CHAPTER 5

Evolution operator and KMS condition

The KMS condition, named after after Kubo, Martin, and Schwinger, is another property that characterises equilibrium states. It is based on the evolution operator instead of the pressure (tangent functional states) or the entropy (Gibbs variational principle). It may appear more cumbersome at first sight, but it has a number of advantages. It does not require translation-invariance (or periodicity) and thus allows the study e.g. of Gibbs states with rigid interfaces (the "Dobrushin states").

5.1. Finite-volume evolution operator

Evolution of quantum systems is given by the Schrödinger equation. It is sometimes convenient to let the observables evolve instead of the system; this is the Heisenberg framework, which is convenient here. With $\mathcal{H}$ a finite-dimensional Hilbert space and H a hermitian operator (the hamiltonian), we consider the family α_t, $t \in \mathbb{R}$, of linear operators $\mathcal{B}(\mathcal{H}) \to \mathcal{B}(\mathcal{H})$ defined by

$$\alpha_t(a) = \mathrm{e}^{\mathrm{i}tH}\, a\, \mathrm{e}^{-\mathrm{i}tH}\,. \tag{5.1}$$

The parameter t represents the time and is normally real. But we allow it to be complex. Then we get another characterisation of finite volume Gibbs states, which is the finite volume version of the KMS condition .

DEFINITION 5.1. *A state $\langle\cdot\rangle$ satisfies the (finite-dimensional)* **KMS conditions** *if for all $a, b \in \mathcal{B}(\mathcal{H})$ we have*

$$\big\langle a\, \alpha_{\mathrm{i}}(b) \big\rangle = \langle ba \rangle.$$

PROPOSITION 5.2. *The Gibbs state is the unique solution to the KMS condition.*

PROOF. Recall that $\rho = \mathrm{e}^{-H}/Z$, with $Z = \mathrm{Tr}\, \mathrm{e}^{-H}$, is the density matrix that corresponds to the Gibbs state. We have

$$\big\langle a\, \alpha_{\mathrm{i}}(b) \big\rangle = \frac{1}{Z} \mathrm{Tr}\, a\, \mathrm{e}^{-H}\, b\, \mathrm{e}^{H}\, \mathrm{e}^{-H} = \frac{1}{Z} \mathrm{Tr}\, ba\, \mathrm{e}^{-H} = \langle ba \rangle. \tag{5.2}$$

We used the cyclicity of the trace. Then the Gibbs state is indeed a solution.

Next, let ρ be the density matrix of a state that satisfies the KMS condition. We have for all $a, b \in \mathcal{B}(\mathcal{H})$,

$$\operatorname{Tr} a\, \mathrm{e}^{-H}\, b\, \mathrm{e}^{H}\, \rho = \operatorname{Tr} ba\rho. \tag{5.3}$$

Since this holds for all $b \in \mathcal{B}(\mathcal{H})$, we have that

$$\mathrm{e}^{H}\, \rho\, a\, \mathrm{e}^{-H} = a\rho \tag{5.4}$$

for all $a \in \mathcal{B}(\mathcal{H})$. Choosing $a = \mathbb{1}$ we get $\mathrm{e}^{H}\rho = \rho\, \mathrm{e}^{H}$, so ρ commutes with e^{H}. Now observe that (5.4) implies that $\rho\, \mathrm{e}^{H}$ commutes with all $a \in \mathcal{B}(\mathcal{H})$. Then $\rho\, \mathrm{e}^{H}$ is proportional to the identity and ρ must be equal to the Gibbs operator. ☐

5.2. Infinite-volume limit of the evolution operator

Given an interaction $\Phi \in \mathcal{I}_r$ we consider the family of evolution operators $\alpha^{\Phi}_{\Lambda,t}$, with $t \in \mathbb{C}$ and $\Lambda \Subset \mathbb{Z}^d$, that acts on local operators of $\mathcal{A}_\Lambda$ as in (5.1)

$$\alpha^{\Phi}_{\Lambda,t}(a) = \mathrm{e}^{\mathrm{i}tH^{\Phi}_{\Lambda}}\, a\, \mathrm{e}^{-\mathrm{i}tH^{\Phi}_{\Lambda}}. \tag{5.5}$$

We first address the question of the existence of the infinite-volume limit of $\alpha^{\Phi}_{\Lambda,t}$. In view of the discussion of KMS states below, we need to consider complex times as well. It turns out that $\alpha^{\Phi}_{\Lambda,t}$ converges uniformly to a bounded operator when $t \in \mathbb{R}$; it converges pointwise when $|\operatorname{Im} t|$ is small; it does not seem to converge otherwise. Recall the definition (2.34) of the norm $\|\Phi\|_r$ of the interaction Φ.

PROPOSITION 5.3 (Infinite-volume limit of the evolution operator). *Assume that $\|\Phi\|_r < \infty$ for some $r > 0$. Then*

(a) *If $t \in \mathbb{C}$ and $|\operatorname{Im} t| < \frac{r}{2\|\Phi\|_r}$, there exists an automorphism $\alpha^{\Phi}_t : \mathcal{A}_{\mathrm{loc}} \to \mathcal{A}$ such that*

$$\lim_{\Lambda \Uparrow \mathbb{Z}^d} \|\alpha^{\Phi}_{\Lambda,t}(a) - \alpha^{\Phi}_t(a)\| = 0$$

for all $a \in \mathcal{A}_{\mathrm{loc}}$. Further, we have

$$\|\alpha^{\Phi}_{\Lambda,t}(a)\| \le \|a\|\, \mathrm{e}^{r|X|} \Big(1 - |\operatorname{Im} t| \frac{2\|\Phi\|_r}{r}\Big)^{-1}$$

whenever $a \in \mathcal{A}_X$.

(b) *For $t \in \mathbb{R}$, α^{Φ}_t is a *isomorphism with $\|\alpha^{\Phi}_t\| = 1$ and it satisfies the group property*

$$\alpha^{\Phi}_{s+t}(a) = \alpha^{\Phi}_s\big(\alpha^{\Phi}_t(a)\big)$$

for all $a \in \mathcal{A}$.

In case clarification is needed, α_t^Φ is an automorphism in the sense that it is linear and $\alpha_t^\Phi(ab) = \alpha_t^\Phi(a)\alpha_t^\Phi(b)$. We also have $\alpha_t^\Phi(a)^* = \alpha_{\bar{t}}^\Phi(a^*)$, so that α_t^Φ is a *automorphism when t is real.

The proof consists of the following steps.

(i) If $|t| < \frac{r}{2\|\Phi\|_r}$, $(\alpha_{\Lambda,t}^\Phi)_{\Lambda\Subset\mathbb{Z}^d}$ is Cauchy for each fixed $a \in \mathcal{A}_{\rm loc}$. We denote the limit $\alpha_t^\Phi(a)$.

(ii) For $t \in \mathbb{R}$, we have $\|\alpha_{\Lambda,t}^\Phi(a)\| = \|a\|$ for all Λ, so $\|\alpha_t^\Phi\| = 1$.

(ii) We use the group property to extend α_t^Φ it to the whole real line, then to the infinite strip.

For the first step, we need the multicommutator expansion. Let $\mathrm{ad}_a(b) = [a, b]$ denote the "adjoint endomorphism".

LEMMA 5.4 (Multicommutator expansion). *Let a and b be two operators on the same finite-dimensional Hilbert space. Then*

$$\mathrm{e}^a\, b\, \mathrm{e}^{-a} = \sum_{n\geq 0} \frac{1}{n!}\mathrm{ad}_a^n(b).$$

PROOF. We show that $\mathrm{e}^{sa}\, b\, \mathrm{e}^{-sa}$ and $\sum_n \frac{s^n}{n!}\mathrm{ad}_a^n(b)$ satisfy the same differential equation. First,

$$\frac{\mathrm{d}}{\mathrm{d}s}\mathrm{e}^{sa}\, b\, \mathrm{e}^{-sa} = \big[a,\, \mathrm{e}^{sa}\, b\, \mathrm{e}^{-sa}\big]. \tag{5.6}$$

Second,

$$\frac{\mathrm{d}}{\mathrm{d}s}\sum_{n\geq 0}\frac{s^n}{n!}\mathrm{ad}_a^n(b) = \sum_{n\geq 1}\frac{s^{n-1}}{(n-1)!}\mathrm{ad}_a\Big(\mathrm{ad}_a^{n-1}(b)\Big) = \Big[a, \sum_{n\geq 0}\frac{s^n}{n!}\mathrm{ad}_a^n(b)\Big]. \tag{5.7}$$

□

PROOF OF PROPOSITION 5.3. Let $a \in \mathcal{B}_Y$ for some $Y \Subset \mathbb{Z}^d$. We show that $(\alpha_{\Lambda,t}^\Phi(a))_{\Lambda\Subset\mathbb{Z}^d}$ is Cauchy. By Lemma 5.4, we have

$$\begin{aligned}\alpha_{\Lambda,t}^\Phi(a) &= \sum_{n\geq 0}\frac{(\mathrm{i}t)^n}{n!}\mathrm{ad}_{H_\Lambda^\Phi}^n(a)\\ &= \sum_{n\geq 0}\frac{(\mathrm{i}t)^n}{n!}\sum_{X_1,\dots,X_n\subset\Lambda}[\Phi_{X_n}, [\Phi_{X_{n-1}}, \dots [\Phi_{X_1}, a]\dots]].\end{aligned} \tag{5.8}$$

We show that this series converges absolutely for small $|t|$. In order for the commutators to differ from zero, the sets must satisfy

$$\begin{aligned} & X_1 \cap Y \neq \emptyset, \\ & X_2 \cap (X_1 \cup Y) \neq \emptyset, \\ & \quad \vdots \\ & X_n \cap (X_1 \cup \dots \cup X_{n-1} \cup Y) \neq \emptyset. \end{aligned} \tag{5.9}$$

The sum over such sets can be realised by first summing over sets that contain the origin, then by summing over translations so that (5.9) is satisfied. One needs to divide by the cardinality of the set in order not to over-count. Then

$$\begin{aligned} \alpha^{\Phi}_{\Lambda,t}(a) = \sum_{n \geq 0} \frac{(\mathrm{i}t)^n}{n!} \sum_{X_1,\dots,X_n \ni 0} \Bigl(\prod_{i=1}^n \frac{1}{|X_i|} \Bigr) \\ \sum_{\substack{x_1,\dots,x_n \in \mathbb{Z}^d \\ X_i + x_i \subset \Lambda \; \forall i}} [\Phi_{X_n + x_n}, [\Phi_{X_{n-1}+x_{n-1}}, \dots [\Phi_{X_1+x_1}, a] \dots]]. \end{aligned} \tag{5.10}$$

For given $X_1, \dots, X_n$ there are no more than

$$\begin{aligned} & |Y| \cdot |X_1| \text{ possible choices for } x_1, \\ & \bigl(|Y| + |X_1|\bigr) \cdot |X_2| \text{ possible choices for } x_2, \\ & \quad \vdots \\ & \bigl(|Y| + |X_1| + \dots + |X_{n-1}|\bigr) \cdot |X_n| \text{ possible choices for } x_n. \end{aligned} \tag{5.11}$$

We get

$$\begin{aligned} & \Bigl\| \sum_{X_1,\dots,X_n} \Bigl(\prod_{i=1}^n \frac{1}{|X_i|} \Bigr) \sum_{x_1,\dots,x_n} [\Phi_{X_n+x_n}, \dots [\Phi_{X_1+x_1}, a] \dots] \Bigr\| \\ & \qquad \leq \|a\| 2^n \sum_{X_1,\dots,X_n \ni 0} \bigl(|X_1| + \dots + |X_n| + |Y| \bigr)^n \prod_{i=1}^n \|\Phi_{X_i}\| \\ & \qquad \leq \|a\| \, \mathrm{e}^{r|Y|} \, n! \Bigl(\frac{2\|\Phi\|_r}{r} \Bigr)^n. \end{aligned} \tag{5.12}$$

We used $c^n \leq n! r^{-n} \, \mathrm{e}^{rc}$, which is obvious from the Taylor series of e^{rc}. The factor 2^n is due to the n commutators. It follows that $\alpha^{\Phi}_{\Lambda,t}(a)$ is absolutely convergent whenever $|t| < \frac{r}{2\|\Phi\|_r}$ if $\|\Phi\|_r < \infty$. Notice the bound

$$\|\alpha^{\Phi}_{\Lambda,t}(a)\| \leq \|a\| \, \mathrm{e}^{r|Y|} \Bigl(1 - |t| \frac{2\|\Phi\|_r}{r} \Bigr)^{-1}. \tag{5.13}$$

for all $a \in \mathcal{A}_Y$. It is uniform in Λ but not in Y.

If $\Lambda' \supset \Lambda$, we have

$$\alpha^\Phi_{\Lambda',t}(a) - \alpha^\Phi_{\Lambda,t}(a) = \sum_{n\geq 0} \frac{(\mathrm{i}t)^n}{n!} \sum_{\substack{X_1,\dots,X_n:Y \\ \cup X_i \not\subset \Lambda}} [\Phi_{X_n}, [\Phi_{X_{n-1}}, \dots [\Phi_{X_1}, a] \dots]]. \tag{5.14}$$

The second sum is over sets in Λ' that satisfy the constraint (5.9) and whose union is not contained in Λ. For small $|t|$, it follows from the absolute convergence of the series that (5.14) is as small as we want by taking Λ large enough. Hence $(\alpha^\Phi_{\Lambda,t}(a))_\Lambda$ is Cauchy, and it converges since $\mathcal{A}$ is complete. We define $\alpha^\Phi_t(a)$ to be equal to the limit.

The map α^Φ_t is clearly an automorphism and it satisfies $\alpha^\Phi_t(a^*) = \alpha^\Phi_t(a)^*$.

For $t \in \mathbb{R}$ we have $\|\alpha^\Phi_{\Lambda,t}\| = 1$ which extends to α^Φ_t. Now assume that $\|\alpha^\Phi_{\Lambda,s} - \alpha^\Phi_s\| \to 0$ and $\|\alpha^\Phi_{\Lambda,t} - \alpha^\Phi_t\| \to 0$; we define $\alpha^\Phi_{s+t} = \alpha^\Phi_s \circ \alpha^\Phi_t$ and we have, for all $\|a\| = 1$,

$$\begin{aligned} \|\alpha^\Phi_{\Lambda,s+t}(a) - \alpha^\Phi_{s+t}(a)\| &\leq \|\alpha^\Phi_{\Lambda,s}\big(\alpha^\Phi_{\Lambda,t}(a) - \alpha^\Phi_t(a)\big)\| + \|(\alpha^\Phi_{\Lambda,s} - \alpha^\Phi_s)(\alpha^\Phi_t(a))\| \\ &\leq \|\alpha^\Phi_{\Lambda,t} - \alpha^\Phi_t\| + \|\alpha^\Phi_{\Lambda,s} - \alpha^\Phi_s\|, \end{aligned} \tag{5.15}$$

which goes to 0 as $\Lambda \Uparrow \mathbb{Z}^d$. This allows to extend α^Φ_t to the whole real line. Finally, if $z = t + \mathrm{i}\beta$ with $|\beta| < \frac{r}{2\|\Phi\|_r}$, we have

$$\alpha^\Phi_{\Lambda,t+\mathrm{i}\beta}(a) = \alpha^\Phi_{\Lambda,t}\big(\alpha^\Phi_{\Lambda,\mathrm{i}\beta}(a)\big) \to \alpha^\Phi_t\big(\alpha^\Phi_{\mathrm{i}\beta}(a)\big). \tag{5.16}$$

This allows to define $\alpha^\Phi_{t+\mathrm{i}\beta} = \alpha^\Phi_t \circ \alpha^\Phi_{\mathrm{i}\beta}$. □

5.3. The KMS condition

One would like to state the KMS condition of Definition 5.1 with the help of the infinite-volume evolution operator α^Φ_t to operators of $\mathcal{A}$. The "time" needs to be $t = \mathrm{i}$. If the norm of the interaction is small, namely if $\|\Phi\|_r < \frac{r}{2}$ for some r, the evolution operator is bounded, is defined on the whole of $\mathcal{A}$, and it is possible to generalise Definition 5.1 in a straightforward fashion. But the case of strong interactions, which corresponds to low temperatures, is especially interesting and justifies more efforts to extend the KMS condition.

We rely on complex analysis. More precisely, we rely on an extension of complex analysis that involves maps from $\mathbb{C}$ to an algebra of bounded operators. Convergent series can be defined in the same way, so the notion of analytic functions and their extensions still makes sense.

Consider a function f on $\mathbb{R}$ whose Fourier transform $\widehat{f}$ belongs to $C^\infty_{\mathrm{c}}(\mathbb{R})$ (the space of smooth functions with compact support). Using the inverse Fourier transform, we have

$$f(t) = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} \mathrm{e}^{\mathrm{i}\xi t}\, \widehat{f}(\xi) \mathrm{d}\xi. \tag{5.17}$$

It is clear that f is analytic in the whole complex plane (it is "entire"). Further, $f(t+\mathrm{i}s)$ is a Schwartz function of t for fixed s. We use such functions to identify a suitable class of operators. Namely, given $a \in \mathcal{A}$, let

$$a_f = \int_{\mathbb{R}} f(t)\alpha_t^{\Phi}(a)\mathrm{d}t. \tag{5.18}$$

If $|\mathrm{Im}\, z|$ is small we have

$$\alpha_z^{\Phi}(a_f) = \int_{\mathbb{R}} f(t)\alpha_{t+z}^{\Phi}(a)\mathrm{d}t = \int_{\mathbb{R}-\mathrm{i}\mathrm{Im}\, z} f(t-z)\alpha_t^{\Phi}(a)\mathrm{d}t = \int_{\mathbb{R}} f(t-z)\alpha_t^{\Phi}(a)\mathrm{d}t. \tag{5.19}$$

As an operator-valued function of z, the last expression is entire and we can use it to define $\alpha_z^{\Phi}(a_f)$ for all $z \in \mathbb{C}$. We let $\tilde{\mathcal{A}}$ denote the following dense subspace of $\mathcal{A}$:

$$\tilde{\mathcal{A}} = \big\{a_f : a \in \mathcal{A}, \widehat{f} \in C_{\mathrm{c}}^{\infty}(\mathbb{R})\big\}. \tag{5.20}$$

Definition 5.5. *We say that the state $\langle\cdot\rangle$ on $\mathcal{A}$ satisfies* **the KMS condition** *for the interaction Φ if either of the two following conditions is satisfied:*

(a) *For all $a \in \tilde{\mathcal{A}}$ and all $b \in \mathcal{A}$, we have*

$$\langle ab\rangle = \langle b\alpha_{\mathrm{i}}^{\Phi}(a)\rangle.$$

Equivalently, we have for all $a, b \in \mathcal{A}$, and all functions f such that $\widehat{f} \in C_{\mathrm{c}}^{\infty}$, that

$$\int_{\mathbb{R}} f(t)\langle\alpha_t^{\Phi}(a)b\rangle\mathrm{d}t = \int_{\mathbb{R}} f(t-\mathrm{i})\langle b\,\alpha_t^{\Phi}(a)\rangle\mathrm{d}t. \tag{5.21}$$

(b) *For each $a, b \in \mathcal{A}$ there is a function $F_{a,b}(z)$ that is analytic in the interior of the strip $0 \le \mathrm{Im}\, z \le 1$, and that is continuous and bounded on the whole strip. Moreover, we have for all $t \in \mathbb{R}$ that*

$$\langle a\,\alpha_t^{\Phi}(b)\rangle = F_{a,b}(t), \qquad \langle\alpha_t^{\Phi}(b)a\rangle = F_{a,b}(t+\mathrm{i}).$$

Proposition 5.6. *Both definitions above are equivalent.*

Proof. We first show that (a) implies (b). If $a \in \mathcal{A}$ and $b_f \in \tilde{\mathcal{A}}$, we take $F_{a,b_f}(z) = \langle a\,\alpha_z^{\Phi}(b_f)\rangle$. Then $F_{a,b_f}(z)$ is indeed analytic and it satisfies the first identity by definition. As for the second identity, we have using (5.21):

$$\begin{aligned} F_{a,b_f}(t+\mathrm{i}) = \langle a\,\alpha^\Phi_{t+\mathrm{i}}(b_f)\rangle &= \int_\mathbb{R} f(s-t-\mathrm{i})\langle a\,\alpha^\Phi_s(b)\rangle \mathrm{d}s \\ &= \int_\mathbb{R} f(u-\mathrm{i})\langle a\,\alpha^\Phi_u(\alpha^\Phi_t(b))\rangle \mathrm{d}u = \int_\mathbb{R} f(u)\langle \alpha^\Phi_u(\alpha^\Phi_t(b))a\rangle \mathrm{d}u \\ &= \Big\langle \alpha^\Phi_t\Big(\int_\mathbb{R} f(u)\alpha^\Phi_u(b)\mathrm{d}u\Big)a\Big\rangle = \langle \alpha^\Phi_t(b_f)a\rangle. \end{aligned} \tag{5.22}$$

This extends to arbitrary $b \in \mathcal{A}$ by continuity. Indeed, if $b_{f_n} \to b$, then $\alpha^\Phi_t(b_{f_n}) \to \alpha^\Phi_t(b)$ and $\langle a\,\alpha^\Phi_t(b_{f_n})\rangle$, $\langle \alpha^\Phi_t(b_{f_n})a\rangle$ converge. Further, the functions $F_{a,b_{f_n}}(z)$ converge by the Pragmén-Lindelöf theorem.

We now show that (b) implies (a). If $a \in \mathcal{A}$ and $b \in \tilde{\mathcal{A}}$, the function $\langle a\,\alpha^\Phi_z(b)\rangle$ is entire. We have $\langle a\,\alpha^\Phi_z(b)\rangle = F_{a,b}(z)$ for all $z \in \mathbb{R}$. The identity is then also valid for z in the strip, so that

$$\langle a\,\alpha^\Phi_\mathrm{i}(b)\rangle = F_{a,b}(\mathrm{i}) = \langle ba\rangle. \tag{5.23}$$

□

Remark: If a state $\langle\cdot\rangle$ satisfies the KMS condition, then it is invariant under the time evolution given by α^Φ_t: $\langle \alpha^\Phi_t(a)\rangle = \langle a\rangle$ for all $t \in \mathbb{R}$. To see this, let $b \in \tilde{\mathcal{A}}$ and observe that the complex function $\langle \alpha^\Phi_z(b)\rangle$ is entire and bounded in the strip $0 \le \operatorname{Im} z \le 1$. The function $F_{\mathbb{1},b}(z)$ in the KMS condition (b) must be equal to $\langle \alpha^\Phi_z(b)\rangle$. Further, the KMS condition states that $F_{\mathbb{1},b}(t+\mathrm{i}) = F_{\mathbb{1},b}(t)$, so this function is periodic in the imaginary direction. Then $F_{\mathbb{1},b}(z)$ is bounded in the whole complex plane, and is therefore constant by Liouville's theorem. Then $\langle \alpha^\Phi_t(b)\rangle = F_{\mathbb{1},b}(t)$ is constant.

5.4. Equivalence with tangent functional states

PROPOSITION 5.7. *Assume that $\langle\cdot\rangle$ satisfies the tangent functional property at $\Phi \in \mathcal{I}$ (Definition 3.10). Then $\langle\cdot\rangle$ satisfies the KMS condition.*

It is possible to prove a converse statement, namely that translation invariant KMS states satisfy the variational principle (and are therefore tangent to the pressure); see Araki [1974].

PROOF. Given $a, b \in \mathcal{A}_\mathrm{loc}$ and f such that $\widehat{f} \in C_\mathrm{c}(\mathbb{R})$, let us introduce the following quasi-local operator:

$$F = \operatorname{Re} \int_\mathbb{R} \big[f(t-\mathrm{i})a\alpha^\Phi_t(b) - f(t)\alpha^\Phi_t(b)a\big]\mathrm{d}t. \tag{5.24}$$

The symbol Re means that we consider the self-adjoint part of the operator, $\operatorname{Re} a = \frac{1}{2}(a+a^*)$. We prove that $\langle F\rangle = 0$ for any a, b. Replacing a by $\mathrm{i}a$, we get that $\langle \mathrm{Im}\, F\rangle = 0$ as well, so the expectation of the original operator also vanishes. This gives the equivalent KMS condition (5.21).

We approximate F by a local observable; with Λ a finite domain that contains the supports of a, b, let

$$F_\Lambda = \mathrm{Re} \int_{\mathbb{R}} \big[f(t-\mathrm{i}) a \alpha^\Phi_{\Lambda,t}(b) - f(t) \alpha^\Phi_{\Lambda,t}(b) a \big] \mathrm{d}t. \tag{5.25}$$

It is clear that $\|F - F_\Lambda\| \to 0$ as $\Lambda \Subset \mathbb{Z}^d$. We prove that $\langle F_\Lambda\rangle$ is arbitrarily small. Let Ψ_{F_Λ} be the interaction that corresponds to the local observable F_Λ, see (3.12). From the tangent functional property, we have for $\lambda \in \mathbb{R}$ that

$$p(\Phi + \lambda \Psi_{F_\Lambda}) \geq p(\Phi) - \lambda \langle F_\Lambda \rangle. \tag{5.26}$$

(We used translation invariance so that $\langle a_{\Psi_{F_\Lambda}}\rangle = \langle F_\Lambda\rangle$.) Then

$$-\lambda \langle F_\Lambda\rangle \leq p(\Phi + \lambda\Psi_{F_\Lambda}) - p(\Phi). \tag{5.27}$$

The right side is clearly less than const$|\lambda|$, but we need to show that the preconstant is arbitrarily small. This may well be true since the finite volume pressure satisfies

$$\frac{\mathrm{d}}{\mathrm{d}\lambda} p_\Lambda(\Phi + \lambda \Psi_{F_\Lambda}) \Big|_{\lambda=0} = \langle F_\Lambda \rangle_\Lambda, \tag{5.28}$$

which is close to 0 since the finite volume Gibbs state satisfies the KMS condition. Then $p(\Phi + \lambda\Psi_{F_\Lambda}) - p(\Phi) = O(\lambda^2)$. But this may be incorrect as the pressure is not necessarily differentiable. To overcome this difficulty we rely on the finite volume tangent functional property. We also work with T_n, the box of size n with periodic boundary conditions, so we can use translation invariance. We have

$$p_{T_n}(\underbrace{\Phi + \lambda\Psi_{F_\Lambda} - \lambda\Psi_{F_\Lambda}}_{=\Phi}) \geq p_{T_n}(\Phi + \lambda\Psi_{F_\Lambda}) + \lambda \langle F_\Lambda \rangle^{\Phi+\lambda\Psi_{F_\Lambda}}_{T_n}. \tag{5.29}$$

We now consider another modification of the operator F, namely

$$F_{\Lambda,T_n,\lambda} = \mathrm{Re} \int_{\mathbb{R}} \big[f(t-\mathrm{i}) a \alpha^{\Phi+\lambda\Psi_{F_\Lambda}}_{T_n,t}(b) - f(t) \alpha^{\Phi+\lambda\Psi_{F_\Lambda}}_{T_n,t}(b) a \big] \mathrm{d}t. \tag{5.30}$$

The motivation behind this operator is that $\langle F_{\Lambda,T_n,\lambda}\rangle^{\Phi+\lambda\Psi_{F_\Lambda}}_{T_n} = 0$ by the KMS condition in finite volume. Since $\|\Psi_{F_\Lambda}\|_r = \mathrm{const}\, \mathrm{e}^{r|\Lambda|}$ we have

$$\begin{aligned} \|F_{\Lambda,T_n,\lambda} - F_{\Lambda,T_n,0}\| &\leq \mathrm{const} |\lambda|\, \mathrm{e}^{r|\Lambda|}, \\ \|F_{\Lambda,T_n,0} - F_\Lambda\| &\leq \mathrm{const}\, \mathrm{e}^{-\mathrm{const}\, \mathrm{dist}(0,\Lambda^c)}. \end{aligned} \tag{5.31}$$

The constants depend on a, b but not on Λ, n, λ. Together with (5.29), we get

$$p_{T_n}(\Phi + \lambda\Psi_{F_\Lambda}) - p_{T_n}(\Phi) \leq \mathrm{const}\, \lambda^2\, \mathrm{e}^{r|\Lambda|} + \mathrm{const} |\lambda|\, \mathrm{e}^{-\mathrm{const}\, \mathrm{dist}(0,\Lambda^c)} \tag{5.32}$$

for all T_n large enough. Taking $n \to \infty$ and using (5.27), we get that for all $\lambda \in \mathbb{R}$,

$$-\lambda \langle F_\Lambda \rangle \leq \mathrm{const}\, \lambda^2\, \mathrm{e}^{r|\Lambda|} + \mathrm{const} |\lambda|\, \mathrm{e}^{-\mathrm{const}\, \mathrm{dist}(0,\Lambda^c)}. \tag{5.33}$$

Then

$$|\langle F_\Lambda \rangle| \leq \text{const}|\lambda|\, \mathrm{e}^{r|\Lambda|} + \text{const}\, \mathrm{e}^{-\text{const}\,\text{dist}(0,\Lambda^c)}\,. \tag{5.34}$$

We can take λ arbitrarily small and Λ arbitrarily large, so that $\langle F \rangle = 0$. □

5.5. The classical KMS condition

Discrete classical systems, such as the Ising model, can be viewed as a special case of quantum systems, where the interactions involve operators that are diagonal in a given basis. The states of classical systems are given by measures and the usual definition of equilibrium state involves the "DLR condition", named after Dobrushin, Lanford, and Ruelle. A natural question is the relation between DLR and KMS in the case where the system is classical. As it turns out, the conditions are equivalent.

We start with the choice of a suitable basis. Let Ω_0 be a discrete set and assume that $\{|\omega_0\rangle : \omega_0 \in \Omega_0\}$ is a basis for $\mathcal{H}_0$. This gives a basis for $\mathcal{H}_\Lambda = \otimes_{x\in\Lambda}\mathcal{H}_x$, namely

$$\big\{|\omega_\Lambda\rangle = \otimes_{x\in\Lambda}|\omega_x\rangle : \omega_\Lambda \in \Omega_0^\Lambda\big\}. \tag{5.35}$$

This is the basis of "classical configurations". A **classical interaction** is a collection of local operators $\Phi = (\Phi_X)_{X \Subset \mathbb{Z}^d}$ such that each Φ_X is diagonal in the basis of classical configurations.

Among the many simplifications, the evolution operator exists more generally. For $\Lambda \Subset \mathbb{Z}^d$, let us introduce the local operator

$$K_\Lambda^\Phi = \sum_{X : X\cap\Lambda\neq\emptyset} \Phi_X. \tag{5.36}$$

The difference with H_Λ^Φ is that K_Λ^Φ also involves the sets X that are partially located outside Λ. It is not hard to check that the sum over X is absolutely convergent for $\Phi \in \mathcal{I}$.

LEMMA 5.8. *If $\Phi \in \mathcal{I}$ is a classical interaction, there exists an evolution operator $\alpha_t^\Phi : \mathcal{A} \to \mathcal{A}$ such that for all $\Lambda \Subset \mathbb{Z}^d$ and all $a \in \mathcal{A}_\Lambda$, we have*

$$\alpha_t^\Phi(a) = \mathrm{e}^{\mathrm{i}tK_\Lambda^\Phi}\, a\, \mathrm{e}^{-\mathrm{i}tK_\Lambda^\Phi}\,.$$

Here the parameter t can be complex.

PROOF. For $a \in \mathcal{A}_\Lambda$ one would like to consider $\alpha_{\Lambda',t}^\Phi(a) = \mathrm{e}^{\mathrm{i}tH_{\Lambda'}^\Phi}\, a\, \mathrm{e}^{-\mathrm{i}H_{\Lambda'}^\Phi}$ and take the limit $\Lambda' \Uparrow \mathbb{Z}^d$. Let

$$K_{\Lambda,\Lambda'}^\Phi = \sum_{\substack{X\subset\Lambda' \\ X\cap\Lambda\neq\emptyset}} \Phi_X. \tag{5.37}$$

Observe that $\alpha_{\Lambda',t}^\Phi(a) = \mathrm{e}^{\mathrm{i}tK_{\Lambda,\Lambda'}^\Phi}\, a\, \mathrm{e}^{-\mathrm{i}tK_{\Lambda,\Lambda'}^\Phi}$ and that $\|K_{\Lambda,\Lambda'}^\Phi - K_\Lambda^\Phi\| \to 0$ as $\Lambda' \Uparrow \mathbb{Z}^d$. Then the evolution operator indeed converges. Moreover, one can check that it is consistent, namely that if $a \in \mathcal{A}_\Lambda$ and $a' \in \mathcal{A}_{\Lambda'}$ are such that $a' = a \otimes 1\!\!1_{\Lambda' \setminus \Lambda}$, then $\alpha_t^\Phi(a) = \alpha_t^\Phi(a')$. □

Let us recall the notion of DLR conditions. For diagonal operators such as H_Λ^Φ, we write $H_\Lambda^\Phi(\omega_\Lambda)$ instead of $\langle \omega_\Lambda | H_\Lambda^\Phi | \omega_\Lambda \rangle$. If $\Lambda \subset \Lambda' \Subset \mathbb{Z}^d$, the classical Gibbs measure in Λ' satisfies

$$\mu_{\Lambda'}(\sigma_\Lambda \omega_{\Lambda' \setminus \Lambda}) = \frac{1}{Z_{\Lambda'}} \, \mathrm{e}^{-K_\Lambda^\Phi(\sigma_\Lambda \omega_{\Lambda' \setminus \Lambda}) - H_{\Lambda' \setminus \Lambda}^\Phi(\omega_{\Lambda' \setminus \Lambda})}, \tag{5.38}$$

for all $\sigma_\Lambda \in \Omega_0^\Lambda$ and $\omega_{\Lambda' \setminus \Lambda} \in \Omega_0^{\Lambda' \setminus \Lambda}$. Then $\mathrm{e}^{K_\Lambda^\Phi(\sigma_\Lambda \omega_{\Lambda' \setminus \Lambda})} \, \mu_{\Lambda'}(\sigma_\Lambda \omega_{\Lambda' \setminus \Lambda})$ does not depend on σ_Λ. This is equivalent to stating that for all functions that depend on spins on $\Lambda' \setminus \Lambda$, we have that

$$\sum_{\omega_{\Lambda' \setminus \Lambda} \in \Omega_0^{\Lambda' \setminus \Lambda}} f(\omega_{\Lambda' \setminus \Lambda}) \, \mathrm{e}^{K_\Lambda^\Phi(\sigma_\Lambda \omega_{\Lambda' \setminus \Lambda})} \, \mu_{\Lambda'}(\sigma_\Lambda \omega_{\Lambda' \setminus \Lambda}) \tag{5.39}$$

is independent of σ_Λ. One can take the limit $\Lambda' \Uparrow \mathbb{Z}^d$ above. Then we rewrite this with the notation of quantum statistical mechanics. With $P_{\sigma_\Lambda} = |\sigma_\Lambda\rangle\langle\sigma_\Lambda|$, this gives

$$\langle P_{\sigma_\Lambda} f \, \mathrm{e}^{K_\Lambda^\Phi} \rangle = \langle P_{\sigma'_\Lambda} f \, \mathrm{e}^{K_\Lambda^\Phi} \rangle \tag{5.40}$$

for all $\sigma_\Lambda, \sigma'_\Lambda \in \Omega_0^\Lambda$. These are the **DLR equations**. We also call a state "classical" if it is assigns the value 0 to all off-diagonal operators. That is, $\langle \cdot \rangle$ is a classical state if

$$\langle P_{\sigma_\Lambda, \sigma'_\Lambda} \rangle = 0 \tag{5.41}$$

for all $\Lambda \Subset \mathbb{Z}^d$ and all $\sigma_\Lambda \neq \sigma'_\Lambda$ in Ω_0^Λ. Here, we defined

$$P_{\sigma_\Lambda, \sigma'_\Lambda} = |\sigma_\Lambda\rangle\langle\sigma'_\Lambda|; \tag{5.42}$$

notice that $P_{\sigma_\Lambda, \sigma_\Lambda} = P_{\sigma_\Lambda}$.

THEOREM 5.9. *Let $\Phi \in \mathcal{I}$ be a classical interaction. Then if a state $\langle \cdot \rangle$ satisfies the KMS condition, it is classical and it satisfies the DLR equations (5.40).*

PROOF. We first show that the KMS state $\langle \cdot \rangle$ is classical. Using the KMS condition we have

$$\begin{aligned} \langle P_{\sigma_\Lambda, \sigma'_\Lambda} \rangle &= \langle P_{\sigma_\Lambda, \sigma'_\Lambda} P_{\sigma'_\Lambda} \rangle = \langle \alpha_{-\mathrm{i}}^\Phi(P_{\sigma'_\Lambda}) P_{\sigma_\Lambda, \sigma'_\Lambda} \rangle \\ &= \langle \, \mathrm{e}^{K_\Lambda^\phi} \, P_{\sigma'_\Lambda} \, \mathrm{e}^{-K_\Lambda^\Phi} \, P_{\sigma_\Lambda, \sigma'_\Lambda} \rangle = \langle P_{\sigma'_\Lambda} P_{\sigma_\Lambda, \sigma'_\Lambda} \rangle = 0 \end{aligned} \tag{5.43}$$

since $[K_\Lambda^\Phi, P_{\sigma'_\Lambda}] = 0$. Next, if $f \in \mathcal{A}_{\Lambda'}$ with $\Lambda' \cap \Lambda = \emptyset$,

$$\begin{aligned} \langle \, \mathrm{e}^{K_\Lambda^\Phi} \, f P_{\sigma_\Lambda} \rangle &= \langle \, \mathrm{e}^{K_\Lambda^\Phi} \, f P_{\sigma_\Lambda, \sigma'_\Lambda} P_{\sigma'_\Lambda, \sigma_\Lambda} \rangle = \langle \alpha_{-\mathrm{i}}(P_{\sigma'_\Lambda, \sigma_\Lambda}) \, \mathrm{e}^{K_\Lambda^\Phi} \, f P_{\sigma_\Lambda, \sigma'_\Lambda} \rangle \\ &= \langle \, \mathrm{e}^{K_\Lambda^\Phi} \, P_{\sigma'_\Lambda, \sigma_\Lambda} f P_{\sigma_\Lambda, \sigma'_\Lambda} \rangle = \langle \, \mathrm{e}^{K_\Lambda^\Phi} \, f P_{\sigma'_\Lambda} \rangle. \end{aligned} \tag{5.44}$$

□

BIBLIOGRAPHICAL REFERENCES

Proposition 5.7 was first proved by Lanford and Robinson [1968] using indirect methods. The proof here is due to Israel [1979]. Equivalence between DLR and KMS was noticed by Brascamp [1970].