Assignment 6

Due Monday 20 November 15:00. Two problems will be selected for marking among those with boxed numbers. There are **4 points** for each of the two problems, and **2 points** for presentation. Do not forget to write **your name** and **your group** (**1**: Tuesday 12-1/Atkinson-Williams; **2**: Tuesday 2-3/Vasdekis; **3**: Thursday 1-2/Archer; **4**: Friday 12-1/Bowditch).

1. Use definite integrals to find the limits of the following sums:

(a) $\lim_{n\to\infty}\left(\frac{1}{n^2}+\frac{2}{n^2}+\dots\frac{n-1}{n^2}\right)$

(b) $\lim_{n\to\infty}\left(\frac{1}{n+1}+\frac{1}{n+2}+\dots\frac{1}{n+n}\right)$

(c) $\lim_{n\to\infty}\frac{1^p+2^p+\dots n^p}{n^{p+1}}$, $p>0$

2. Determine the sign of the improper integral

$$\int_0^{2\pi}\frac{\sin(x)}{x}\,\mathrm{d}x.$$

3. Determine (without evaluating) which of the following integrals is greater.

(a) $\int_0^1\sqrt{1+x^2}\,\mathrm{d}x$ or $\int_0^1 1\,\mathrm{d}x$

(b) $\int_0^1 x^2\sin^2(x)\,\mathrm{d}x$ or $\int_0^1 x\sin^2(x)\,\mathrm{d}x$

(c) $\int_1^2 \mathrm{e}^{x^2}\,\mathrm{d}x$ or $\int_1^2 \mathrm{e}^{x}\,\mathrm{d}x$

4. Let f a function $[0,1]\times[0,1]\to\mathbb{R}$ that is continuously differentiable with respect to both parameters, and let $g(x)=\int_0^x f(x,t)\,\mathrm{d}t$. Show that

$$g'(x)=f(x,x)+\int_0^x\frac{\partial f}{\partial x}(x,t)\,\mathrm{d}t.$$

Hint: Start with the definition of the derivative, $g'(x)=\lim_{h\to 0}\frac{g(x+h)-g(x)}{h}$, and reorganise the terms so the equation above appears.

5. Applying differentiation with respect to a parameter α, evaluate the following integral

$$\int_0^\infty\frac{\mathrm{e}^{-\alpha x}-\mathrm{e}^{-\beta x}}{x}\mathrm{d}x,\qquad \alpha,\beta>0.$$

Hint. You may exchange the order of integration and differentiation without a justification.

6. The Laplace transform of $f:[0,\infty)\to\mathbb{R}$ is a function $F:(0,\infty)\to\mathbb{R}$ defined by the formula

$$F(p)=\int_0^\infty \mathrm{e}^{-pt}f(t)\,\mathrm{d}t,\ p>0.$$

Find the Laplace transform of the following functions:

(a) $f(t)=1$

(b) $f(t)=\mathrm{e}^{-\alpha t}$, $\alpha>0$

(c) $f(t)=\cos(\beta t),\ \beta\in\mathbb{R}$

Notice that the Laplace transform of a bounded function is not necessarily bounded.