

Assignment 5

Due Monday 14 November 15:00. Two problems will be selected for marking among those with boxed numbers. There are **4 points** for each of the two problems, and **2 points** for presentation. Do not forget to write **your name** and **your group** (**1:** Tuesday 12-1/Buze; **2:** Tuesday 2-3/Vogel; **3:** Thursday 1-2/Khor; **4:** Friday 12-1/Matejczyk).

1. Let f be the following function on $[0, 2] \times [0, 1]$:

$$f(x, y) = \begin{cases} \frac{xy(x^2 - y^2)}{(x^2 + y^2)^3} & \text{if } (x, y) \neq (0, 0), \\ 0 & \text{if } x = y = 0. \end{cases}$$

Compute $\int_0^2 dx \int_0^1 dy f(x, y)$ and $\int_0^1 dy \int_0^2 dx f(x, y)$.

2. Prove that the Euler integrals of the first kind (beta-function) converge when $p > 0$ and $q > 0$:

$$B(p, q) = \int_0^1 x^{p-1}(1-x)^{q-1} dx.$$

3. Let $I_n = \int_0^{\pi/2} \cos^n(x) dx$, for $n \in \mathbb{N}$. Use integration by parts to show that

$$I_n = \frac{n-1}{n} I_{n-2},$$

for all $n \geq 2$. Use this to find an explicit expression for I_n , for arbitrary n . (Notation: For $n \in \mathbb{N}$, $n!! = n(n-2)(n-4)\dots$, ending at 1 or 2, depending on whether n is odd or even.)

4.

(a) Show that $\sin t \leq t$ for all $t \geq 0$. For this, define $g(t) = t - \sin t$; use the fundamental theorem of calculus to write $g(t) = \int_0^t g'(s) ds$, and check that $g'(s) \geq 0$ for all s .

(b) Show that $\int_0^\infty x^{-a} \sin \frac{1}{x} dx$ exists as an improper integral for $a \in (0, 1)$.

(c) Show that $\int_0^\infty x^{-a} \cos \frac{1}{x} dx$ exists as an improper integral for $a \in (1, 2)$.

(Hint: Use integration by parts to get an expression involving $\int_0^\infty x^{-b} \sin \frac{1}{x} dx$ for some b , and use (b).)

5. Use integration by parts to show that $\int_0^\pi f(x) \sin(nx) dx$ goes to 0 as $n \rightarrow \infty$, for any differentiable function f .

6. The devil's staircase as limit of differentiable functions. Let $f_0(x) = x^2[1 - (1-x)^2]$, and for $n \geq 0$ and $x \in [0, 1]$, let

$$f_{n+1}(x) = \begin{cases} \frac{1}{2} f_n(3x) & \text{if } x \in [0, \frac{1}{3}), \\ \frac{1}{2} & \text{if } x \in [\frac{1}{3}, \frac{2}{3}], \\ \frac{1}{2} + \frac{1}{2} f_n(3x-2) & \text{if } x \in (\frac{2}{3}, 1]. \end{cases}$$

(a) Check that f_0 is differentiable, increasing, and that $f(0) = 0, f(1) = 1, f'(0) = f'(1) = 0$.

(b) Draw f_0, f_1 , and f_2 on the interval $[0, 1]$.

(c) Show by induction that f_n is continuous and differentiable on $[0, 1]$.

(d) Check that $(f_n)_{n \geq 1}$ is a Cauchy sequence with respect to the sup norm.

Can you conclude that the limiting function $f = \lim_n f_n$ exists and is differentiable?