Assignment 4

Due Monday 6 November 15:00. Two problems will be selected for marking among those with boxed numbers. There are **4 points** for each of the two problems, and **2 points** for presentation. Do not forget to write **your name** and **your group** (**1**: Tuesday 12-1/Atkinson; **2**: Tuesday 2-3/Vasdekis; **3**: Thursday 1-2/Archer; **4**: Friday 12-1/Bowditch).

1. This problem is motivated by the Gamma function $\Gamma(x) = \int_0^\infty t^{x-1} \mathrm{e}^{-t} \mathrm{d}t$. For given $x \in \mathbb{R}$, consider $f(t) = t^{x-1}\mathrm{e}^{-t}$.

1. For which values of x does the integral $\int_0^1 f(t)\mathrm{d}t$ exist?
2. For which values of x does the Riemann integral $\int_0^1 f(t)\mathrm{d}t$ exist?
3. For which values of x does the improper integral $\int_0^\infty f(t)\mathrm{d}t$ exist?

2. Find the following integrals.

1. $\displaystyle\int_0^1 \frac{\cos\sqrt{t}}{\sqrt{t}}\,\mathrm{d}t.$
2. $\displaystyle\int_2^\infty t^{-2}\log t\,\mathrm{d}t.$
3. $\displaystyle\int_2^\infty \mathrm{e}^{-\sqrt{t}}\mathrm{d}t.$
4. $\displaystyle\int_0^\pi \mathrm{e}^{\sin^2 t}\sin t\cos t\,\mathrm{d}t.$

3. Let $f(x) = \displaystyle\int_x^{x^2} \mathrm{e}^{-t^2}\mathrm{d}t$. Find its derivative $f'(x)$, and draw it for $x \in (-\infty, \infty)$. For approximately which x is f maximum?

4. Find the pointwise limits of the following functions as $n \to \infty$. Is the convergence uniform? Prove it!

1. $f_n(x) = x^{1/n}$ for $x \in [0, 1]$.
2. $f_n(x) = \sin(x + \frac{1}{n})$ for $x \in \mathbb{R}$.
3. $f_n(x) = \mathrm{e}^{n(\cos x - 1)}$ for $x \in \mathbb{R}$.
4. $f_n(x) = \mathrm{e}^{x/n}$ for $x \in [0, 2\pi]$.

5. Find the pointwise limits of the following functions as $n \to \infty$. Is the convergence uniform? Prove it!

1. $f_n(x) = \min(\cos x, 1 - \frac{1}{n})$ for $x \in \mathbb{R}$.
2. $f_n(x) = n\sin\frac{x}{n}$ for $x \in \mathbb{R}$.
3. $f_n(x) = \mathrm{e}^{-x/n}$ for $x \in [0, \infty)$.
4. $f_n(x) = \lim_{m\to\infty}[\cos(n!\pi x)]^{2m}$ for $x \in [0, 1]$.

6. Consider the functions $f_n(x) = n^a x\,\mathrm{e}^{-n^b x}$ on $[0, \infty)$, where a, b are fixed numbers. Find the pointwise limit as $n \to \infty$. Draw a few functions, calculate the derivatives, and find the values of a, b for which the convergence is uniform.