Assignment 2

Due Monday 24 October 15:00. Two problems will be selected for marking among those with boxed numbers. There are **4 points** for each of the two problems, and **2 points** for presentation. Do not forget to write **your name** and **your group** (**1**: Tuesday 12-1/Buze; **2**: Tuesday 2-3/Vogel; **3**: Thursday 1-2/Khor; **4**: Friday 12-1/Matejczyk).

1. (a) Let $f(x) = x \sin \frac{1}{x}$ for $x \in (0, 2\pi]$ and $f(0) = 0$. Prove that f is regulated.
(b) Let $f(x) = \frac{1}{x}$ for $x \in (0, 1]$ and $f(0) = 0$. Prove that f is not regulated.

2. (a) Give an example of a regulated function f on $[0, 1]$ such that $f(x) \geq 0$ for all $x \in [0, 1]$, $\int_0^1 f(x)\mathrm{d}x = 0$, and such that $f(x_0) > 0$ for at least one $x_0 \in [0, 1]$.
(b) Show that if f is a continuous function on $[a, b]$ that satisfies $f(x) \geq 0$ for all $x \in [a, b]$ and $\int_a^b f(x)\mathrm{d}x = 0$, then f is identically zero.

3. Let $f : [a, b] \to \mathbb{R}$ be regulated and non-negative. Prove that $g : [a, b] \to \mathbb{R}$ defined by $g(x) = \sqrt{f(x)}$ is regulated.

4. Let $a > 0$. By using a suitable sequence of step functions on $[0, a]$, show directly from the definition that $\int_0^a x\mathrm{d}x = \frac{1}{2}a^2$.

5. By using a suitable sequence of step functions φ_n with partition $P = \{0, \frac{1}{n}, \frac{2}{n}, \dots, \frac{n-1}{n}, 1\}$, show directly from the definition that $\int_0^1 \mathrm{e}^x \, dx = \mathrm{e} - 1$.

6. The goal is to show the *Riemann-Lebesgue lemma*, namely that $\int_a^b f(x) \sin(nx)\mathrm{d}x \to 0$ as $n \to \infty$, for any regulated function f. For this, show that
(a) For all $a < b$, we have $\int_a^b \sin(nx)\mathrm{d}x \to 0$ as $n \to \infty$.
(b) By considering separately each interval of the partition, show that $\int_a^b \varphi(x) \sin(nx)\mathrm{d}x \to 0$ as $n \to \infty$ for all $\varphi \in S[a, b]$.
(c) Extend this to all $f \in R[a, b]$.

7. Give an example of a function $f : [0, 1]$ that is bounded, piecewise continuous, and not regulated. Prove that it is impossible to approximate this function: There exists $\varepsilon > 0$ such that any step function φ satisfies $\|f - \varphi\|_\infty > \varepsilon$.

8. Let $f \in R[-1, 1]$ be an *odd function*, meaning that $f(-x) = -f(x)$. Show that $\int_{-1}^1 f(x)\mathrm{d}x = 0$.