

Assignment 6

Due Monday 21 November 15:00. Two problems will be selected for marking among those with boxed numbers. There are **4 points** for each of the two problems, and **2 points** for presentation. Do not forget to write **your name** and **your group** (**1:** Tuesday 12-1/Buze; **2:** Tuesday 2-3/Vogel; **3:** Thursday 1-2/Khor; **4:** Friday 12-1/Matejczyk).

IMPORTANT: Because of a confusion, everybody will receive **full credit** for Problem 1 of Assignment 4. Please show your assignment to the tutor if needed!

1. Use definite integrals to find the limits of the following sums:

(a) $\lim_{n \rightarrow \infty} \left(\frac{1}{n^2} + \frac{2}{n^2} + \dots + \frac{n-1}{n^2} \right)$

(b) $\lim_{n \rightarrow \infty} \left(\frac{1}{n+1} + \frac{1}{n+2} + \dots + \frac{1}{n+n} \right)$

(c) $\lim_{n \rightarrow \infty} \frac{1^p + 2^p + \dots + n^p}{n^{p+1}}, p > 0$

2. Determine the sign of the improper integral

$$\int_0^{2\pi} \frac{\sin(x)}{x} dx.$$

3. Determine (without evaluating) which of the following integrals is greater.

(a) $\int_0^1 \sqrt{1+x^2} dx$ or $\int_0^1 1 dx$

(b) $\int_0^1 x^2 \sin^2(x) dx$ or $\int_0^1 x \sin^2(x) dx$

(c) $\int_1^2 e^{x^2} dx$ or $\int_1^2 e^x dx$

4. Let f a function $[0, 1] \times [0, 1] \rightarrow \mathbb{R}$ that is continuously differentiable with respect to both parameters, and let $g(x) = \int_0^x f(x, t) dt$. Show that

$$g'(x) = f(x, x) + \int_0^x \frac{\partial f}{\partial x}(x, t) dt.$$

Hint: Start with the definition of the derivative, $g'(x) = \lim_{h \rightarrow 0} \frac{g(x+h) - g(x)}{h}$, and reorganise the terms so the equation above appears.

5. Applying differentiation with respect to a parameter α , evaluate the following integral

$$\int_0^\infty \frac{e^{-\alpha x} - e^{-\beta x}}{x} dx, \quad \alpha, \beta > 0.$$

Hint. You may exchange the order of integration and differentiation without a justification.

6. The Laplace transform of $f : [0, \infty) \rightarrow \mathbb{R}$ is a function $F : (0, \infty) \rightarrow \mathbb{R}$ defined by the formula

$$F(p) = \int_0^\infty e^{-pt} f(t) dt, \quad p > 0.$$

Find the Laplace transform of the following functions:

(a) $f(t) = 1$

(b) $f(t) = e^{-\alpha t}, \alpha > 0$

(c) $f(t) = \cos(\beta t), \beta \in \mathbb{R}$

Notice that the Laplace transform of a bounded function is not necessarily bounded.