CHAPTER 10

Two-dimensional systems with continuous symmetry

We consider translation-invariant Gibbs states for two-dimensional models, on $\mathbb{Z}^2$, for an interaction Φ which is invariant under a continuous group of rotations. We prove two claims about the absence of symmetry breaking. In Section 10.1 we show that all infinite-volume Gibbs states retain the continuous symmetry. In Section 10.2 we show that the correlations associated with the continuous symmetry are short-range.

10.1. All Gibbs states retain the continuous symmetry

More precisely, we assume that there are operators S_x, $x \in \mathbb{Z}^2$, such that $[\Phi_X, \sum_{x\in X} S_x] = 0$ for all $X \Subset \mathbb{Z}^2$. For an angle $\theta \in [0, 2\pi]$ define the rotation $U_X(\theta) = \exp(\sum_{x\in X} \mathrm{i}\theta S_x)$. The condition $[\Phi_X, \sum_{x\in X} S_x] = 0$ means that $U_X(\theta)\Phi_X U_X^{-1}(\theta) = \Phi_X$ for any angle θ, thus the interaction is invariant under rotations.

THEOREM 10.1. *Under the assumptions above, let $\langle\cdot\rangle \in \mathcal{G}^{\Phi}_{\text{tr.inv.}}$. Then for all $\Lambda \Subset \mathbb{Z}^2$ and all $a \in \mathcal{A}_\Lambda$ we have that*

$$\langle U_\Lambda(\theta) a U_\Lambda^{-1}(\theta)\rangle = \langle a\rangle. \tag{10.1}$$

For the proof we rely on two lemmas. The first tells us that we can approximate the expectation $\langle a\rangle$ using finite-volume states. This holds without assumptions on the dimension d or the interaction Φ. Recall that $\Lambda_n = \{1, 2, \dots, n\}^d$.

LEMMA 10.2. *Let $\langle\cdot\rangle \in \mathcal{G}^{\Phi}_{\text{tr.inv.}}$ be a translation-invariant Gibbs state for an interaction $\Phi \in \mathcal{I}$ and let $a \in \mathcal{A}_\Lambda$ where $\Lambda \Subset \mathbb{Z}^d$. There is a sequence of real numbers $s_n \to 0$ such that*

$$\langle a\rangle = \lim_{n\to\infty} \langle a\rangle^{\Phi+s_n\Psi_a}_{\Lambda_n^{\text{per}}}.$$

It should be noted that the sequence (s_n) depends on a; the lemma shows that $\langle a\rangle$ can be approximated by a finite volume expectation, but it does not give an approximation for the Gibbs state $\langle\cdot\rangle$.

PROOF. We use the tangent-functional characterisation of $\langle\cdot\rangle$, Definition 3.10. Consider the finite-volume pressure $p_n(s) = p_{\Lambda_n^{\text{per}}}(\Phi + s\Psi_a)$ as a function of $s \in \mathbb{R}$. These functions are convex, continuously differentiable (in fact, smooth) and form a sequence converging pointwise to $p(s) = p(\Phi + s\Psi_a)$ by Corollary 3.4. We have $p(s) \geq p(0) - s\langle a \rangle$ for all $s \in \mathbb{R}$. The right and left derivatives of $p(s)$ at $s = 0$ are given respectively by $\partial_+ p(0) = \inf_{s>0} \frac{p(s)-p(0)}{s}$ and $\partial_- p(0) = \sup_{s<0} \frac{p(s)-p(0)}{s}$. Thus $-\langle a \rangle \in [\partial_- p(0), \partial_+ p(0)]$ and by Proposition A.5, there is a sequence $s_n \to 0$ such that $-\langle a \rangle = \lim_{n\to\infty} \frac{\mathrm{d}}{\mathrm{d}s} p_n(s_n)$. Similarly to (3.19) we have that $\frac{\mathrm{d}}{\mathrm{d}s} p_n(s_n) = -\langle a \rangle^{\Phi+s_n\Psi_a}_{\Lambda_n^{\mathrm{per}}}$, which gives the claim. □

The second lemma we rely on also plays an important role in quantum information theory:

Lemma 10.3 (Quantum Pinsker's inequality). *For two density-matrices $\rho, \sigma \in \mathcal{B}(\mathcal{H})$ on a finite-dimensional Hilbert space $\mathcal{H}$, define the* relative entropy

$$S(\rho\|\sigma) := \operatorname{Tr} \rho(\log \rho - \log \sigma). \tag{10.2}$$

Then $S(\rho\|\sigma) \geq \frac{1}{2}\|\rho - \sigma\|_1^2$. In particular $S(\rho\|\sigma) \geq 0$.

For the proof of Pinsker's inequality, see Section A.4.

Proof of Theorem 10.1. Note that $U_X^{-1}(\theta) = U_X^*(\theta) = U_X(-\theta)$. The key idea of the proof is to introduce 'gradual' rotations and then to transfer the rotations from the observable to the interactions. Let $\boldsymbol{\theta} = (\theta_x)$ to be angles such that $\theta_x = \theta$ for $x \in X$, and will otherwise be chosen below (see Eq. (10.20)); let

$$V(\boldsymbol{\theta}) = \exp\left(\mathrm{i} \textstyle\sum_{x\in\Lambda_n} \theta_x S_x\right). \tag{10.3}$$

Notice that $U_X(\theta) a U_X^{-1}(\theta) = V(\boldsymbol{\theta}) a V^{-1}(\boldsymbol{\theta})$ since $a \in \mathcal{A}_X$. In what follows we simply write V for $V(\boldsymbol{\theta})$ to lighten the notation.

Let us write $Z^{\Phi+\Psi_n}_{\Lambda_n} = \operatorname{Tr} \mathrm{e}^{-H^\Phi - H^{\Psi_n}}$ and

$$\rho_{\Lambda_n} = \frac{\mathrm{e}^{-H^\Phi_{\Lambda_n} - H^{\Psi_n}_{\Lambda_n}}}{Z^{\Phi+\Psi_n}_n} \tag{10.4}$$

for the density-matrix associated with the finite-volume Gibbs-state $\langle\cdot\rangle^{\Phi+\Psi_n}_{\Lambda_n}$. Let us also write

$$\rho^V_{\Lambda_n} = \frac{\mathrm{e}^{-V^{-1}H^\Phi_{\Lambda_n}V - V^{-1}H^{\Psi_n}_{\Lambda_n}V}}{Z^{\Phi+\Psi_n}_n} \tag{10.5}$$

for the density-matrix associated with $\langle\cdot\rangle^{V^{-1}\Phi V + V^{-1}\Psi_n V}_{\Lambda_n}$. (The partition-functions are the same as the trace is invariant under conjugation.) We have that

$$|\langle U_{\Lambda_n}(\theta) a U^{-1}_{\Lambda_n}(\theta)\rangle^{\Phi+\Psi_n}_{\Lambda_n} - \langle a \rangle^{\Phi+\Psi_n}_{\Lambda_n}| = |\langle V a V^{-1}\rangle^{\Phi+\Psi_n}_{\Lambda_n} - \langle a \rangle^{\Phi+\Psi_n}_{\Lambda_n}| = |\operatorname{Tr} a(\rho_{\Lambda_n} - \rho^V_{\Lambda_n})| \tag{10.6}$$

Applying first Hölder's inequality followed by Pinsker's inequality we find that

$$|\operatorname{Tr} a(\rho_{\Lambda_n} - \rho^V_{\Lambda_n})| \leq \|a\|_\infty \|\rho_{\Lambda_n} - \rho^V_{\Lambda_n}\|_1 \leq \|a\|_\infty \sqrt{2S(\rho_{\Lambda_n} \mid \rho^V_{\Lambda_n})}. \tag{10.7}$$

Now we apply a strange-looking trick which turns out to be crucial. By the non-negativity of the relative entropy,

$$S(\rho_{\Lambda_n} \mid \rho^V_{\Lambda_n}) \le S(\rho_{\Lambda_n} \mid \rho^V_{\Lambda_n}) + S(\rho_{\Lambda_n} \mid \rho^{V^{-1}}_{\Lambda_n}) \tag{10.8}$$

where $\rho^{V^{-1}}_{\Lambda_n}$ is the density-matrix as in (10.5) but with V^{-1} replacing V, or equivalently, with rotations $-\theta_x$ replacing θ_x. The reason for doing this is that it leads to a cancellation of terms which are linear in the angle differences $\theta_x - \theta_y$, leaving only second-order terms.

Now note that

$$\begin{aligned} S(\rho_{\Lambda_n} \mid \rho^V_{\Lambda_n}) + S(\rho_{\Lambda_n} \mid \rho^{V^{-1}}_{\Lambda_n}) =& \big\langle 2H^{\Phi}_{\Lambda_n} - H^{V\Phi V^{-1}}_{\Lambda_n} - H^{V^{-1}\Phi V}_{\Lambda_n} \big\rangle^{\Phi+\Psi_n}_{\Lambda_n} \\ &+ \big\langle 2H^{\Psi_n}_{\Lambda_n} - H^{V\Psi_n V^{-1}}_{\Lambda_n} - H^{V^{-1}\Psi_n V}_{\Lambda_n} \big\rangle^{\Phi+\Psi_n}_{\Lambda_n}. \end{aligned} \tag{10.9}$$

The right side is smaller than

$$\|2H^{\Phi}_{\Lambda_n} - H^{V\Phi V^{-1}}_{\Lambda_n} - H^{V^{-1}\Phi V}_{\Lambda_n}\| + \|2H^{\Psi_n}_{\Lambda_n} - H^{V\Psi_n V^{-1}}_{\Lambda_n} - H^{V^{-1}\Psi_n V}_{\Lambda_n}\|. \tag{10.10}$$

For given rotations $\boldsymbol{\theta} = (\theta_x)$, the last term is less than $4\|\!|\Psi_n|\!\|\,|Y|$ which goes to 0 as $n \to \infty$. We now check that the first term is as small as we wish by taking n and Y large.

For each set $Z \subseteq \Lambda_n$, we let $z_0(Z)$ be one of its sites. We have

$$V\Phi_Z V^{-1} = \mathrm{e}^{-\mathrm{i}T_Z}\,\Phi_Z\,\mathrm{e}^{-\mathrm{i}T_Z}\,, \tag{10.11}$$

where

$$T_Z = \sum_{z\in Z} (\theta_z - \theta_{z_0(Z)}) S_z. \tag{10.12}$$

Using the multi-commutator expansion we get

$$\begin{aligned} H^{V\Phi V^{-1}}_{\Lambda_n} &= \sum_{Z\subset\Lambda_n} \Phi_Z + \sum_{j\ge1}\sum_{Z\subset\Lambda_n} \frac{(-1)^j}{j!}\,\mathrm{ad}^j_{T_Z}(\Phi_Z) \\ &= H_\Lambda + B(\boldsymbol{\theta}) + C(\boldsymbol{\theta}), \end{aligned} \tag{10.13}$$

where

$$B(\boldsymbol{\theta}) = -\sum_{j\ge1}\sum_{Z\subset\Lambda_n} \frac{1}{(2j-1)!}\mathrm{ad}^{2j-1}_{T_Z}(\Phi_Z), \tag{10.14}$$

and

$$C(\boldsymbol{\theta}) = \sum_{j\ge1}\sum_{Z\subset\Lambda_n} \frac{1}{(2j)!}\mathrm{ad}^{2j}_{T_Z}(\Phi_Z). \tag{10.15}$$

Since B is odd and C is even we have

$$2H^{\Phi}_{\Lambda_n} - H^{V\Phi V^{-1}}_{\Lambda_n} - H^{V^{-1}\Phi V}_{\Lambda_n} = 2C(\boldsymbol{\theta}). \tag{10.16}$$

We now estimate $\|C(\boldsymbol{\theta})\|$. Using $\|[a,b]\| \leq 2\|b\|\,\|b\|$ we obtain

$$\begin{aligned} \|C(\boldsymbol{\theta})\| &\leq \sum_{j\geq 1} \sum_{Z\subset\Lambda} \frac{2^{2j}}{(2j)!} \|T_Z\|^{2j} \|\Phi_Z\| \\ &= \sum_{Z\subset\Lambda} \|\Phi_Z\| \big(\cosh\left(2\|T_Z\|\right) - 1\big) \\ &\leq 2 \sum_{Z\subset\Lambda_n} \|\Phi_Z\| \|T_Z\|^2 \, \mathrm{e}^{2\|T_Z\|} \,. \end{aligned} \tag{10.17}$$

We used the inequality $\cosh u - 1 \leq \frac{1}{2} u^2 \, \mathrm{e}^u$, which is easily verified for all $u \geq 0$. We now bound $\mathrm{e}^{2\|T_Z\|} \leq \mathrm{e}^{4|\theta||Z|}$ and

$$\|T_Z\|^2 \leq \Big(\sum_{z\in Z} |\theta_z - \theta_{z_0(Z)}|\Big)^2 \leq \Big(|Z| \sum_{\substack{\{z,z\}\subset Z \\ \|z-z'\|=1}} |\theta_z - \theta_{z'}|\Big)^2 \leq d|Z|^3 \sum_{\substack{\{z,z\}\subset Z \\ \|z-z'\|=1}} |\theta_z - \theta_{z'}|^2. \tag{10.18}$$

The first inequality above involves summing over nearest-neighbour paths from $z \to z_0(Z)$ and the triangle inequality; a given edge appears no more than $|Z|$ times. The second inequality is Cauchy-Schwarz, using that the set $Z \subset \mathbb{Z}^2$ has no more than $2|Z|$ nearest-neighbour edges. We obtain

$$\begin{aligned} \|C(\boldsymbol{\theta})\| &\leq 4 \sum_{Z\subset\Lambda_n} \|\Phi_Z\| |Z|^3 \, \mathrm{e}^{4|\theta||Z|} \sum_{\substack{\{z,z\}\subset Z \\ \|z-z'\|=1}} |\theta_z - \theta_{z'}|^2 \\ &\leq 4 \sum_{\substack{\{z,z\}\subset \Lambda_n \\ \|z-z'\|=1}} |\theta_z - \theta_{z'}|^2 \sum_{Z\supset\|z,z'\|} \|\Phi_Z\| |Z|^3 \, \mathrm{e}^{4|\theta||Z|} \\ &\leq 4 \|\Phi\|_{4|\theta|+3} \sum_{\substack{\{z,z\}\subset \Lambda_n \\ \|z-z'\|=1}} |\theta_z - \theta_{z'}|^2. \end{aligned} \tag{10.19}$$

We now choose the angles $\boldsymbol{\theta} = (\theta_x)$. Let m_0 be large enough so that X is included in the box of size $2m_0$ around the origin. Then we take

$$\theta_x = \begin{cases} \theta & \text{if } \|x\|_1 \leq m_0, \\ \theta\big(1 - \frac{\log(\|x\|_1 - m_0)}{\log m}\big) & \text{if } m_0 < \|x\|_1 < m_0 + m, \\ 0 & \text{if } \|x\|_1 \geq m. \end{cases} \tag{10.20}$$

Summing over edges at distance r from the box of size m_0, we get

$$\begin{aligned} \|C(\boldsymbol{\theta})\| &\leq 4\|\Phi\|_{4|\theta|+3} \sum_{r=1}^{m} 8(m_0 + r) \Big(\frac{\log(r+1) - \log r}{\log m}\Big)^2 \\ &\leq \frac{32 m_0 \|\Phi\|_{4|\theta|+3}}{(\log m)^2} \sum_{r=1}^{m} \frac{1}{r}. \end{aligned} \tag{10.21}$$

The sum diverges as $\log m$, but it is controlled by the denominator. Then $\|C(\boldsymbol{\theta})\|$ can be made as small as we wish by taking m large (this requires the domain Λ_n to be large). Using our bound in (10.16) then (10.10), we see that the difference of the expectation of the local observable a in (10.7) is vanishingly small. □

10.2. Decay of correlations

To be added.

Bibliographical references

Mermin and Wagner [1966] proved that the quantum Heisenberg has no spontaneous magnetisation at all positive temperatures. Such is the importance of this result that the name "Mermin-Wagner theorem" has come to designate *all* results about absence of continuous symmetry breaking in two dimensions, although the mathematical setting and the methods of proofs of further results are very different. The fact that all Gibbs states remain symmetric was first proved in classical systems by Dobrushin and Shlosman [1975]. The extension to quantum systems was obtained by Fröhlich and Pfister [1981]; the present proof uses similar ideas but is more elementary.

The decay of correlations was first addressed in Fisher and Jasnow (1971). McBryan and Spencer (1977) proved polynomial decay in classical systems. The extension to quantum systems is due to Koma and Tasaki (1992); the general setting considered here was proposed in Benassi, Fröhlich, Ueltschi (2017).