CHAPTER 9

The Ising model

9.1. The classical Ising model

9.2. The Ising regime of the xxz-model

The XXZ-model has hamiltonian which is usually written

$$H_\Lambda = -\sum_{xy\in\mathcal{E}(\Lambda)} \big(S_x^{(1)}S_y^{(1)} + S_x^{(2)}S_y^{(2)} + \Delta S_x^{(3)}S_y^{(3)}\big) \tag{9.1}$$

where $\Delta \in \mathbb{R}$. For simplicity, we assume that $\Lambda = \{-n, -n+1, \dots, n\}^d$ is a box of side-length $2n+1$. We restrict here to the case of spin $S = \frac{1}{2}$, and as basis for the Hilbert space $\mathcal{H}_\Lambda = (\mathbb{C})^{\otimes\Lambda}$ we use the product basis $\{|\sigma\rangle : \sigma \in \{-1,+1\}^\Lambda\}$.

The *Ising regime* of this model is when $|\Delta| > 1$. Then, the interactions in the third direction of spin are stronger than the other two directions. At low enough temperature, one may expect the $S^{(3)}$-interaction to be dominant. Moreover, if $\Delta > 1$ the model has ferromagnetic properties, respectively antiferromagnetic if $\Delta < -1$. These claims, as well as the term *Ising regime*, are justified in the following:

EXERCISE 9.1. *Show that the ground state of the model (9.1) with $\Delta > 1$ is the projector onto the subspace spanned by the constant vectors $|+\rangle$ and $|-\rangle$. Similarly, when $\Delta < -1$ show that the ground state is the projection onto the subspace associated with the antiferromagnetic classical Ising model, i.e. $|\sigma\rangle$ where $\sigma_x = (-1)^{\|x\|_1}$ or $\sigma_x = -(-1)^{\|x\|_1}$.*

The goal of this section is to show that the model (9.1) with $\Delta > 1$ behaves like a ferromagnetic Ising model also at positive (low) temperature. Specifically, we will prove the following theorem due to Tom Kennedy:

THEOREM 9.1. *Consider the model (9.1) with $\Delta > 1$ and spin $\frac{1}{2}$, for fixed $d \geq 2$. For each $\Delta > 1$, there is $\beta_c(\Delta) < \infty$ such that for each $\beta > \beta_c(\Delta)$ and some $c(\beta,\delta) > 0$*

$$\liminf_{n\to\infty} \langle S_x^{(3)}S_y^{(3)}\rangle_\Lambda \geq c, \qquad \textit{for all } x, y \in \mathbb{Z}^d. \tag{9.2}$$

In other words, at low enough temperature the model exhibits long-range order in the $S^{(3)}$ spin direction.

The proof uses a so-called *Peierls argument*. The goal is to express deviations from the ground-state vectors $|+\rangle$ or $|-\rangle$ in terms of *contours* separating $+$ and $-$ entries, and to show that these contours are 'costly' when β is large. Readers familiar with the classical Ising model will recognise this reasoning, which goes back to a ground-breaking paper of Rudolf Peierls from 1936. The starting-point for the quantum model (9.1) is to write the Gibbs factor $\mathrm{e}^{-\beta H}$ as a sequence of classical Ising models evolving over time. We are then faced with the task of controlling an *evolving* sequence of Peierls contours.

Before diving into the proof, we will spend some time discussing how to express $\mathrm{e}^{-\beta H}$ in terms of evolving Ising-models. The technique, which relies on the Lie–Trotter expansion, is frequently used in studying quantum spin systems.

9.2.1. Lie–Trotter expansion. Rather than working with the hamiltonian in the form (9.1), it will be convenient to introduce the parameter $\delta = 1 - \frac{1}{\Delta} \in (0,1)$ and consider the hamiltonian

$$H_\Lambda = -2 \sum_{xy \in \mathcal{E}(\Lambda)} \big[(1-\delta) S_x^{(1)} S_y^{(1)} + (1-\delta) S_x^{(2)} S_y^{(2)} + S_x^{(3)} S_y^{(3)} - \tfrac{1}{4}\big]. \tag{9.3}$$

Up to an additive constant (which does not change the Gibbs states) and a rescaling of the inverse-temperature ($\beta \leftrightarrow 2(1-\delta)\beta$) this does not change the model. Next, instead of the spin-matrices we use the Pauli matrices $\sigma^{(i)} = 2S^{(i)}$. We may then write (9.3) as

$$H = H^{\mathrm{Z}} + (1-\delta) H^{\mathrm{XX}} \tag{9.4}$$

where

$$H^{\mathrm{Z}} = \sum_{xy \in \mathcal{E}(\Lambda)} \tfrac{1}{2}(1 - \sigma_x^{(3)} \sigma_y^{(3)}) \tag{9.5}$$

and (recalling that $\sigma^{\pm} = \frac{1}{2}(\sigma^{(1)} \pm \mathrm{i}\sigma^{(2)})$)

$$H^{\mathrm{XX}} = -\sum_{xy \in \mathcal{E}(\Lambda)} \tfrac{1}{2}\big(\sigma_x^{(1)}\sigma_y^{(1)} + \sigma_x^{(2)}\sigma_y^{(2)}\big) = -\sum_{xy \in \mathcal{E}(\Lambda)} \big(\sigma_x^+ \sigma_y^- + \sigma_x^- \sigma_y^+\big). \tag{9.6}$$

The form (9.4) emphasises that we view our model as a perturbation of the classical Ising model, which is recovered in the case $\delta = 1$. Note that we have dispensed with the subscript Λ; indeed we will keep Λ fixed for the time being.

We now introduce an expansion for the exponential of the sum of two non-commuting matrices, which will then allow us to write the Gibbs factor $\mathrm{e}^{-\beta H}$ as a sequence of classical Ising models.

Proposition 9.2 (Lie–Trotter formula). *Let a, b be $n \times n$ matrices. Then*

$$\mathrm{e}^{a+b} = \lim_{N\to\infty} \Big(\mathrm{e}^{\frac{1}{N}a}\, \mathrm{e}^{\frac{1}{N}b}\Big)^N = \lim_{N\to\infty} \Big[\mathrm{e}^{\frac{1}{N}a}\big(1 + \tfrac{1}{N}b\big)\Big]^N.$$

PROOF. We prove the second formula — the mild changes for the other formula are straightforward. Let K_N be the matrix such that

$$e^{\frac{1}{N}a}\left(1+\tfrac{1}{N}b\right)=1+\tfrac{1}{N}(a+b)+K_N. \tag{9.7}$$

It is clear that $\|K_N\|=O(\frac{1}{N^2})$. We have

$$\left[e^{\frac{1}{N}a}\left(1+\tfrac{1}{N}b\right)\right]^N=\left(1+\tfrac{1}{N}(a+b)\right)^N+R_N, \tag{9.8}$$

where R_N is a matrix whose norm satisfies

$$\|R_N\|\le\sum_{k=0}^{N-1}\binom{N}{k}\|1+\tfrac{1}{N}(a+b)\|^k\,\|K_N\|^{N-k}=O(\tfrac{1}{N}). \tag{9.9}$$

The first term in the right side of (9.8) converges to e^{a+b}. □

Using Proposition 9.2 with $a=-\beta H^{\mathrm{z}}$ and $b=-\beta(1-\delta)H^{\mathrm{xx}}$ we obtain:

$$e^{-\beta H}=\lim_{N\to\infty}\left[e^{-\frac{\beta}{N}H^{\mathrm{z}}}\left(1-\tfrac{\beta(1-\delta)}{N}H^{\mathrm{xx}}\right)\right]^N. \tag{9.10}$$

One of the ways in which we will use the expansion (9.10) is to rewrite the partition function $Z=\mathrm{Tr}\,e^{-\beta H}$. First note that, for finite Λ, the trace is a finite sum over the basis vectors for $\mathcal{H}_\Lambda$:

$$\mathrm{Tr}\,e^{-\beta H}=\sum_\sigma\left(e^{-\beta H}\right)_{\sigma,\sigma}=\sum_\sigma\langle\sigma|\,e^{-\beta H}\,|\sigma\rangle. \tag{9.11}$$

A finite sum can always be interchanged with a limit. Consider therefore a fixed N and the expression

$$\langle\sigma|\left[e^{-\frac{\beta}{N}H^{\mathrm{z}}}\left(1-\tfrac{\beta(1-\delta)}{N}H^{\mathrm{xx}}\right)\right]^N|\sigma\rangle. \tag{9.12}$$

First note that

$$e^{-\frac{\beta}{N}H^{\mathrm{z}}}\left(1-\tfrac{\beta(1-\delta)}{N}H^{\mathrm{xx}}\right)|\sigma\rangle=e^{-\frac{\beta}{N}H^{\mathrm{z}}}\,|\sigma\rangle+\sum_{xy\in\mathcal{E}(\Lambda)}(\sigma_x^+\sigma_y^-+\sigma_x^-\sigma_y^+)|\sigma\rangle. \tag{9.13}$$

Let us write $H^{\mathrm{z}}(\sigma)$ for the *function*

$$H^{\mathrm{z}}(\sigma)=\tfrac{1}{2}\#\{xy\in\mathcal{E}(\Lambda):\sigma_x\neq\sigma_y\}. \tag{9.14}$$

Since H^{z} is diagonal we have

$$e^{-\frac{\beta}{N}H^{\mathrm{z}}}\,|\sigma\rangle=e^{-\frac{\beta}{N}H^{\mathrm{z}}(\sigma)}\,|\sigma\rangle. \tag{9.15}$$

Next, $\sigma_x^+\sigma_y^-|\sigma\rangle$ is $=0$ unless $\sigma_x=-1$ and $\sigma_y=+1$, in which case it is obtained from $|\sigma\rangle$ by swapping the entries at x and y (and similarly for $\sigma_x^-\sigma_y^+|\sigma\rangle$).

To see what this gives in (9.12), we use the usual rules for matrix-multiplication (alternatively, the identity $\mathbb{1} = \sum_\tau |\tau\rangle\langle\tau|$) to write

$$\langle\sigma|\Big[\,\mathrm{e}^{-\frac{\beta}{N}H^{\mathrm{z}}}\big(1-\tfrac{\beta(1-\delta)}{N}H^{\mathrm{xx}}\big)\Big]^2|\sigma\rangle$$
$$= \sum_{\tau\in\{-1,+1\}^\Lambda} \langle\sigma|\,\mathrm{e}^{-\frac{\beta}{N}H^{\mathrm{z}}}\big(1-\tfrac{\beta(1-\delta)}{N}H^{\mathrm{xx}}\big)|\tau\rangle\langle\tau|\,\mathrm{e}^{-\frac{\beta}{N}H^{\mathrm{z}}}\big(1-\tfrac{\beta(1-\delta)}{N}H^{\mathrm{xx}}\big)|\sigma\rangle. \tag{9.16}$$

Repeating this leads to the following way of describing (9.12): we get a weighted sum over all N-step trajectories $\sigma = \tau^0, \tau^1, \dots, \tau^{N-1}, \tau^N = \sigma$ of classical Ising-configurations τ^i, starting and ending at σ, such that at each step

- either $\tau^i = \tau^{i-1}$, in which case we pick up a factor $\mathrm{e}^{-\frac{\beta}{N}H^{\mathrm{z}}(\tau^i)}$;
- or τ^i is obtained from τ^{i-1} by swapping *neighbouring* entries + and −, in which case we pick up the factor $\mathrm{e}^{-\frac{\beta}{N}H^{\mathrm{z}}(\tau^i)}\big(\frac{\beta(1-\delta)}{N}\big)$.

To summarize:

$$Z = \lim_{N\to\infty}\sum_{\tau:\sigma\to\sigma} \mathrm{e}^{-\beta\overline{H}^{\mathrm{z}}(\tau)}\big(\tfrac{\beta(1-\delta)}{N}\big)^{n(\tau)}, \tag{9.17}$$

where the sum is over N-step trajectories τ as described above, $n(\tau)$ is the number of times $\tau^i \neq \tau^{i-1}$, and

$$\overline{H}^{\mathrm{z}}(\tau) = \frac{1}{N}\sum_{i=1}^N H^{\mathrm{z}}(\tau^i) = \frac{1}{N}\sum_{i=1}^N \tfrac{1}{2}\#\{xy\in\mathcal{E}(\Lambda) : \tau^i_x \neq \tau^i_y\} \tag{9.18}$$

is half the average number of edges xy where the neighbouring values differ.

It is worth noting that the total number of + and − entries in any τ^i appearing in the sum above is the same as in σ, indeed the operation of swapping neighbouring + and − entries conserves these numbers. In contrast, the number of edges where the values differ (and thus $H^{\mathrm{z}}(\tau^i)$) may change.

9.2.2. Contours and weights. The next step is to re-interpret the expression (9.17) in terms of *contours* separating + and − values in the configurations τ^i. This is easiest to visualise in two dimensions, so let us start with that case. For a box $\Lambda = \{-n, -n+1, \dots, n\}^2$ in $\mathbb{Z}^2$, we fix an embedding of it in $\mathbb{R}^2$, where the edges $e \in \mathcal{E}(\Lambda)$ are embedded as straight lines of unit length. To any edge $e \in \mathcal{E}(\Lambda)$ we associate a *dual edge* $e^\star$, which we think of as a straight line of unit length bisecting e perpendicularly. If e is not at the boundary of Λ, it separates two faces inside Λ, and $e^\star$ connects the centres of these two faces. If e is on the boundary, $e^\star$ connects the centre of one face to a point outside Λ. The usual way to define the dual of a planar graph in graph theory would involve connecting all the 'sticking out' edges; however, we do not do this here. In fact, we do not even need to define the dual graph properly, as we only work with the dual *edges* $e^\star$ as defined above.

In higher dimensions $d \geq 3$, we would proceed similarly except that the dual $e^\star$ of an edge e is no longer an edge but a *plaquette*, meaning a $(d-1)$-dimensional unit square perpendicularly bisected by e. In general, we define the *contour* $G(\sigma)$ associated with $\sigma \in \{-1,+1\}^\Lambda$ as the collection of plaquettes $e^\star$, where the edge $e = xy$ satisfies $\sigma_x \neq \sigma_y$. Note that $G(\sigma) = G(-\sigma)$, and conversely if $G(\sigma) = G(\tau)$ then $\tau = -\sigma$. Since the pair $\{\sigma, -\sigma\}$ can be recovered from the contour $G(\sigma)$, we may formally identify a contour G with the corresponding pair of configurations; more useful for us, however, is to identify it with the *sum* $G(\sigma) \equiv |\sigma\rangle + |-\sigma\rangle$. We also define

$$H^{\rm z}(G) = \#\{xy \in \mathcal{E}(\Lambda) : \sigma_x \neq \sigma_y\} = |G|, \tag{9.19}$$

the number of plaquettes in G. Note that this is a factor 2 larger than $H^{\rm z}(\sigma)$, as is consistent with our identification $G = |\sigma\rangle + |-\sigma\rangle$.

Let us now make the connection with (9.17). As the configurations τ^i evolve, so do the contours $G(\tau^i)$. We define a *quantum contour* $\Gamma = (\Gamma^0, \Gamma^1, \dots, \Gamma^N)$ as the sequence of contours $\Gamma^i = G(\tau^i) = |\tau^i\rangle + |-\tau^i\rangle$ associated with a sequence $\tau^0, \tau^1, \dots, \tau^N$ as appearing in (9.17). We define the *weight* of a quantum contour Γ as

$$w(\Gamma) = \, \mathrm{e}^{-\beta \overline{H}^{\rm z}(\Gamma)} \left(\tfrac{\beta(1-\delta)}{N}\right)^{n(\mathbf{\Gamma})}, \tag{9.20}$$

where $n(\Gamma) = n(\tau)$ is the number of 'flips' and

$$\overline{H}^{\rm z}(\Gamma) = \frac{1}{N} \sum_{i=1}^{N} H^{\rm z}(\Gamma^i) = \frac{1}{N} \sum_{i=1}^{N} |\Gamma^i| \tag{9.21}$$

is the average number of plaquettes in the contours Γ^i.

For now on, we focus on the case $d = 2$ where contours are easier to visualise. The extension to $d \geq 3$ is straightforward. Let us fix two vertices $x, y \in \Lambda$ (which we think of as being far apart). Given these two vertices, we define the *sign* of a contour G as $\mathrm{sgn}(G) = (-1)^{k(x,y)}$ where $k(x,y)$ is the number of edges of G traversed on a path from x to y in Λ. Although $k(x,y)$ is not well-defined (it depends on the path), $(-1)^{k(x,y)}$ is. Equivalently, one may define $\mathrm{sgn}(G)$ by the relation

$$\sigma_x^{(3)} \sigma_y^{(3)} G = \mathrm{sgn}(G) G, \qquad G = G(\sigma) = |\sigma\rangle + |-\sigma\rangle. \tag{9.22}$$

For a quantum contour $\Gamma = (\Gamma^0, \Gamma^1, \dots, \Gamma^N)$ we define $\mathrm{sgn}(\Gamma) = \mathrm{sgn}(\Gamma^0)$.

PROPOSITION 9.3.

$$Z = \mathrm{Tr}\, \mathrm{e}^{-\beta H} \, = \lim_{N\to\infty} 2 \sum_{\Gamma} w(\Gamma), \tag{9.23}$$

$$\mathrm{Tr}\,(1 - \sigma_x^{(3)} \sigma_y^{(3)})\, \mathrm{e}^{-\beta H} \, = \lim_{N\to\infty} 4 \sum_{\Gamma : \mathrm{sgn}(\Gamma) = -1} w(\Gamma). \tag{9.24}$$

PROOF. The formula (9.23) follows from (9.17) by reorganising the sum $\sum_{\sigma\in\{-1,+1\}^\Lambda}$ in the trace, specifically grouping each σ with $-\sigma$ so that we sum over $\Gamma^0 = |\sigma\rangle + |-\sigma\rangle$. The factor 2 appears because there are two configurations $\sigma, -\sigma$ for each choice of Γ^0 (or more explicitly, $(\langle\sigma| + \langle -\sigma|)(|\sigma\rangle + |-\sigma\rangle) = 2$). The formula (9.24) is similar, except that there is another factor $1 - \mathrm{sgn}(\Gamma^0) \in \{0, 2\}$ due to the factor $(1 - \sigma_x^{(3)}\sigma_y^{(3)})$. □

Next, we introduce the notions of *support* and *connectedness* of quantum contours. For each i, the contour Γ^i is a union of line segments which assemble into rectangular loops. As contours evolve, it will be useful to keep track of how they have changed. For that reason, we associate to each vertex $x \in \Lambda$ the unit square $\square(x) \subseteq \mathbb{R}^2$ centered at x and with sides parallell to the edges $\mathcal{E}(\Lambda)$ (as embedded in $\mathbb{R}^2$). If the $\pm$-value at x changes at some step in the evolution τ, we say that $\square(x)$ is a *flipped square.* We let $F(\Gamma)$ be the union of flipped squares, and define the *support* of Γ as

$$S(\Gamma) = F(\Gamma) \cup \Gamma^0 \subseteq \mathbb{R}^2, \tag{9.25}$$

which is then a union of squares and line segments in $\mathbb{R}^2$. Note that we have $S(\Gamma) = F(\Gamma) \cup \Gamma^i$ for any $0 \le i \le N$, as an edge $e^\star \in \Gamma^0$ either becomes part of a flipped square, or stays in Γ^i for all i. The size $|S(\Gamma)|$ of the support is defined as the number of squares plus the number of dual edges (not counting those that are part of a square).

We say that Γ is *connected* if $S(\Gamma)$ is connected as a subset of $\mathbb{R}^2$. Note that connectedness of a quantum contour Γ takes into account its whole evolution, and it is allowed that Γ is connected while any individual Γ^i is not. For notational convenience, we use lowercase γ to denote connected quantum contours. An arbitrary quantum contour Γ can be written as a union $\Gamma = \cup_{j=1}^k \gamma_j$ of connected quantum contours, which are disjoint subsets of $\mathbb{R}^2$. Note that the signs and weights then factorise as

$$\mathrm{sgn}(\Gamma) = \prod_{j=1}^k \mathrm{sgn}(\gamma_j), \qquad w(\Gamma) = \prod_{j=1}^k w(\gamma_j), \qquad \Gamma = \cup_{j=1}^k \gamma_j. \tag{9.26}$$

PROPOSITION 9.4. *We have that*

$$\langle 1 - \sigma_x^{(3)}\sigma_y^{(3)}\rangle_\Lambda \le 2 \limsup_{N\to\infty} \sum_{\gamma:\mathrm{sgn}(\gamma)=-1} w(\gamma), \tag{9.27}$$

where the sum is restricted to connected *quantum contours.*

PROOF. Using Proposition 9.3 we can write

$$\langle 1-\sigma_x^{(3)}\sigma_y^{(3)}\rangle_\Lambda = \frac{4}{Z}\lim_{N\to\infty}\sum_{k\geq 1}\frac{1}{k!}\sum_{\substack{\gamma_1,\dots,\gamma_k\\ \mathrm{sgn}(\gamma_1\cup\dots\cup\gamma_k)=-1}}\prod_{j=1}^{k} w(\gamma_j) \tag{9.28}$$

where we organised the sum over Γ according to the number k of connected components. The factor $1/k!$ appears as we are also summing over arbitrary labellings of the k components.

For the condition $\mathrm{sgn}(\gamma_1\cup\cdots\cup\gamma_k)=-1$ to hold, there must be some $1\leq i\leq k$ such that $\mathrm{sgn}\,(\gamma_i)=-1$. Summing over the possibilities for i and writing γ for γ_i we obtain

$$\langle 1-\sigma_x^{(3)}\sigma_y^{(3)}\rangle_\Lambda \leq \frac{4}{Z}\limsup_{N\to\infty}\sum_{\gamma:\mathrm{sgn}(\gamma)=-1} w(\gamma)\sum_{k\geq 1}\frac{1}{(k-1)!}\sum_{\gamma_1,\dots,\gamma_{k-1}}\prod_{j=1}^{k-1} w(\gamma_j). \tag{9.29}$$

Using Proposition 9.3 again,

$$\lim_{N\to\infty}\sum_{k\geq 1}\frac{1}{(k-1)!}\sum_{\gamma_1,\dots,\gamma_{k-1}}\prod_{j=1}^{k-1} w(\gamma_j) = \lim_{N\to\infty}\sum_{k\geq 1}\frac{1}{k!}\sum_{\gamma_1,\dots,\gamma_k}\prod_{j=1}^{k} w(\gamma_j) = \frac{Z}{2}, \tag{9.30}$$

which gives the result. □

They following key estimate says that the sum over contours, which are 'tied down' to a given square, is small, even when the weight $w(\gamma)$ is further penalised using a factor $\mathrm{e}^{\varepsilon|S(\gamma)|}$ (provided $\varepsilon>0$ is small enough). The point of introducing the additional factor $\mathrm{e}^{\varepsilon|S(\gamma)|}$ is that it gives the leverage required to control the 'entropy' (i.e. multitide of possibilities) associated with the choice of the given square.

PROPOSITION 9.5. *There is $\beta_0>0$ and $\varepsilon>0$ as well as a function $r(\beta)\to 0$, all depending on δ, such that for any square $\square=\square(z)$ in Λ,*

$$\limsup_{N\to\infty}\sum_{\gamma:\gamma\cap\square\neq\varnothing} w(\gamma)\,\mathrm{e}^{\varepsilon|S(\gamma)|}\leq r(\beta) \tag{9.31}$$

where the sum is over connected quantum contours whose support intersect the square $\square$.

Before proving this, we show how to deduce long-range-order:

PROOF OF THEOREM 9.1. We may assume that Λ is large enough so that the distance between the boundary $\partial\Lambda$ and a given shortest path between x and y is at least $\|x-y\|_1$. In (9.27), the fact that $\mathrm{sgn}(\gamma)=-1$ means that γ separates x from y: either there is a circuit surrounding one but not the other, or a path starting and ending at $\partial\Lambda$ separating them. In either case, such a circuit or path must intersect some square $\square$ at distance $\leq |S(\gamma)|$ from either x or y. This will allow us to use Proposition 9.5. Indeed,

$$\begin{aligned}\sum_{\gamma:\mathrm{sgn}(\gamma)=-1} w(\gamma) &= \sum_{m=1}^{M} \mathrm{e}^{-\varepsilon m} \sum_{\substack{\gamma:\mathrm{sgn}(\gamma)=-1\\|S(\gamma)|=m}} w(\gamma)\,\mathrm{e}^{\varepsilon|S(\gamma)|}\\ &\leq \sum_{m=1}^{M} \mathrm{e}^{-\varepsilon m} \sum_{\square:\max(\|\square-x\|_1,\|\square-y\|_1)\leq m}\ \sum_{\gamma:\gamma\cap\square\neq\varnothing} w(\gamma)\,\mathrm{e}^{\varepsilon|S(\gamma)|}\,,\end{aligned} \tag{9.32}$$

where M is some constant depending on Λ. Then by estimating the number of squares in the second sum,

$$\limsup_{N\to\infty} \sum_{\gamma:\mathrm{sgn}(\gamma)=-1} w(\gamma) \leq r(\beta) \sum_{m=1}^{M} \mathrm{e}^{-\varepsilon m}\, cm^2 \tag{9.33}$$

for some absolute constant $c > 0$. Since the sum at the end is convergent as $M \to \infty$, and $r(\beta) \to 0$ as $\beta \to \infty$, for large enough β we have that $\langle 1 - \sigma_x^{(3)}\sigma_y^{(3)}\rangle_\Lambda$ is bounded away from 1, uniformly in Λ. This gives the result. $\square$

9.2.3. Proof of Proposition 9.5. We now turn to the most technical part of the argument, the bound (9.31). Let $\rho = \frac{1-\delta}{1-\delta/2} \in (0,1)$ and define

$$w_0(\gamma) = w(\gamma)\rho^{-n(\gamma)}\,\mathrm{e}^{-\beta\frac{\delta}{2}\overline{H}^{\mathrm{z}}(\gamma)} = \left(\tfrac{\beta(1-\delta/2)}{N}\right)^{n(\gamma)} \mathrm{e}^{-\beta(1-\frac{\delta}{2})\overline{H}^{\mathrm{z}}(\gamma)}\,. \tag{9.34}$$

We will rewrite the left-hand-side of (9.31) using the weights $w_0(\gamma)$ rather than $w(\gamma)$. By definition,

$$\sum_{\gamma:\gamma\cap\square\neq\varnothing} w(\gamma)\,\mathrm{e}^{\varepsilon|S(\gamma)|} = \sum_{\gamma:\gamma\cap\square\neq\varnothing} w_0(\gamma)\rho^{n(\gamma)}\,\mathrm{e}^{\varepsilon|S(\gamma)|}\,\mathrm{e}^{-\beta\frac{\delta}{2}\overline{H}^{\mathrm{z}}(\gamma)}\,. \tag{9.35}$$

Since $\overline{H}^{\mathrm{z}}(\gamma)$ is an average, and the function $x \mapsto \mathrm{e}^{-cx}$ is convex for any $c > 0$, Jensen's inequality gives

$$\begin{aligned}\sum_{\gamma:\gamma\cap\square\neq\varnothing} w(\gamma)\,\mathrm{e}^{\varepsilon|S(\gamma)|} &\leq \frac{1}{N}\sum_{i=1}^{N} \sum_{\gamma:\gamma\cap\square\neq\varnothing} w_0(\gamma)\rho^{n(\gamma)}\,\mathrm{e}^{\varepsilon|S(\gamma)|}\,\mathrm{e}^{-\beta\frac{\delta}{2}|\gamma^i|}\\ &= \sum_{\gamma:\gamma\cap\square\neq\varnothing} w_0(\gamma)\rho^{n(\gamma)}\,\mathrm{e}^{\varepsilon|S(\gamma)|}\,\mathrm{e}^{-\beta\frac{\delta}{2}|\gamma^0|}\,.\end{aligned} \tag{9.36}$$

For the equality, we noted that all values of i are equivalent by symmetry under 'time-rotation'. explain better

Next note that $|S(\gamma)| \leq |\gamma^0| + 2n(\gamma)$ since each flip adds at most two squares to the support. Then

$$\begin{aligned}\rho^{n(\gamma)}\, \mathrm{e}^{\varepsilon|S(\gamma)|}\, \mathrm{e}^{-\beta\frac{\delta}{2}|\gamma^0|} &= \rho^{\frac{1}{2}n(\gamma)}\, \mathrm{e}^{-\beta\frac{\delta}{4}|\gamma^0|} \exp\big((\varepsilon - \beta\tfrac{\delta}{4})|\gamma^0| + (\varepsilon + \tfrac{1}{2}\log\rho)n(\gamma)\big) \\ &\leq \rho^{\frac{1}{2}n(\gamma)}\, \mathrm{e}^{-\beta\frac{\delta}{4}|\gamma^0|}\,,\end{aligned} \tag{9.37}$$

where the last inequality holds for $\varepsilon > 0$ small enough. So far we have obtained

$$\sum_{\gamma:\gamma\cap\square\neq\varnothing} w(\gamma)\, \mathrm{e}^{\varepsilon|S(\gamma)|} \leq \sum_{\gamma:\gamma\cap\square\neq\varnothing} w_0(\gamma)\rho^{\frac{1}{2}n(\gamma)}\, \mathrm{e}^{-\beta\frac{\delta}{4}|\gamma^0|}\,. \tag{9.38}$$

We will now sum over the possibilities G for γ^0. We then want to replace $n(\gamma)$ by something that only depends on G. For this, let $D_\square(G)$ be the minimal size of a set of squares F such that $\square \cup G \cup F$ is connected. If $G = \gamma^0$ where $\gamma \cap \square \neq \varnothing$, then $D_\square(G) \leq |F(\gamma)| = 2n(\gamma)$, the number of flipped squares in γ. We deduce from this and (9.38) that

$$\sum_{\gamma:\gamma\cap\square\neq\varnothing} w(\gamma)\, \mathrm{e}^{\varepsilon|S(\gamma)|} \leq \sum_{G\neq\varnothing} \rho^{\frac{1}{4}D_\square(G)}\, \mathrm{e}^{-\beta\frac{\delta}{4}|G|} \sum_{\gamma:\gamma^0=G} w_0(\gamma). \tag{9.39}$$

We can now remove the constraint the γ be connected:

$$\sum_{\gamma:\gamma^0=G} w_0(\gamma) \leq \sum_{\Gamma:\Gamma^0=G} w_0(\Gamma) \tag{9.40}$$

where the sum is now over all contours. Moreover, since a limsup of a sum is at most the sum of limsups,

$$\limsup_{N\to\infty} \sum_{\gamma:\gamma\cap\square\neq\varnothing} w(\gamma)\, \mathrm{e}^{\varepsilon|S(\gamma)|} \leq \sum_{G\neq\varnothing} \rho^{\frac{1}{4}D_\square(G)}\, \mathrm{e}^{-\beta\frac{\delta}{4}|G|} \limsup_{N\to\infty} \sum_{\Gamma:\Gamma^0=G} w_0(\Gamma). \tag{9.41}$$

What have we gained? The limsup in the last expression is reminiscent of the formula (9.23) for the partition function. At this point, it is useful to go back to the linear-algebraic formulation we started with. The constraint $\Gamma^0 = G$ means that we are summing only over the vector $|G\rangle$, or equivalently

$$\limsup_{N\to\infty} \sum_{\Gamma:\Gamma^0=G} w_0(\Gamma) = \tfrac{1}{2}\mathrm{Tr}\ \mathrm{e}^{-\beta H_0}\, P_G \tag{9.42}$$

where P_G is the projector on the two-dimensional subspace spanned by $|\sigma\rangle$ and $|-\sigma\rangle$ where $G = G(\sigma)$, and $H_0 = (1-\frac{\delta}{2})(H^{\mathrm{xx}} + H^{\mathrm{z}})$. Thus, H_0 is the hamiltonian for the Heisenberg XXX-model. We only need one very simple fact about H_0, namely that $H_0 \geq 0$ (Exercise 9.2). Thus $\mathrm{e}^{-\beta H_0} \leq 1$ so that $\frac{1}{2}\mathrm{Tr}\ \mathrm{e}^{-\beta H_0}\, P_G \leq 1$.

Thus, our estimates boil down to

$$\limsup_{N\to\infty} \sum_{\gamma:\gamma\cap\square\neq\varnothing} w(\gamma)\, \mathrm{e}^{\varepsilon|S(\gamma)|} \leq \sum_{G\neq\varnothing} \rho^{\frac{1}{4}D_\square(G)}\, \mathrm{e}^{-\beta\frac{\delta}{4}|G|} = r(\beta). \tag{9.43}$$

To show that $r(\beta) \to 0$ we sum over possible connections. Indeed, we can write $G = \{\gamma_1, \dots, \gamma_k\}$ for some $k \geq 1$ where the γ_i are connected. Then $D_\square(G)$ counts the number of squares needed to connect the γ_i as well as $\square$ together using paths of squares. The minimal way of connecting them together is to form a tree-structure.

Define a sequence $A_1, A_2, \dots$ inductively as follows. First

$$A_1 = \sum_{k \geq 1} \frac{1}{k!} \sum_{\gamma_1,\dots,\gamma_k} \mathrm{e}^{-\beta \frac{\delta}{4} \sum |\gamma_i|} \sum_{p_1,\dots,p_k} \rho^{\frac{1}{2} \sum |p_i|}, \tag{9.44}$$

where $p_1, \dots, p_k$ are paths (possibly empty) linking $\square$ to $\gamma_1, \dots, \gamma_k$ respectively. Next,

$$A_\ell = \sum_{k \geq 1} \frac{1}{k!} \sum_{\gamma_1,\dots,\gamma_k} \mathrm{e}^{-\beta \frac{\delta}{4} \sum |\gamma_i|} \sum_{p_1,\dots,p_k} \rho^{\frac{1}{2} \sum |p_i|} \prod_{i=1}^{k} \Big(\sum_{x_i \in \gamma_i} A_{\ell-1} \Big). \tag{9.45}$$

Then

$$\sum_{G \neq \varnothing} \rho^{\frac{1}{4} D_\square(G)} \, \mathrm{e}^{-\beta \frac{\delta}{4} |G|} = r(\beta) \leq A_k - 1. \tag{9.46}$$

This expresses the fact that the connections form a tree of depth at most k.

It is not hard to see that

$$A_1 \leq \exp\big(\tfrac{2}{1-\rho^{1/2}} \varepsilon(\beta) \big), \quad \text{where} \quad \varepsilon(\beta) = \sum_{\gamma \ni 0} \mathrm{e}^{-\beta \frac{\delta}{4} |\gamma|} \, |\gamma|. \tag{9.47}$$

Then $\varepsilon(\beta)$ is arbitrarily small for β large. Then we can show by induction that if $\eta > 1$ then $A_\ell \leq \eta$ for all $\ell \geq 1$, provided β is large enough. Indeed, the definition of A_ℓ and induction implies that

$$A_\ell \leq \exp\big(\tfrac{2}{1-\rho^{1/2}} \varepsilon(\beta) \eta \big) \leq \eta \tag{9.48}$$

when β is large enough. Since G is nonempty (thus $k \geq 1$) we finally get

$$\sum_{G \neq \varnothing} \rho^{\frac{1}{4} D_\square(G)} \, \mathrm{e}^{-\beta \frac{\delta}{4} |G|} = r(\beta) \leq A_k - 1 \leq \eta - 1 \tag{9.49}$$

which can be made arbitrarily small since $\eta > 1$ was arbitrary. $\square$

EXERCISE 9.2. *Show that* $H^{\mathrm{xxx}} = H^{\mathrm{xx}} + H^{\mathrm{z}}$ *is non-negative definite.*