorname{Tr} \exp \beta \Big\{ \sum_{i=1}^{2} \sum_{x,y} J^{(i)}_{x-y} S^{(i)}_x S^{(i)}_y + \sum_{x,y} J^{(3)}_{x-y} \big(S^{(3)}_x + \tfrac{v_x}{2}\big)\big(S^{(3)}_y + \tfrac{v_y}{2}\big) \\ &\qquad\qquad - \widehat{J}^{(3)}(0) \sum_{x} \big(S^{(3)}_x v_x + \tfrac{v_x^2}{4}\big) \Big\}.\end{aligned} \tag{11.76}$$

We used $J^{(i)}_x = J^{(i)}_{-x}$. This formula holds for general couplings and we will also use it in the long-range case (with $J^{(i)}_{x,\text{per}}$). We now assume that $J^{(i)}_x = 0$ except when $\|x\|_1 = 1$, in which case it equals a constant $J^{(i)}$. Then the above expression has the form of Lemma 11.9 by choosing

$$\begin{aligned}A &= \beta \sum_{x,y \in \Lambda_{\text{left}}} \Big[\sum_{i=1}^{2} J^{(i)}_{x-y} S^{(i)}_x S^{(i)}_y + J^{(3)}_{x-y} \big(S^{(3)}_x + \tfrac{v_x}{2}\big)\big(S^{(3)}_y + \tfrac{v_y}{2}\big) \Big] \\ &\qquad - \widehat{J}^{(3)}(0) \sum_{x \in \Lambda_{\text{left}}} \big(S^{(3)}_x v_x + \tfrac{v_x^2}{4}\big) \\ B &= \beta \sum_{x,y \in \Lambda_{\text{right}}} \Big[\sum_{i=1}^{2} J^{(i)}_{x-y} S^{(i)}_x S^{(i)}_y + J^{(3)}_{x-y} \big(S^{(3)}_x + \tfrac{v_x}{2}\big)\big(S^{(3)}_y + \tfrac{v_y}{2}\big) \Big] \\ &\qquad - \widehat{J}^{(3)}(0) \sum_{x \in \Lambda_{\text{right}}} \big(S^{(3)}_x v_x + \tfrac{v_x^2}{4}\big) \\ \int C_i D_i \mathrm{d}\mu(i) &= \beta \sum_{\substack{x \in \Lambda_{\text{left}} \\ y \in \Lambda_{\text{right}} \\ \|x-y\|=1}} \Big[J^{(1)} S^{(1)}_x S^{(1)}_y - J^{(2)} (\mathrm{i} S^{(2)}_x)(\mathrm{i} S^{(2)}_y) \\ &\qquad + J^{(3)} \big(S^{(3)}_x + \tfrac{v_x}{2}\big)\big(S^{(3)}_y + \tfrac{v_y}{2}\big) \Big].\end{aligned} \tag{11.77}$$

In the usual basis where all $S_x^{(3)}$ are diagonal, we have $\overline{S_x^{(1)}} = S_x^{(1)}$, $\overline{\mathrm{i}S_x^{(2)}} = \mathrm{i}S_x^{(2)}$, $\overline{S_x^{(3)}} = S_x^{(3)}$. Then $\overline{A} = A$ and $\overline{B} = B$. We have multiplied $S_x^{(2)}$ by i, so taking the complex conjugate gives the operator back. Then $\overline{C_i} = C_i$ and $\overline{D_i} = D_i$. Moreover, when $x \in \Lambda_{\text{left}}$ and $y \in \Lambda_{\text{right}}$ with $\|x-y\| = 1$, the reflection interchanges x and y. In order to use Lemma 11.9 the measure μ needs to be positive, which is guaranteed by $J^{(1)}, J^{(3)} \geq 0$ and $J^{(2)} \leq 0$. □

An important observation is that if certain interactions can be cast in the form above, then this can also be done with convex combinations of these interactions. We use this property below.

Proof of Lemma 11.10 for long-range couplings. We now consider the case when $J_x^{(i)} = \int_{\mathbb{R}^d} \mathrm{d}\nu^{(i)}(k)\, \mathrm{e}^{ik\cdot x}$ where $\nu^{(i)}$ is a positive, finite measure on $\mathbb{R}^d$. We see from (11.76) that it suffices to consider a fixed $i \in \{1,2,3\}$ and to simplify the notation we dispense with the superscript $^{(i)}$. We use the decomposition (11.77) but with $J_{x,\text{per}}$ in place of J_x. It suffices to consider the cross-term

$$\sum_{\substack{x\in\Lambda_{\text{left}}\\ y\in\Lambda_{\text{right}}}} J_{x-y,\text{per}} T_x T_y \tag{11.78}$$

where $T_x \in \{S_x^{(1)}, \mathrm{i}S_x^{(2)}, S_x^{(3)} + \frac{v_x}{2}\}$. We aim to write this in the form $\int_I C_i D_i \,\mathrm{d}\mu(i)$ in order to apply Lemma 11.9. We expand

$$J_{x-y,\text{per}} = \sum_{z\in\mathbb{Z}^d} J_{x-y+\ell z} = \sum_{z\in\mathbb{Z}^d} \int_{\mathbb{R}^d} \mathrm{d}\nu(k)\, \mathrm{e}^{\mathrm{i}k\cdot(x-y+\ell z)} \tag{11.79}$$

to write

$$\sum_{\substack{x\in\Lambda_{\text{left}}\\ y\in\Lambda_{\text{right}}}} J_{x-y,\text{per}} T_x T_y = \sum_{z\in\mathbb{Z}^d} \int_{\mathbb{R}^d} \mathrm{d}\nu(k) \Big(\sum_{x\in\Lambda_{\text{left}}} \mathrm{e}^{\mathrm{i}k\cdot(x+\ell z/2)}\, T_x \Big) \Big(\sum_{y\in\Lambda_{\text{right}}} \mathrm{e}^{-\mathrm{i}k\cdot(y-\ell z/2)}\, T_y \Big). \tag{11.80}$$

This is of the required form $\int_I C_i D_i \,\mathrm{d}\mu(i)$ with index set $I = \mathbb{Z}^d \times \mathbb{R}^d$, except that we need the measure μ to be finite. In order to achieve this, we may approximate the sum over $z \in \mathbb{Z}^d$ by a sum over $z \in \Lambda'$ and then let $\Lambda' \Uparrow \mathbb{Z}^d$. The rest of the argument follows as in the nearest-neighbour case. □

Corollary 11.11. *For all $v \in \mathbb{R}^{\Lambda_\ell}$, we have $\tilde{Z}(v) \leq \tilde{Z}(0)$.*

Proof. Without loss of generality we can assume that $v_0 = 0$. We observe that $\tilde{Z}(\lambda v) \to 0$ as $|\lambda| \to \infty$, so that $\tilde{Z}(v)$ is maximised for finite v. Indeed, in the expression (11.73) we have $\mathrm{e}^{\frac14 \beta\lambda^2 (v,\Delta v)} \sim \mathrm{e}^{-c\lambda^2}$ and $Z(\lambda v) \leq \mathrm{e}^{C|\lambda|}$.

Then let (v_1, v_2) be a maximiser with $v_0 = 0$. Using Lemma 11.10 with a plane crossing the edge $(0, e_1)$, we have that $(v_1, \mathcal{R}v_1)$ is also a maximiser, with

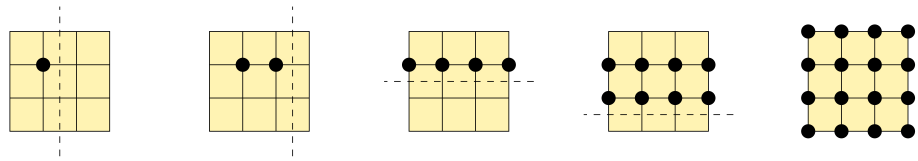

FIGURE 11.2. Starting with a maximiser, reflections yield further maximisers where more and more values are identical.

$v_0 = v_{e_1} = 0$. Using a plane crossing the edge $(e_1, 2e_1)$, we get a maximiser with more zeros. Iterating, we get a maximiser with a whole line of zeros. We then consider reflection planes in another direction to get a maximiser with a plane of zeros. We then consider reflection planes in further directions. See Fig. 11.2 for an illustration. □

PROOF OF LEMMA 11.5. From Corollary 11.11 and Eq. (11.73), we have the "Gaussian domination" bound

$$\frac{Z(sv)}{Z(0)} \leq \mathrm{e}^{-\frac{1}{4}s^2\beta(v,\Delta v)}\,. \tag{11.81}$$

The derivative of $Z(sv)$ with respect to s is equal to 0 at $s = 0$ because of symmetries (for instance, a rotation around the 3rd spin axis by angle π, which takes $S_x^{(i)}$ to $-S_x^{(i)}$, $i = 1, 2$, and leaves $S_x^{(3)}$ invariant). The second derivative can be calculated e.g. using the Duhamel formula (11.65) and translation-invariance. Recalling the Duhamel correlation function η from (11.22), we get

$$\frac{1}{Z(0)}\frac{\mathrm{d}^2}{\mathrm{d}s^2}Z(sv)\bigg|_{s=0} = \beta^2 \sum_{x,y\in\Lambda} h_x h_y \eta(x-y), \tag{11.82}$$

where we recall that $h_x = (\Delta v)_x$. We now choose the field v to be

$$v_x = \cos(kx), \qquad k \in \Lambda_\ell^*. \tag{11.83}$$

Observe that $\Delta v_x = -\varepsilon(k) v_x$. The order s^2 of the inequality (11.81) gives

$$\tfrac{1}{2}\beta^2\varepsilon(k)^2 \sum_{x,y\in\Lambda_\ell} \cos(kx)\cos(ky)\eta(x-y) \leq \tfrac{1}{4}\beta\varepsilon(k) \sum_{x\in\Lambda_\ell} \cos(kx)^2. \tag{11.84}$$

Since $\eta(x)$ and $\widehat{\eta}(k)$ are both real, the left-hand-side satisfies

$$\begin{aligned}\sum_{x,y\in\Lambda_\ell}\cos(kx)\cos(ky)\eta(x-y) &= \sum_{x\in\Lambda_\ell}\cos(kx)\sum_{y\in\Lambda_\ell}\mathrm{e}^{\mathrm{i}ky}\,\eta(x-y)\\ &= \sum_{x\in\Lambda_\ell}\cos(kx)\sum_{z\in\Lambda_\ell}\mathrm{e}^{\mathrm{i}k(x-z)}\,\eta(z)\\ &= \sum_{x\in\Lambda_\ell}\cos(kx)\,\mathrm{e}^{\mathrm{i}kx}\,\widehat{\eta}(k)\\ &= \widehat{\eta}(k)\sum_{x\in\Lambda_\ell}\cos(kx)^2.\end{aligned} \tag{11.85}$$

Inserting this in Eq. (11.84) we obtain Lemma 11.5. □

Bibliographical references

The first proof of continuous symmetry breaking is due to Fröhlich, Simon, and Spencer [1976]; they established that the classical Heisenberg model undergoes a phase transition in dimensions three and higher. Their work was inspired by ideas from quantum field field theory, specifically by the Källén–Lehmann representation of two-point Green functions in relativistic quantum field theory, which suggested the right form of infrared bounds, and by reflection positivity, as formulated in the works of Jost [1965], Osterwalder and Schrader [1973], and Glaser [1974]. (Furthermore, bounds in Glimm and Jaffe [1970] and Fröhlich [1974] inspired the exponential infrared bounds proved in Fröhlich, Simon, and Spencer [1976].)

The extension of these ideas to quantum spin systems was achieved in another groundbreaking article, by Dyson, Lieb, and Simon [1978]. The method was then further extended and streamlined in Fröhlich, Israel, Lieb, and Simon [1978] and [1980]. Further refinements include an extension to the ground states in two dimensions Neves, Perez [1986] and improved conditions that establish long-range order in the XY model in two dimensions (Kennedy, Lieb, Shastry [1988a] and [1988b]; Kubo, Kishi [1988]).

It should be pointed out that the method does not apply to models where all coupling constants are positive (Speers [1985]). An important problem, which remains open to this day, is to prove spontaneous magnetisation or long-range order in the Heisenberg ferromagnet.

A beautiful account of the method of reflection positivity in statistical mechanics (restricted to classical systems) has been written by Biskup [2009]. The handwritten notes of Tóth [1996] for his Prague lectures give a clear account of the method. And an extensive overview, which retraces the origin of the key ideas, can be found in the handwritten notes of Fröhlich [2011] for his Vienna lectures.