Assignment 8

Due Monday 5 December 15:00. Two problems will be selected for marking among those with boxed numbers. There are **4 points** for each of the two problems, and **2 points** for presentation. Do not forget to write **your name** and **your group** (**1**: Tuesday 12-1/Buze; **2**: Tuesday 2-3/Vogel; **3**: Thursday 1-2/Khor; **4**: Friday 12-1/Matejczyk).

1. (a) Let $\|x\|_p = \big(\sum_{k=1}^n |x_k|^p\big)^{1/p}$ for $x = (x_1, \dots, x_n)$. For which $p \in [1, \infty]$ does this norm satisfies the parallelogram identity?
(b) Let $\|x\|_p = \big(\sum_{k=1}^\infty |x_k|^p\big)^{1/p}$ for sequences $x = (x_1, x_2, \dots)$. For which $p \in [1, \infty]$ does this norm satisfies the parallelogram identity?

2. Let S_f denote the set of all finite sequences, i.e. sequences $a = (a_n)_{n \geq 0}$ such that $a_n = 0$ for all n larger than some N (that depends on the sequence).

(a) Show that S_f is a vector space and that its dimension is infinite.

(b) Show that the map $T : S_\mathrm{f} \to \mathbb{R}$ defined by $Ta = \sum_{n=0}^\infty a_n$ is linear.

(c) We now equip S_f with the $\|\cdot\|_p$ norm. For which $p \in [1, \infty]$ is T bounded? Justify your answer.

3. Let A, B be two bounded operators on the normed space $(V, \|\cdot\|)$. Show that $\|AB\| \leq \|A\| \, \|B\|$.

4. Let $f \in C[0, 2\pi]$ and a_k, b_k its Fourier coefficients. Assume for simplicity that there are only finitely many nonzero coefficients. Show that

$$\|f\|_2^2 = 2\pi a_0^2 + \pi \sum_{k=1}^\infty \big(a_k^2 + b_k^2\big).$$

It turns out that this identity holds for any function on $[0, 2\pi]$ such that $\int f^2 < \infty$.

5. Consider the function $f(x) = x$ on $[0, 2\pi]$. Calculate $\|f\|_2^2$ and its Fourier coefficients. Use the identity of Problem 4 to find the value of the series $\sum_{k=1}^\infty \frac{1}{k^2}$. functions such as $f(x) = x^2$.

6. Repeat the previous exercise with $f(x) = x^2$.

7. For the following spaces of sequences, decide whether they are open, or closed, or neither, with respect to the norm $\|\cdot\|_p$, $p \in [1, \infty]$.

(a) The space S_f of finite sequences, defined in Problem 2.

(b) The n-dimensional space $\big\{(a_k)_{k \geq 1} : a_k = 0 \text{ for } k > n\big\}$.

(c) The space $\big\{(a_k)_{k \geq 1} : \lim_{k \to \infty} k^2 a_k = 0\big\}$.