Assignment 7

Due Monday 28 November 15:00. Two problems will be selected for marking among those with boxed numbers. There are **4 points** for each of the two problems, and **2 points** for presentation. Do not forget to write **your name** and **your group** (**1**: Tuesday 12-1/Buze; **2**: Tuesday 2-3/Vogel; **3**: Thursday 1-2/Khor; **4**: Friday 12-1/Matejczyk).

1. Consider the space of sequences $(x_k)_{k\geq 1}$, $x_k \in \mathbb{R}$, and the following norm:

$$\|x\| = \sum_{k=1}^{\infty} 2^{-k} |x_k|.$$

Show that $\|\cdot\|$ is a norm.

2. Consider the space of sequences $(x_k)_{k\geq 1}$, $x_k \in \mathbb{R}$, and the following norm:

$$\|x\| = \Big(\sum_{k=1}^{\infty} k |x_k|^2 \Big)^{1/2}.$$

Show that $\|\cdot\|$ is a norm. (Problems 5 and 6 may actually help.)

3. Are the norms in Problems 1 and 2 equivalent? Give justifications.

4. Let $a = (a_k)_{k\geq 1}$ be a fixed sequence of real numbers, and let $X = \{(x_k)_{k\geq 1} : \sum_k |a_k x_k| < \infty\}$. For $x \in X$, define

$$\|x\|_a = \sum_{k=1}^{\infty} a_k |x_k|.$$

(a) Give sufficient and necessary conditions for a so that the function above is a norm.

Let $a = (a_k)$ and $b = (b_k)$ be two decreasing sequences of positive numbers.

(b) Assume that there exists $\varepsilon > 0$ such that $a_k/b_k \geq \varepsilon$ and $b_k/a_k \geq \varepsilon$ for all k. Prove that $\|\cdot\|_a$ and $\|\cdot\|_b$ are equivalent.

(c) Assume that $a_k/b_k \to 0$ as $k \to \infty$. Prove that $\|\cdot\|_a$ and $\|\cdot\|_b$ are not equivalent.

5. Check that the following are inner products:

(a) On $\mathbb{R}^n$, let $\langle x, y\rangle = \sum_{i=1}^n x_i\, y_i$.

(b) Let ℓ be the space of sequences $(x_k)_{k\geq 1}$ such that $\sum_k k|x_k|^2 < \infty$. Show that $\langle x, y\rangle = \sum_{k=1}^{\infty} k\, x_k\, y_k$ is an inner product on ℓ.

6. Let $\langle \cdot, \cdot \rangle$ be an inner product, and $\|\cdot\| = \sqrt{\langle \cdot, \cdot\rangle}$ be its induced norm.

(a) Show that $\|\cdot\|$ is a norm indeed.

(b) Show that $\|\cdot\|$ satisfies the parallelogram identity.

(c) Show that the Euclidean norm on $\mathbb{R}^n$, $\|x\|_2 = \big(\sum_{i=1}^n |x_i|^2\big)^{1/2}$, satisfies the triangle inequality.