Assignment 1

Due Monday 17 October 15:00. Two problems will be selected for marking among those with boxed numbers. There are **4 points** for each of the two problems, and **2 points** for presentation.

1. Give an example of a step function $[0,1] \to \mathbb{R}$ that takes four different values and that is discontinuous at three points. Is there a step function $[0,1] \to \mathbb{R}$ which takes two different values and is discontinuous at three million points?

2. Let $\varphi, \psi : [a,b] \to \mathbb{R}$ be step functions.
(a) Prove that $|\varphi|$ is a step function and that $|\int_a^b \varphi(x)\mathrm{d}x| \le \int_a^b |\varphi(x)|\mathrm{d}x$.
(b) Prove that $\int_a^b |\varphi(x) + \psi(x)|\mathrm{d}x \le \int_a^b |\varphi(x)|\mathrm{d}x + \int_a^b |\psi(x)|\mathrm{d}x$.

3. Let φ, ψ, and hence $\varphi + \psi$, be step functions $[a,b] \to \mathbb{R}$. Write $z(\varphi)$ for the number of discontinuities of φ. Prove or disprove:
(a) $z(\varphi+\psi) \le z(\varphi) + z(\psi)$.
(b) $z(\varphi+\psi) \ge \max\{z(\varphi), z(\psi)\}$.

4. Let $\varphi, \psi : [a,b] \to \mathbb{R}$ be step functions.
(a) Prove that their product $\varphi\psi : [a,b] \to \mathbb{R}$ is a step function.
(b) Prove that $(\int_a^b \varphi(x)\psi(x)\mathrm{d}x)^2 \le \int_a^b \varphi(x)^2\mathrm{d}x \int_a^b \psi(x)^2\mathrm{d}x$.
[This is called the Cauchy-Schwarz inequality for the integral of step functions.
Suggestion: Consider the quadratic function of t defined by $\int_a^b (t\varphi + \psi)^2$.]

5. Draw a qualitative plot of the following functions on the interval $(0, \infty)$:
(a) $f(x) = \sin \frac{1}{x}$.
(b) $g(x) = \sqrt{x} \sin \frac{1}{x}$.
(c) $h(x) = \frac{\sin x}{x}$.

6. Give a proof of the additivity of integrals of step functions. Namely, prove that for any $\varphi \in S[a,b]$ and any $c \in (a,b)$, we have

$$\int_a^b \varphi(x)\mathrm{d}x = \int_a^c \varphi(x)\mathrm{d}x + \int_c^b \varphi(x)\mathrm{d}x.$$

7. Consider the step function $f \in S[0,2]$:

$$f(x) = \begin{cases} -1 & \text{if } 0 \le x < \frac{1}{2}; \\ -2 & \text{if } x = \frac{1}{2}; \\ \frac{1}{2} & \text{if } \frac{1}{2} < x \le \frac{4}{3}; \\ 1 & \text{if } \frac{4}{3} < x \le 2. \end{cases}$$

Define $F(x) = \int_0^x f(t)\mathrm{d}t$, with $x \in [0,2]$. Draw f and F.

8. For $s, t \in \mathbb{R}$, let us define the step function $f \in S[0,2]$:

$$f(x) = \begin{cases} 1 & \text{if } 0 \le x < \frac{3}{4}; \\ s & \text{if } x = \frac{3}{4}; \\ t & \text{if } \frac{3}{4} < x \le \frac{3}{2}; \\ 2 & \text{if } \frac{3}{2} < x \le 2. \end{cases}$$

Find all values of s, t for which the integral $\int_0^2 f$ is equal to 0.