Lecture notes on quantum spin systems[1]

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Contents

CHAPTER 3

Mathematical setting

1. Tensor products (of Hilbert spaces)

Let $\mathcal{H}$ be a separable Hilbert space and $\{e_i\}_{i\geq 1}$ be a finite or countable orthonormal basis. Recall that $\text{span}\{e_i\}$ denotes the space of finite linear combinations of $\{e_i\}$, and that its completion is isomorphic to $\mathcal{H}$.

DEFINITION 3.1 (Tensor product). *Let $\mathcal{H}_1, \mathcal{H}_2$ be two Hilbert spaces with respective bases $\{e_i^1\}, \{e_i^2\}$. The* **tensor product** *of $\mathcal{H}_1$ and $\mathcal{H}_2$, denoted $\mathcal{H}_1 \otimes \mathcal{H}_2$, is the completion of the linear span of $\{(e_i^1, e_j^2)\}_{i,j\geq 1}$.*

The dimension of the tensor product space satisfies

$$\dim \mathcal{H}_1 \otimes \mathcal{H}_2 = \dim \mathcal{H}_1 \cdot \dim \mathcal{H}_2. \tag{3.1}$$

Given two vectors $\varphi_1 \in \mathcal{H}_1$ and $\varphi_2 \in \mathcal{H}_2$, we can construct the element $\varphi_1 \otimes \varphi_2$ as follows. Let $\sum_i a_i e_i^1$ and $\sum_j b_j e_j^2$ be the decompositions of φ_1, φ_2 in the bases $\{e_i^1\}$ and $\{e_j^2$, respectively. Then

$$\varphi_1 \otimes \varphi_2 = \sum_{i,j\geq 1} a_i b_j e_i^1 \otimes e_j^2. \tag{3.2}$$

Notice that $\varphi_1 \otimes 2\varphi_2 = 2\varphi_1 \otimes \varphi_2 = 2(\varphi_1 \otimes \varphi_2)$. Not all elements of $\mathcal{H}_1 \otimes \mathcal{H}_2$ can be written as tensor product vectors (see Exercise 3.3).

The inner product on $\mathcal{H}_1 \otimes \mathcal{H}_2$ is

$$\langle \varphi_1 \otimes \varphi_2, \psi_1 \otimes \psi_2 \rangle = \langle \varphi_1, \psi_1 \rangle \cdot \langle \varphi_2, \psi_2 \rangle, \tag{3.3}$$

where the inner products in the right side are in $\mathcal{H}_1$ and $\mathcal{H}_2$, respectively. This extends to general elements of $\mathcal{H}_1 \otimes \mathcal{H}_2$ by linearity.

Let $A_1 \in \mathcal{B}(\mathcal{H}_1)$ and $A_2 \in \mathcal{B}(\mathcal{H}_2)$ be two bounded operators. The tensor product operator $A_1 \otimes A_2$ is an operator acting on $\mathcal{H}_1 \otimes \mathcal{H}_2$ and its action on tensor product vectors is

$$(A_1 \otimes A_2)(\varphi_1 \otimes \varphi_2) = A_1\varphi_1 \otimes A_2\varphi_2. \tag{3.4}$$

Its action on general vectors is obtained by linearity.

This construction is easily generalised to more than two Hilbert spaces. The tensor product space $\mathcal{H}_1 \otimes \cdots \otimes \mathcal{H}_n$ is the completion of the linear span of $\{(e_{j_1}^1, \ldots, e_{j_n}^n)\}_{j_1,\ldots,j_n\geq 1}$, where $\{e_j^i\}_{j\geq 1}$ is an orthonormal basis of $\mathcal{H}_i$.

2. Direct sums

DEFINITION 3.2. *The* **direct sum** *of two Hilbert spaces* $\mathcal{H}_1$ *and* $\mathcal{H}_2$*, denoted* $\mathcal{H}_1 \oplus \mathcal{H}_2$*, is the space of pairs* (φ_1, φ_2) *with* $\varphi_1 \in \mathcal{H}_1$*,* $\varphi_2 \in \mathcal{H}_2$*, with operations*

$$\alpha(\varphi_1, \varphi_2) + \beta(\psi_1, \psi_2) = (\alpha\varphi_1 + \beta\psi_1, \alpha\varphi_2 + \beta\psi_2), \qquad \alpha, \beta \in \mathbb{C}.$$

If $\{e_i^1\}$ and $\{e_j^2\}$ are bases of $\mathcal{H}_1$ and $\mathcal{H}_2$, then $\{(e_i^1, 0)\} \cup \{(0, e_j^2)\}$ is a basis of $\mathcal{H}_1 \oplus \mathcal{H}_2$. It follows that

$$\dim \mathcal{H}_1 \oplus \mathcal{H}_2 = \dim \mathcal{H}_1 + \dim \mathcal{H}_2. \tag{3.5}$$

3. Spin operators

Let $S \in \frac{1}{2}\mathbb{N}$. On $\mathbb{C}^{2S+1}$, let S^1, S^2, S^3 be hermitian matrices that satisfy the following properties:

$$[S^1, S^2] = \mathrm{i}S^3, \quad [S^2, S^3] = \mathrm{i}S^1, \quad [S^3, S^1] = \mathrm{i}S^2, \tag{3.6}$$

$$[S^1]^2 + [S^2]^2 + [S^3]^2 = S(S+1)\mathtt{Id}. \tag{3.7}$$

The existence of such matrices follows by construction: Let $|a\rangle$, $a \in \{-S, -S+1, \dots, S\}$ denote an orthonormal basis of $\mathbb{C}^{2S+1}$, and define $S^3|a\rangle = a|a\rangle$. Next, let S^+, S^- be defined by

$$S^+|a\rangle = \sqrt{S(S+1) - a(a+1)}\;|a+1\rangle, \quad S^-|a\rangle = \sqrt{S(S+1) - (a-1)a}\;|a-1\rangle. \tag{3.8}$$

Then we set $S^1 = \frac{1}{2}(S^+ + S^-)$ and $S^2 = \frac{1}{2\mathrm{i}}(S^+ - S^-)$.

LEMMA 3.1. *The operators* S^1, S^2, S^3 *constructed above satisfy the relations (3.6) and (3.7).*

PROOF. One can check the following commutation relations:

$$[S^3, S^+] = S^+, \qquad [S^3, S^-] = -S^-, \qquad [S^+, S^-] = 2S^3. \tag{3.9}$$

The relations (3.6) follow. Finally,

$$[S^1]^2 + [S^2]^2 + [S^3]^2 = S^+S^- + [S^3]^2 - S^3 = S(S+1)\mathtt{Id}. \tag{3.10}$$

□

For $S = \frac{1}{2}$, the choice above gives the Pauli matrices

$$S^1 = \tfrac{1}{2}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \qquad S^2 = \tfrac{1}{2}\begin{pmatrix} 0 & -\mathrm{i} \\ \mathrm{i} & 0 \end{pmatrix}, \qquad S^3 = \tfrac{1}{2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \tag{3.11}$$

For $S = 1$, we get

$$S^1 = \frac{1}{\sqrt{2}}\begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}, \qquad S^2 = \frac{1}{\sqrt{2}}\begin{pmatrix} 0 & -\mathrm{i} & 0 \\ \mathrm{i} & 0 & -\mathrm{i} \\ 0 & \mathrm{i} & 0 \end{pmatrix}, \qquad S^3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix}. \tag{3.12}$$

Notice that, for $S > 1$, the matrix of S^1 is not proportional to $\delta_{|i-j|,1}$. Spin operators are not unique, but their spectrum is uniquely determined by the commutation relations.

LEMMA 3.2. *Assume that S^1, S^2, S^3 are hermitian matrices in $\mathbb{C}^{2S+1}$ that satisfy the relations (3.6) and (3.7). Then each S^i has eigenvalues $\{-S, -S+1, \dots, S\}$.*

PROOF. It is enough to prove the claim for S^3. Define $S^+ = S^1 + \mathrm{i}S^2$ and $S^- = S^1 - \mathrm{i}S^2$. One can check that

$$\begin{aligned} S^+S^- &= S(S+1)\mathtt{Id} - [S^3]^2 + S^3, \\ S^-S^+ &= S(S+1)\mathtt{Id} - [S^3]^2 - S^3. \end{aligned} \tag{3.13}$$

Let $|a\rangle$ be an eigenvector of S^3 with eigenvalue a. It follows from Eq. (3.13) that

$$\begin{aligned} \big\| S^+|a\rangle \big\|^2 &= \langle a|S^-S^+|a\rangle = S(S+1) - a^2 - a \geq 0, \\ \big\| S^-|a\rangle \big\|^2 &= \langle a|S^+S^-|a\rangle = S(S+1) - a^2 + a \geq 0. \end{aligned} \tag{3.14}$$

Then $|a| \leq S$, and $S^+|a\rangle \neq 0$ if $a \neq S$. Next, observe that $[S^3, S^+] = S^+$. Then

$$S^3 S^+|a\rangle = (a+1)S^+|a\rangle. \tag{3.15}$$

Then if $a \neq S$ is an eigenvalue, $a+1$ is also an eigenvalue. There are similar relations with S^-, so that if $a \neq -S$ is an eigenvalue, $a-1$ is also an eigenvalue. It follows that $\{-S, -S+1, \dots S\}$ is the set of eigenvalues. □

Notice that the relations (3.8) always hold; this follows from (3.15) and (3.14). It follows from the parallelogram identity that $\|S^\pm\| = \sqrt{2}S$:

$$\begin{aligned} \|S^+\|^2 &= \tfrac14 (2\|S^+\|^2 + 2\|S^-\|^2) = \tfrac14 (\|S^+ + S^-\|^2 + \|S^+ - S^-\|^2) \\ &= \tfrac14 (4\|S^1\|^2 + 4\|S^2\|^2) = 2S^2. \end{aligned} \tag{3.16}$$

Spin operators are related to rotations in $\mathbb{R}^3$. Let $\vec{S} = (S^1, S^2, S^3)$. Given $\vec{a} \in \mathbb{R}^3$, let

$$S^{\vec{a}} = \vec{a} \cdot \vec{S} = a_1 S^1 + a_2 S^2 + a_3 S^3. \tag{3.17}$$

By linearity, the commutation relations (3.6) generalize as

$$[S^{\vec{a}}, S^{\vec{b}}] = \mathrm{i} S^{\vec{a} \times \vec{b}}. \tag{3.18}$$

Finally, let $R_{\vec{a}}\vec{b}$ denote the vector $\vec{b}$ rotated around $\vec{a}$ by the angle $\|\vec{a}\|$.

LEMMA 3.3.

$$\mathrm{e}^{-\mathrm{i}S^{\vec{a}}}\, S^{\vec{b}}\, \mathrm{e}^{\mathrm{i}S^{\vec{a}}} = S^{R_{\vec{a}}\vec{b}}.$$

PROOF. We replace $\vec{a}$ by $s\vec{a}$, and we check that both sides of the identity satisfy the same differential equation. We find

$$\frac{\mathrm{d}}{\mathrm{d}s}\, \mathrm{e}^{-\mathrm{i}S^{s\vec{a}}}\, S^{\vec{b}}\, \mathrm{e}^{\mathrm{i}S^{s\vec{a}}} = -\mathrm{i}[S^{\vec{a}}, \mathrm{e}^{-\mathrm{i}S^{s\vec{a}}}\, S^{\vec{b}}\, \mathrm{e}^{\mathrm{i}S^{s\vec{a}}}], \tag{3.19}$$

and

$$\frac{\mathrm{d}}{\mathrm{d}s} S^{R_{s\vec{a}}\vec{b}} = \Big(\frac{\mathrm{d}}{\mathrm{d}s} R_{s\vec{a}}\vec{b}\Big) \cdot \vec{S} = \Big(\vec{a} \times R_{s\vec{a}}\vec{b}\Big) \cdot \vec{S} = -\mathrm{i}[S^{\vec{a}}, S^{R_{s\vec{a}}\vec{b}}]. \tag{3.20}$$

We used (3.18) for the last identity. □

It also follows from Lemmas 3.2 and 3.3 that any matrix $S^{\vec{a}}$, $\vec{a} \in \mathbb{R}^3$ with $\|\vec{a}\| = 1$, has eigenvalues $\{-S, -S+1, \dots, S\}$.

Corollary 3.4. *Let $\psi_{\vec{b},c}$ be the eigenvector of $S^{\vec{b}}$ with eigenvalue c. Then $\mathrm{e}^{-\mathrm{i}S^{\vec{a}}}\,\psi_{\vec{b},c}$ is eigenvector of $S^{R_{\vec{a}}\vec{b}}$ with eigenvalue c.*

Proof. Using Lemma 3.3,

$$S^{R_{\vec{a}}\vec{b}}\,\mathrm{e}^{-\mathrm{i}S^{\vec{a}}}\,\psi_{\vec{b},c} = \mathrm{e}^{-\mathrm{i}S^{\vec{a}}}\,S^{\vec{b}}\psi_{\vec{b},c} = c\,\mathrm{e}^{-\mathrm{i}S^{\vec{a}}}\,\psi_{\vec{b},c}. \tag{3.21}$$

□

Finally, let us note the following useful relations:

$$\begin{aligned} \mathrm{e}^{-\mathrm{i}aS^3}\,S^+\,\mathrm{e}^{\mathrm{i}aS^3} &= \mathrm{e}^{-\mathrm{i}a}\,S^+, \\ \mathrm{e}^{-\mathrm{i}aS^3}\,S^-\,\mathrm{e}^{\mathrm{i}aS^3} &= \mathrm{e}^{\mathrm{i}a}\,S^-. \end{aligned} \tag{3.22}$$

Exercise 3.1. *For $S = 1$, check that the following matrices satisfy the spin relations.*

$$S^1 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -\mathrm{i} \\ 0 & \mathrm{i} & 0 \end{pmatrix}, \qquad S^2 = \begin{pmatrix} 0 & 0 & -\mathrm{i} \\ 0 & 0 & 0 \\ \mathrm{i} & 0 & 0 \end{pmatrix}, \qquad S^3 = \begin{pmatrix} 0 & -\mathrm{i} & 0 \\ \mathrm{i} & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

Exercise 3.2. *For $S = 1$, check that the following matrices do not satisfy the spin relations.*

$$S^1 = \tfrac{1}{2}\begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \qquad S^2 = \tfrac{1}{2}\begin{pmatrix} 0 & -\mathrm{i} & 0 \\ \mathrm{i} & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \qquad S^3 = \tfrac{1}{2}\begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

Exercise 3.3. *Show that there exist no $\varphi_1 \in \mathcal{H}_1$, $\varphi_2 \in \mathcal{H}_2$ such that*

$$e_1^1 \otimes e_1^2 + e_2^1 \otimes e_2^2 = \varphi_1 \otimes \varphi_2.$$

Exercise 3.4. *Show that*

(a) $\|\varphi_1 \otimes \varphi_2\| = \|\varphi_1\| \cdot \|\varphi_2\|$ *for all* $\varphi_1 \in \mathcal{H}_1$, $\varphi_2 \in \mathcal{H}_2$.
(b) $\|A_1 \otimes A_2\| = \|A_1\| \cdot \|A_2\|$ *for all* $A_1 \in \mathcal{B}(\mathcal{H}_1)$, $A_2 \in \mathcal{B}(\mathcal{H}_2)$.

Exercise 3.5. *Show that*

$$\underbrace{\mathcal{H} \oplus \cdots \oplus \mathcal{H}}_{n \text{ times}} \simeq \mathcal{H} \otimes \mathbb{C}^n.$$

CHAPTER 4

Models of quantum spins

1. Origin and motivation

The electron is a particle that possesses a mass m, a charge $-e$, and also a spin. In quantum mechanics, the state space for an electron in domain Ω is the Hilbert space $\mathcal{H}_1 = L^2(\Omega) \otimes \mathbb{C}^2$. The description of an atom with Z protons and N electrons turns out to be very complicated (except for $N = 1$). The Hilbert space is the antisymmetric subspace of $\mathcal{H}_1^{\otimes N}$ and the Hamiltonian is the operator

$$H = -\tfrac{\hbar^2}{2m} \sum_{i=1}^{N} \Delta_i - Ze^2 \sum_{i=1}^{N} \frac{1}{\|X_i\|} + e^2 \sum_{1 \leq i < j \leq N} \frac{1}{\|X_i - X_j\|}. \tag{4.1}$$

Here, Δ_i is the Laplacian for the ith particle, that is

$$\Delta_i = \Big(\mathbb{1}_{L^2(\Omega)} \otimes \mathbb{1}_{\mathbb{C}^2} \Big) \otimes \cdots \otimes \Big(\Delta_{L^2(\Omega)} \otimes \mathbb{1}_{\mathbb{C}^2} \Big) \otimes \cdots \otimes \Big(\mathbb{1}_{L^2(\Omega)} \otimes \mathbb{1}_{\mathbb{C}^2} \Big). \tag{4.2}$$

The position operator of the ith particle, $X_i = (X_i^1, X_i^2, X_i^3)$, is defined similarly. We assumed that the nucleus is located at the origin. A system of condensed matter is even more complicated, as it consists of many atoms and many electrons. Evidence shows that, in many cases, atoms arrange themselves in periodic lattices. This is ill-understood but we accept it, so we assume that the positions of the atoms are given by the vertices of a regular lattice. Here, "lattice" means a graph with a periodic structure.

Our goal is to understand the behaviour of the electrons, that is, to understand the electronic properties of the system. The evolution of the system is formidably complex. However, a large system at equilibrium is described by statistical mechanics. The expectation of the observable A is given by

$$\langle A \rangle = \frac{\mathrm{Tr}\, A\, \mathrm{e}^{-\beta H}}{\mathrm{Tr}\ \mathrm{e}^{-\beta H}}. \tag{4.3}$$

Here, β is a parameter that is equal to the inverse temperature of the system. This linear functional is called a *finite-volume Gibbs state*. Its justification is physically and mathematically delicate, but we accept it.

Eq. (4.3) is still intractable and we are led to the notion of *models*. Models are grossly simplified systems that nevertheless capture several relevant mechanisms at work in the original systems. The main approach of theoretical condensed matter physics consists in introducing interesting models, to work out their properties, and to link them with actual physical systems.

2. Models of quantum spin systems

We obtain an important class of models by assuming that only one electron per atom is relevant, and by restricting our attention to its spin. We assume that the total Hamiltonian is the sum of two-body interactions. If $S = \frac{1}{2}$ and if we also assume that the interaction is rotation invariant, it necessarily is of the form $\pm \vec{S}_x \cdot \vec{S}_y$. We actually consider the following more general class of models.

Let Λ be the (finite) set of vertices. The Hilbert space is

$$\mathcal{H}_\Lambda = \otimes_{x \in \Lambda} \mathbb{C}^{2S+1}, \tag{4.4}$$

where $S \in \frac{1}{2}\mathbb{N}$ is a fixed parameter. The Hamiltonian is

$$H_\Lambda = -\tfrac{1}{2} \sum_{x,y \in \Lambda} \Big(J^1_{xy} S^1_x S^1_y + J^2_{xy} S^2_x S^2_y + J^3_{xy} S^3_x S^3_y \Big). \tag{4.5}$$

Here, $J^i_{xy} = J^i_{yx}$ are real parameters. The spin operator S^i_x is equal to

$$S^i_x = S^i \otimes \mathbb{1}_{\Lambda \setminus \{x\}} \tag{4.6}$$

where $\mathbb{1}_{\Lambda \setminus \{x\}}$ is the identity in $\otimes_{y \in \Lambda \setminus \{x\}} \mathbb{C}^{2S+1}$.

It is natural to choose Λ to be a box in $\mathbb{Z}^d$ and to set $J^i_{xy} = 0$ unless x, y are nearest-neighbours. Several famous models belong to the general class:

- The case $J^3_{xy} = J$ for all neighbours x, y, and $J^i_{xy} = 0$ for $i = 1, 2$ or x, y not neighbours. This is the Ising model, invented by Lenz in 1920. All relevant operators commute with one another and the quantum setting is superfluous.
- The case $J^1_{xy} = J^2_{xy} = J$ for all neighbours x, y, and $J^i_{xy} = 0$ for $i = 3$ or x, y not neighbours. This model is known as the quantum XY model, or model of quantum rotators.
- The case $J^1_{xy} = J^2_{xy} = J^3_{xy} = J$ for all neighbours x, y, and $J^i_{xy} = 0$ for x, y not neighbours. This is the ferromagnetic Heisenberg model when $J > 0$ and the antiferromagnetic Heisenberg model when $J < 0$.

3. System of two spins

Systems of two spins are relevant for interaction operators. The Hilbert space is $\mathbb{C}^{2S+1} \otimes \mathbb{C}^{2S+1}$, and the spin operators are $S^i_1 = S^i \otimes \mathbb{1}$, $S^i_2 = \mathbb{1} \otimes S^i$, $i = 1, 2, 3$.

LEMMA 4.1.

- *The eigenvalues of* $(\vec{S}_1 + \vec{S}_2)^2$ *are* $J(J+1)$, *with* $J = 0, 1, \dots, 2S$. *The degeneracy of* J *is* $2J + 1$.
- $[S^i_1 + S^i_2, (\vec{S}_1 + \vec{S}_2)^2] = 0$, $i = 1, 2, 3$, *and the eigenvalues of* $S^i_1 + S^i_2$ *in the sector* J *are* $-J, -J + 1, \dots, J$.

In the sequel we call J an eigenvalue of $(\vec{S}_1 + \vec{S}_2)^2$, even though the actual eigenvalue is $J(J + 1)$.

Proof. It is enough to consider the case $i = 3$. We already have a basis of eigenvectors of $S_1^3 + S_2^3$, namely $|a\rangle \otimes |b\rangle$ with $a, b \in \{-S, -S+1, \dots, S\}$, so that the eigenvalues of $S_1^3 + S_2^3$ are $m = -2S, -2S+1, \dots, 2S$, with degeneracy $2S+1-|m|$. The following relations are easily checked:

$$\begin{aligned}(\vec{S}_1 + \vec{S}_2)^2 &= 2S(S+1) + 2\vec{S}_1 \cdot \vec{S}_2 \\ &= \tfrac{1}{2}(S_1^+ + S_2^+)(S_1^- + S_2^-) + \tfrac{1}{2}(S_1^- + S_2^-)(S_1^+ + S_2^+) + (S_1^3 + S_2^3)^2,\end{aligned} \tag{4.7}$$

and

$$2\vec{S}_1 \cdot \vec{S}_2 = S_1^+ S_2^- + S_1^- S_2^+ + 2S_1^3 S_2^3. \tag{4.8}$$

The commutation relation of the lemma follows. Furthermore,

$$[S_1^+ + S_2^+, \vec{S}_1 \cdot \vec{S}_2] = [S_1^- + S_2^-, \vec{S}_1 \cdot \vec{S}_2] = 0. \tag{4.9}$$

The idea of the proof is similar to that of Lemma 3.2. Let ψ be an eigenvector of $(\vec{S}_1 + \vec{S}_2)^2$ and $S_1^3 + S_2^3$ with eigenvalues (J, m), and consider $(S_1^{(\pm)} + S_2^{(\pm)})\psi$. Its norm is nonnegative, so that $|m| \leq J$. It differs from 0 if $m \neq \pm J$ and it is eigenvector with eigenvalues $(J, m \pm 1)$. Then J must be an integer smaller or equal to $2S$ — otherwise, it is possible to construct eigenvectors of $S_1^3 + S_2^3$ with eigenvalues satisfying $|m| > 2S$.

Let $D_{J,m}$ denote the degeneracy of (J, m). If ψ and ψ' are two orthogonal eigenvectors with eigenvalues (J, m), we can check that $(S_1^+ + S_2^+)\psi$ and $(S_1^+ + S_2^+)\psi'$ are also orthogonal — this can be seen with the help of

$$(S_1^- + S_2^-)(S_1^+ + S_2^+) = 2S(S+1)\mathtt{Id} + 2\vec{S}_1 \cdot \vec{S}_2 - (S_1^3 + S_2^3)^2 - (S_1^3 + S_2^3). \tag{4.10}$$

It follows that $D_{J,m+1} \geq D_{J,m}$. A similar argument with $(S_1^- + S_2^-)$ implies that $D_{J,m-1} \geq D_{J,m}$. Then $D_{J,m}$ does not depend on m; call it $\bar{D}_J$.

The degeneracy D_m of m can be written as

$$D_m = \sum_{J=|m|}^{2S} \bar{D}_J. \tag{4.11}$$

Then $\bar{D}_J = D_J - D_{J+1}$. Since $D_m = 2S + 1 - |m|$, we obtain $\bar{D}_J = 1$ and the lemma follows. □

Exercise 4.1. *In each case, show that the following Hamiltonians are related by unitary transformations. Write the unitary matrices explicitly.*

(a) $H_1 = -\frac{1}{2}\sum_{x,y} J_{xy} S_x^1 S_y^1$, $H_2 = -\frac{1}{2}\sum_{x,y} J_{xy} S_x^2 S_y^2$, $H_3 = -\frac{1}{2}\sum_{x,y} J_{xy} S_x^3 S_y^3$.

(b) $H_1 = -\frac{1}{2}\sum_{x,y} J_{xy}\Big(S_x^2 S_y^2 + S_x^3 S_y^3\Big)$, $H_2 = -\frac{1}{2}\sum_{x,y} J_{xy}\Big(S_x^1 S_y^1 + S_x^3 S_y^3\Big)$, $H_3 = -\frac{1}{2}\sum_{x,y} J_{xy}\Big(S_x^1 S_y^1 + S_x^2 S_y^2\Big)$.

(c) *Assume that the graph is bipartite, that is,* $\Lambda = \Lambda_\mathrm{A} \cup \Lambda_\mathrm{B}$ *and* $J_{xy} = 0$ *if* $x, y \in \Lambda_\mathrm{A}$ *or* $x, y \in \Lambda_\mathrm{B}$*. Then let* $H_1 = -\frac{1}{2} \sum_{x,y} J_{xy} \Big(S_x^1 S_y^1 + S_x^2 S_y^2 \Big)$, $H_2 = -\frac{1}{2} \sum_{x,y} J_{xy} \Big(S_x^1 S_y^1 - S_x^2 S_y^2 \Big)$, $H_3 = +\frac{1}{2} \sum_{x,y} J_{xy} \Big(S_x^1 S_y^1 + S_x^2 S_y^2 \Big)$.

EXERCISE 4.2. *Consider the Hilbert space* $\mathbb{C}^2 \otimes \mathbb{C}^2$ *(case* $S = \frac{1}{2}$*) and the basis* $|a\rangle \otimes |b\rangle$, $a, b = \pm\frac{1}{2}$. *Write down the eigenvector of* $(\vec{S}_1 + \vec{S}_2)^2$ *with eigenvalue* $J = 0$.

CHAPTER 5

Gibbs states

The most striking phenomenon of equilibrium statistical mechanics is that of a *phase transition*. We are all familiar with transitions from ice to water and water to vapour; more generally, from solid to liquid to gas. Here, we are concerned with magnetic properties of solids. Ferromagnetism is the ability for a material to remain magnetised, after the external magnetic field has been removed. This actually take place only if the temperature is below the *Curie temperature*. The statistical mechanics explanation is that low temperature ferromagnets can be in several different *Gibbs states* that are characterised by the direction of their magnetisation. Since only one Gibbs state exists at high temperatures, their number must change as the temperature is lowered, which corresponds to the occurrence of a phase transition.

1. Evolution operator

Let $N = 2S+1$. We also use the notation $\Lambda \subset\subset \mathbb{Z}^d$ when Λ is a finite subset of $\mathbb{Z}^d$. Let $\mathcal{H}_\Lambda = \otimes_{x\in\Lambda} \mathbb{C}^N$ and $\mathcal{A}_\Lambda = \mathcal{B}(\mathcal{H}_\Lambda)$ the algebra of linear operators on $\mathcal{H}_\Lambda$. It is a C^* algebra, namely, it possesses of hermitian conjugation and a norm. If $\Lambda \subset \Lambda'$, we view $\mathcal{A}_\Lambda$ as a subalgebra of $\mathcal{A}_{\Lambda'}$ by identifying $A \in \mathcal{A}_\Lambda$ with $A \otimes \mathbb{1}_{\Lambda'\setminus\Lambda} \in \mathcal{A}_{\Lambda'}$.

Let $(\Phi_X)_{X\subset\mathbb{Z}^d}$ denote an "interaction", that is, a collection of operators $\Phi_X \in \mathcal{A}_X$, for any finite subset X of $\mathbb{Z}^d$. The norm of an interaction is defined by

$$\|\Phi\|_r = \sup_{x\in\mathbb{Z}^d} \sum_{X\ni x} \|\Phi_X\| \, \mathrm{e}^{r|X|} \,. \tag{5.1}$$

Here, $\|\Phi_X\|$ denotes the usual operator norm in $\mathcal{A}_X$, and $r \geq 0$ is a parameter. The Hamiltonian associated with a finite domain $\Lambda \subset \mathbb{Z}^d$ is given by

$$H_\Lambda = \sum_{X\subset\Lambda} \Phi_X. \tag{5.2}$$

It is time to give a precise meaning to the notion of infinite-volume limit. The next definition will apply to several different objects such as vectors, operators, states, etc...

Definition 5.1. *Let $(\mathcal{X}, d)$ be a metric space and let (x_Λ) be a family of elements of $\mathcal{X}$ indexed by finite subsets $\Lambda \subset\subset \mathbb{Z}^d$. We say that (x_Λ) converges to $x \in \mathcal{X}$ as $\Lambda \nearrow \mathbb{Z}^d$,*

$$\lim_{\Lambda\nearrow\mathbb{Z}^d} x_\Lambda = x,$$

if for every $\varepsilon > 0$, there exists $\Lambda_\varepsilon \subset\subset \mathbb{Z}^d$ such that $d(x_\Lambda, x) < \varepsilon$ for all finite $\Lambda \supset \Lambda_\varepsilon$.

In order to describe infinite systems, we consider the C^*-algebra $\mathcal{A}$ of quasi-local observables, which is the norm-completion of the usual algebra of local observables

$$\mathcal{A} = \overline{\mathcal{A}_0}, \qquad \text{where} \qquad \mathcal{A}_0 = \bigvee_{\Lambda \nearrow \mathbb{Z}^d} \mathcal{A}_\Lambda. \tag{5.3}$$

For $t \in \mathbb{C}$, let α_t^Λ be the linear automorphism of $\mathcal{A}_\Lambda$ that describes the time evolution of operators ("observables") in $\mathcal{A}_\Lambda$, namely

$$\alpha_t^\Lambda(a) = \mathrm{e}^{\mathrm{i}tH_\Lambda}\, a\, \mathrm{e}^{-\mathrm{i}tH_\Lambda}\,. \tag{5.4}$$

By tensoring with identities, α_t^Λ can be extended as a bounded operator $\mathcal{A} \to \mathcal{A}$. Its norm depends a priori on Λ.

We first address the question of the existence of the infinite-volume limit of α_t^Λ. In view of the discussion of KMS states below, we need to consider complex times as well. It turns out that α_t^Λ converges uniformly to a bounded operator when $t \in \mathbb{R}$; it converges pointwise when $|\mathrm{Im}\, t|$ is small; it is not known otherwise.

Proposition 5.1 (Infinite-volume limit of the evolution operator). *Assume that $\|\Phi\|_r < \infty$ for some $r > 0$. Then*

(a) *If $t \in \mathbb{C}$ and $|\mathrm{Im}\, t| < \frac{r}{2\|\Phi\|_r}$, there exists an automorphism $\alpha_t : \mathcal{A}_0 \to \mathcal{A}$ such that*
$$\lim_{\Lambda \nearrow \mathbb{Z}^d} \|\alpha_t^\Lambda(a) - \alpha_t(a)\| = 0$$
for all $a \in \mathcal{A}_0$. Further, we have
$$\|\alpha_t^\Lambda(a)\| \leq \|a\|\, \mathrm{e}^{r|X|} \Big(1 - |t| \frac{2\|\Phi\|_r}{r}\Big)^{-1}$$
whenever $a \in \mathcal{A}_X$.

(b) *For $t \in \mathbb{R}$, α_t is a *isomorphism with $\|\alpha_t\| = 1$ and it satisfies the group property*
$$\alpha_{s+t}(a) = \alpha_s\big(\alpha_t(a)\big).$$

In case clarification is needed, α_t is an automorphism in the sense that $\alpha_t(ab) = \alpha_t(a)\alpha_t(b)$. We also have $\alpha_t(a)^* = \alpha_{\bar{t}}(a^*)$, so that α_t is a *automorphism for real t.

The proof consists of the following steps.

(i) If $|t| < \frac{r}{2\|\Phi\|_r}$, $(\alpha_t^\Lambda)_{\Lambda \subset\subset \mathbb{Z}^d}$ is Cauchy for each fixed $a \in \mathcal{A}_0$. We denote the limit $\alpha_t(a)$.
(ii) For $t \in \mathbb{R}$, we have $\|\alpha_t^\Lambda(a)\| = \|a\|$ for all Λ, so $\|\alpha_t\| = 1$.
(ii) We use the group property to extend α_t it to the whole real line, then to the infinite strip.

For the first step, we need the multicommutator expansion. Let $\mathrm{ad}_A(B) = [A, B]$ denote the "adjoint endomorphism".

LEMMA 5.2 (Multicommutator expansion). *Let A and B be two operators on the same finite-dimensional Hilbert space. Then*

$$\mathrm{e}^{A}\, B\, \mathrm{e}^{-A} = \sum_{n\geq 0} \frac{1}{n!} \mathrm{ad}_A^n(B).$$

PROOF. We show that $\mathrm{e}^{sA}\, B\, \mathrm{e}^{-sA}$ and $\sum_n \frac{s^n}{n!} \mathrm{ad}_A^n(B)$ satisfy the same differential equation. First,

$$\frac{\mathrm{d}}{\mathrm{d}s} \mathrm{e}^{sA}\, B\, \mathrm{e}^{-sA} = \big[A,\, \mathrm{e}^{sA}\, B\, \mathrm{e}^{-sA}\big]. \tag{5.5}$$

Second,

$$\frac{\mathrm{d}}{\mathrm{d}s} \sum_{n\geq 0} \frac{s^n}{n!} \mathrm{ad}_A^n(B) = \sum_{n\geq 1} \frac{s^{n-1}}{(n-1)!} \mathrm{ad}_A\Big(\mathrm{ad}_A^{n-1}(B)\Big) = \Big[A, \sum_{n\geq 0} \frac{s^n}{n!} \mathrm{ad}_A^n(B)\Big]. \tag{5.6}$$

□

PROOF OF PROPOSITION 5.1. Let $a \in \mathcal{A}_Y$ for some $Y \subset\subset \mathbb{Z}^d$. We show that $(\alpha_t^\Lambda(a))_{\Lambda\subset\subset\mathbb{Z}^d}$ is Cauchy. By Lemma 5.2, we have

$$\begin{aligned} \alpha_t^\Lambda(a) &= \sum_{n\geq 0} \frac{(\mathrm{i}t)^n}{n!} \mathrm{ad}_{H_\Lambda}^n(a) \\ &= \sum_{n\geq 0} \frac{(\mathrm{i}t)^n}{n!} \sum_{X_1,\dots,X_n\subset\Lambda} [\Phi_{X_n}, [\Phi_{X_{n-1}}, \dots [\Phi_{X_1}, a] \dots]. \end{aligned} \tag{5.7}$$

We show that this series converges absolutely for small $|t|$. In order for the commutators to differ from zero, the sets must satisfy

$$\begin{aligned} &X_1 \cap Y \neq \emptyset, \\ &X_2 \cap (X_1 \cup Y) \neq \emptyset, \\ &\qquad\vdots \\ &X_n \cap (X_1 \cup \dots \cup X_{n-1} \cup Y) \neq \emptyset. \end{aligned} \tag{5.8}$$

The sum over such sets can be realised by first summing over sets that contain the origin, then by summing over translations so that (5.8) is satisfied. There are no more than

$$\begin{aligned} |Y| &\text{ translations for } X_1, \\ |X_1| + |Y| &\text{ translations for } X_2, \\ &\vdots \\ |X_1| + \dots + |X_{n-1}| + |Y| &\text{ translations for } X_n. \end{aligned} \tag{5.9}$$

We get

$$\Big\| \sum_{X_1,\dots,X_n} [\Phi_{X_n}, \dots [\Phi_{X_1}, a] \dots] \Big\| \leq \|a\| 2^n \sum_{X_1,\dots,X_n \ni 0} \big(|X_1| + \dots + |X_n| + |Y| \big)^n \prod_{i=1}^n \|\Phi_{X_i}\| \\ \leq \|a\| \, \mathrm{e}^{r|Y|} \, n! \Big(\frac{2\|\Phi\|_r}{r} \Big)^n. \tag{5.10}$$

We used $c^n \leq n! r^{-n} \mathrm{e}^{rc}$, which is obvious from the Taylor series of e^{rc}. It follows that $\alpha_t^\Lambda(a)$ is absolutely convergent whenever $|t| < \frac{r}{2\|\Phi\|_r}$ for some $r > 0$. Notice the bound

$$\|\alpha_t^\Lambda(a)\| \leq \|a\| \, \mathrm{e}^{r|Y|} \Big(1 - |t| \frac{2\|\Phi\|_r}{r} \Big)^{-1}. \tag{5.11}$$

for all $a \in \mathcal{A}_Y$. It is uniform in Λ but not in Y.

If $\Lambda' \supset \Lambda$, we have

$$\alpha_t^{\Lambda'}(a) - \alpha_t^\Lambda(a) = \sum_{n \geq 0} \frac{(\mathrm{i}t)^n}{n!} \sum_{\substack{X_1,\dots,X_n : Y \\ \cup X_i \not\subset \Lambda}} [\Phi_{X_n}, [\Phi_{X_{n-1}}, \dots [\Phi_{X_1}, a] \dots]. \tag{5.12}$$

The second sum is over sets in Λ' that satisfy the constraint (5.8) and whose union is not contained in Λ. For small $|t|$, it follows from the absolute convergence of the series that (5.12) is as small as we want by taking Λ large enough. Hence $(\alpha_t^\Lambda(a))_\Lambda$ is Cauchy, and it converges since $\mathcal{A}$ is complete. We define $\alpha_t(a)$ to be equal to the limit.

The map α_t is clearly linear and it satisfies $\alpha_t(a^*) = \alpha_t(a)^*$.

If $\|\alpha_s^\Lambda - \alpha_s\| \to 0$ and $\|\alpha_t^\Lambda - \alpha_t\| \to 0$, we can define $\alpha_{s+t} = \alpha_s \circ \alpha_t$ and we have, for all $\|a\| = 1$,

$$\|\alpha_{s+t}^\Lambda(a) - \alpha_{s+t}(a)\| \leq \|\alpha_s^\Lambda\big(\alpha_t^\Lambda(a) - \alpha_t(a)\big)\| + \|(\alpha_s^\Lambda - \alpha_s)(\alpha_t(a))\| \\ \leq \|\alpha_t^\Lambda - \alpha_t\| + \|\alpha_s^\Lambda - \alpha_s\|, \tag{5.13}$$

which goes to 0. This allows to extend α_t to the whole real line; the group property is indeed satisfied. Finally, if $z = t + \mathrm{i}\beta$ with $|\beta| < \frac{r}{2\|\Phi\|_r}$, we have

$$\alpha_{t+\mathrm{i}\beta}^\Lambda(a) = \alpha_t^\Lambda\big(\alpha_{\mathrm{i}\beta}^\Lambda(a)\big) \to \alpha_t\big(\alpha_{\mathrm{i}\beta}(a)\big). \tag{5.14}$$

This allows to define $\alpha_{t+\mathrm{i}\beta} = \alpha_t \circ \alpha_{\mathrm{i}\beta}$. □

2. KMS states

A "state" is a bounded, positive, normalised linear functional on $\mathcal{A}$. That is, it satisfies

$$\rho(\mathbb{1}) = 1, \\ \rho(a^* a) \geq 0, \tag{5.15}$$

for all $a \in \mathcal{A}$. Notice that $\rho(a) \in \mathbb{R}$ when a is hermitian, and $\rho(a^*) = \overline{\rho(a)}$ in general, since any operator can be written as the sum of hermitian and anti-hermitian operators, $a = \frac{1}{2}(a + a^*) + \frac{1}{2}(a - a^*)$.

LEMMA 5.3. *States are bounded linear functionals with norm 1.*

PROOF. Since $\rho(\mathbb{1}) = 1$, it is clear that $\|\rho\| \geq 1$. We show that $|\rho(a)| \leq 1$ for all $\|a\| = 1$. If $a = a^*$, we have $1 \pm a \geq 0$ and $\rho(1 \pm a) = 1 \pm \rho(a) \geq 0$, so that $|\rho(a)| \leq 1$. For general a, we have

$$\rho\big((a^* - \overline{\rho(a)})(a - \rho(a))\big) \geq 0 \tag{5.16}$$

and linearity implies that $|\rho(a)|^2 \leq \rho(a^* a) \leq 1$. □

For finite volumes, equilibrium states of quantum statistical mechanics are given by the Gibbs states

$$\rho_\Lambda(a) = \frac{1}{\operatorname{Tr} \mathrm{e}^{-\beta H_\Lambda}} \operatorname{Tr} a \, \mathrm{e}^{-\beta H_\Lambda}\,, \tag{5.17}$$

where the parameter β represents the inverse temperature of the system. Using a compactness argument, one obtains the existence of cluster points in the infinite-volume limit. But this limit is rather delicate.An alternative approach is to seek a property that directly deals with states on $\mathcal{A}$ and that characterises equilibrium. This is the motivation for the *KMS condition*, named after Kubo, Martin, and Schwinger. It is not straightforward and it requires a discussion.

The starting point is the following identity for finite volumes, that follows from cyclicity of the trace:

$$\begin{aligned} \rho_\Lambda\big(ab\big) &= \frac{1}{\operatorname{Tr} \mathrm{e}^{-\beta H_\Lambda}} \operatorname{Tr} ab \, \mathrm{e}^{-\beta H_\Lambda} \\ &= \frac{1}{\operatorname{Tr} \mathrm{e}^{-\beta H_\Lambda}} \operatorname{Tr} b \, \mathrm{e}^{\mathrm{i}(\mathrm{i}\beta) H_\Lambda} \, a \, \mathrm{e}^{-\mathrm{i}(\mathrm{i}\beta) H_\Lambda} \; \mathrm{e}^{-\beta H_\Lambda} \\ &= \rho_\Lambda\big(b \, \alpha^\Lambda_{\mathrm{i}\beta}(a)\big). \end{aligned} \tag{5.18}$$

When the infinite volume limit of $\alpha^\Lambda_{\mathrm{i}\beta}$ exists, which the case for β small, we get the following rather simple condition.

DEFINITION 5.2. *Assume that the inverse temperature satisfies* $0 \leq \beta \leq \frac{r}{2\|\Phi\|_r}$. *A state* ρ *satisfies the KMS condition if*

$$\rho\big(ab\big) = \rho\big(b \, \alpha_{\mathrm{i}\beta}(a)\big), \tag{5.19}$$

for all a, b *in* $\mathcal{A}_0$.

Condensed matter physics is much more interesting at low temperatures, and it is worth extending this definition to large β. A key ingredient is the Three-Line Lemma of complex analysis; recall that if $f(z)$ is analytic in the strip $0 \leq \operatorname{Im} z \leq \beta$, the lemma states that for all $0 \leq c \leq 1$, we have

$$\sup_{t \in \mathbb{R}} |f(t + \mathrm{i}c\beta)| \leq \big(\sup_{t \in \mathbb{R}} |f(t)|\big)^{1-c} \big(\sup_{t \in \mathbb{R}} |f(t + \mathrm{i}\beta)|\big)^{c}. \tag{5.20}$$

For finite volume, we have

$$\rho_\Lambda\big(b \, \alpha^\Lambda_{t+\mathrm{i}\beta}(a)\big) = \rho_\Lambda\big(\alpha^\Lambda_t(a) \, b\big). \tag{5.21}$$

Since $\|\alpha_t^\Lambda\| = 1$ and $\|\rho\| = 1$, we have

$$\big|\rho_\Lambda\big(b\alpha_t^\Lambda(a)\big)\big|,\ \big|\rho_\Lambda\big(b\alpha_{t+\mathrm{i}\beta}^\Lambda(a)\big)\big| \leq \|a\|\,\|b\|. \tag{5.22}$$

It follows from the Three-Line Lemma that $|\rho_\Lambda(\alpha_z^\Lambda(a)\,b)| \leq \|a\|\,\|b\|$ for z in the strip $0 \leq \operatorname{Im} z \leq \beta$. This holds uniformly in Λ. Further, the sequence $\rho_\Lambda(\alpha_z^\Lambda(a)\,b)$ converges uniformly as $\Lambda \nearrow \mathbb{Z}^d$, so the limit is analytic by Weierstrass convergence theorem. These properties suggest the following generalisation of the KMS condition

Definition 5.3. *The state ρ satisfies the KMS condition at inverse temperature β if for any $a, b \in \mathcal{A}$, there exists an analytic function $f(z)$ in the strip $0 \leq \operatorname{Im} z \leq \beta$ such that*

$$f(t) = \rho(b\,\alpha_t(a)) \quad \text{and} \quad f(t+\mathrm{i}\beta) = \rho(\alpha_t(a)\,b). \tag{5.23}$$

If KMS states represent equilibrium, they should be invariant under time evolution. This is easy to check in finite volume Gibbs states, but it remains true in general.

Proposition 5.4. *If ρ is a KMS state, we have $\rho(\alpha_t(a)) = \rho(a)$ for all $a \in \mathcal{A}$ and all $t \in \mathbb{R}$.*

Proof. Choose $b = \mathbb{1}$ in the KMS definition. Then $f(t) = f(t+\mathrm{i}\beta)$, so $f(z)$ can be extended to an analytic function in the whole of $\mathbb{C}$. It is periodic in the imaginary time direction with period β. Since it is bounded in the strip $0 \leq \operatorname{Im} z \leq \beta$, it is bounded everywhere, and Liouville theorem implies that it is constant. □

3. Uniqueness theorem

Theorem 5.5. *Assume that*

$$\beta\,\|\Phi\|_{N+1} < (2N)^{-1}.$$

Then there exists a unique KMS state at inverse temperature β.

We actually prove the theorem under the more general condition that there exists $s < 1/N$ such that $2\beta\|\Phi\|_{N(1+s)} < s$. As mentioned above, the strategy of our proof is to reformulate the KMS condition as an equation for the equilibrium state that has a unique solution when β is small enough. In order to derive this equation, we express observables as commutators of operators. The proof of Theorem 5.5 will be given after the one of Lemma 5.6, which we state next.

Here and in the sequel, $\|\cdot\|_{\mathrm{HS}}$ denotes the normalized Hilbert-Schmidt norm

$$\|A\|_{\mathrm{HS}}^2 = \frac{1}{\dim \mathcal{H}_\Lambda} \operatorname{Tr} A^* A. \tag{5.24}$$

Notice that

$$\frac{1}{\sqrt{\dim \mathcal{H}_\Lambda}} \|A\| \leq \|A\|_{\mathrm{HS}} \leq \|A\| \tag{5.25}$$

for all $A \in \mathcal{A}_\Lambda$.

Lemma 5.6. *Let A be a hermitian $N\times N$ matrix with the property that $\operatorname{Tr} A = 0$. Then there exist hermitian $N\times N$ matrices $B_1,\dots,B_{N-1}$ and $C_1,\dots,C_{N-1}$ such that*

$$A = \sum_{i=1}^{N-1} [B_i, C_i],$$
$$\sum_{i=1}^{N-1} \|B_i\|_{\rm HS}\, \|C_i\|_{\rm HS} \leq \sqrt{N}\, \|A\|_{\rm HS}.$$

Proof. Let $a_1,\dots,a_N$ be the eigenvalues of A (repeated according to their multiplicity). We have that

$$\sum_{i=1}^N a_i = 0, \qquad \sum_{i=1}^N |a_i|^2 = N\, \|A\|_{\rm HS}^2. \tag{5.26}$$

In particular, each $|a_i|$ is bounded above by $\sqrt{N}\|A\|_{\rm HS}$. Let us order the eigenvalues so that

$$\Bigl| \sum_{i=1}^k a_i \Bigr| \leq \sqrt{N}\, \|A\|_{\rm HS} \tag{5.27}$$

for all $1\leq k\leq N-1$. This is indeed possible, as can be seen by induction using $\sum a_i = 0$: If $0 \leq \sum^k a_i \leq \sqrt{N}\|A\|_{\rm HS}$, we can find $a_{k+1}\leq 0$ among the remaining eigenvalues such that $|\sum^{k+1} a_i| \leq \sqrt{N}\|A\|_{\rm HS}$. And if the partial sum is negative, we can find $a_{k+1}\geq 0$ among the remaining eigenvalues, with the same conclusion.

We work in a basis such that A is diagonal and its eigenvalues are ordered so they satisfy the properties above. Let $\tilde a_k = \sum_{i=1}^k a_i$, and let $\sigma^1_{j,j+1}, \sigma^2_{j,j+1}, \sigma^3_{j,j+1}$ be $N\times N$ matrices that are equal to Pauli matrices on the 2×2 block that contains (j,j) and $(j+1,j+1)$, and that are equal to zero everywhere else. It is not hard to check that

$$A = \sum_{j=1}^{N-1} \tilde a_j\, \sigma^3_{j,j+1}. \tag{5.28}$$

We therefore have that

$$A = \tfrac12 \sum_{j=1}^{N-1} \tilde a_j\, [\sigma^1_{j,j+1}, \sigma^2_{j,j+1}], \tag{5.29}$$

which proves the first claim. The bound follows from $|\tilde a_j| \leq \sqrt{N}\|A\|_{\rm HS}$ and $\|\sigma^i_{j,j+1}\|^2_{\rm HS} = 2/N$. □

Proof of Theorem 5.5. Let $(e_i)_{i=0}^{N^2-1}$ be a hermitian basis of $\mathcal{M}_N(\mathbb{C})$, with $e_0 = \mathbb{1}$, $\operatorname{Tr} e_i = 0$ if $1\neq 0$, and $\|e_i\| = 1$, for all i. Let J be the set of multi-indices $j = (j_x)_{x\in\mathbb{Z}^d}$, $0\leq j_x \leq N^2-1$, with finite support

$$\operatorname{supp} j = \{x\in\mathbb{Z}^d | j_x \neq 0\}. \tag{5.30}$$

Given $j \in J$, let $e_j = \otimes_{x \in \operatorname{supp} j} e_{j_x} \in \mathcal{A}_{\operatorname{supp} j}$. The linear span of $\{e_j\}_{j \in J}$ is dense in $\mathcal{A}$.

Let tr denote the normalized trace on $\mathcal{A}$; it is equal to $\frac{1}{\dim \mathcal{H}_\Lambda} \mathrm{Tr}$ on $\mathcal{A}_\Lambda$ and it can be extended to $\mathcal{A}$ by continuity. The state ρ can be written as $\rho = \mathrm{tr} + \varepsilon$ where $\varepsilon(\mathbb{1}) = 0$. We actually have that

$$\varepsilon(e_j) = \begin{cases} \rho(e_j) & \text{if } j \not\equiv 0, \\ 0 & \text{if } j \equiv 0. \end{cases} \tag{5.31}$$

Using Lemma 5.6, we have that

$$e_j = \frac{1}{|\operatorname{supp} j|} \sum_{y \in \operatorname{supp} j} \sum_{i=1}^{N-1} \big[\otimes_{x \neq y} e_{j_x} \otimes b_i^{(j_y)}, \otimes_{x \neq y} \mathbb{1} \otimes c_i^{(j_y)} \big], \tag{5.32}$$

for $j \not\equiv 0$. Here, $b_i^{(k)}, c_i^{(k)}$ are the matrices B_i, C_i of Lemma 5.6 in the case where the matrix A is e_k.

We now use this decomposition and the KMS condition, Definition 5.2, in order to get an equation for ε. For $j \not\equiv 0$,

$$\begin{aligned} \varepsilon(e_j) &= \frac{1}{|\operatorname{supp} j|} \sum_{y \in \operatorname{supp} j} \sum_{i=1}^{N-1} \rho\Big(\big[\otimes_{x \neq y} e_{j_x} \otimes b_i^{(j_y)}, \otimes_{x \neq y} \mathbb{1} \otimes c_i^{(j_y)} \big] \Big) \\ &= \frac{1}{|\operatorname{supp} j|} \sum_{y \in \operatorname{supp} j} \sum_{i=1}^{N-1} \rho\Big(\otimes_{x \neq y} e_{j_x} \otimes b_i^{(j_y)} \cdot (\mathbb{1} - \alpha_{\mathrm{i}\beta}) \otimes_{x \neq y} \mathbb{1} \otimes c_i^{(j_y)} \Big) \\ &= \delta(e_j) + K_\beta \varepsilon(e_j). \end{aligned} \tag{5.33}$$

In the above equation, we set

$$\delta(e_j) = \frac{1}{|\operatorname{supp} j|} \sum_{y \in \operatorname{supp} j} \sum_{i=1}^{N-1} \mathrm{tr}\Big(\otimes_{x \neq y} e_{j_x} \otimes b_i^{(j_y)} \cdot (\mathbb{1} - \alpha_{\mathrm{i}\beta}) \otimes_{x \neq y} \mathbb{1} \otimes c_i^{(j_y)} \Big), \tag{5.34}$$

and the operator K_β is defined by

$$(K_\beta \phi)(e_j) = \frac{1}{|\operatorname{supp} j|} \sum_{y \in \operatorname{supp} j} \sum_{i=1}^{N-1} \phi\Big(\otimes_{x \neq y} e_{j_x} \otimes b_i^{(j_y)} \cdot (\mathbb{1} - \alpha_{\mathrm{i}\beta}) \otimes_{x \neq y} \mathbb{1} \otimes c_i^{(j_y)} \Big). \tag{5.35}$$

Notice that K_β is a linear operator on the Banach space $\mathcal{L}(\mathcal{A})$ of linear functionals on $\mathcal{A}$. Equation (5.33) can be written as

$$(\mathbb{1} - K_\beta)\varepsilon = \delta. \tag{5.36}$$

Let us introduce the following norm on $\mathcal{L}(\mathcal{A})$:

$$|\!|\!|\phi|\!|\!| = \sup_{j \in J} |\phi(e_j)|. \tag{5.37}$$

Because $\|e_j\| = 1$ for all j, we have $|\!|\!|\phi|\!|\!| \leq \|\phi\|$ and $(\mathcal{L}(\mathcal{A}), |\!|\!| \cdot |\!|\!|)$ is a normed vector space. We consider K_β as an operator on $(\mathcal{L}(\mathcal{A}), |\!|\!| \cdot |\!|\!|)$ and we show that its norm is strictly less than 1; the solution of (5.36) is then unique. The norm of K_β is equal to

$$\|K_\beta\| = \sup_{\|\!|\phi|\!\|=1} \sup_{j\in J} |K_\beta\phi(e_j)|. \tag{5.38}$$

Recall that $\alpha_{\mathrm{i}\beta} = \lim_\Lambda \alpha^\Lambda_{\mathrm{i}\beta}$ (with convergence in the operator norm) and that $\alpha^\Lambda_{\mathrm{i}\beta}(A)$, $A \in \mathcal{A}$, has an expansion in multiple commutators. From (5.35), we get

$$\begin{aligned}\big|K_\beta\phi(e_j)\big| \leq \frac{1}{|\mathrm{supp}\, j|} \sum_{y\in\mathrm{supp}\, j} \sum_{i=1}^{N-1} \sum_{n\geq 1} \frac{\beta^n}{n!} \sup_{\Lambda\subset\mathbb{Z}^d} \sum_{X_1,\dots,X_n\subset\Lambda} \\ \Big|\phi\Big(\otimes_{x\neq y} e_{j_x} \otimes b_i^{(j_y)} \big[\Phi_{X_n}, \dots, \big[\Phi_{X_1}, \otimes_{x\neq y}\mathbb{1}\otimes c_i^{(j_y)}\big] \cdots \big]\Big)\Big|. \end{aligned} \tag{5.39}$$

Because of the commutators, the sum over the X_k's is restricted to subsets that satisfy the following constraints, as in (5.8):

$$\begin{aligned} &X_1 \ni y, \\ &X_2 \cap X_1 \neq \emptyset, \\ &\quad\vdots \\ &X_n \cap (X_1 \cup \dots \cup X_{n-1}) \neq \emptyset. \end{aligned} \tag{5.40}$$

Let $A = \sum_{(j'_x)_{x\in X}} a_{j'} e_{j'}$ be an operator in $\mathcal{A}_X$. For any $(j_x)_{x\notin X}$, we have

$$\begin{aligned} \big|\phi\big(\otimes_{x\notin X} e_{j_x} \otimes A\big)\big| &= \Big|\sum_{(j'_x)_{x\in X}} a_{j'}\ \phi\big(\otimes_{x\notin X} e_{j_x} \otimes_{x\in X} e_{j'_x}\big)\Big| \\ &\leq \|\!|\phi|\!\| \sum_{(j'_x)_{x\in X}} |a_{j'}| \\ &\leq \|\!|\phi|\!\| \|A\|_{\mathrm{HS}} N^{|X|}. \end{aligned} \tag{5.41}$$

Using Eq. (5.41) with $\|\!|\phi|\!\| = 1$, $\|AB\|_{\mathrm{HS}} \leq \|A\|\, \|B\|_{\mathrm{HS}}$, and $\|c_i^{(j_y)}\| \leq \sqrt{N}\|c_i^{(j_y)}\|_{\mathrm{HS}}$, we get

$$\begin{aligned} \big|K_\beta\phi(e_j)\big| &\leq \sqrt{N} \sup_{y\in\mathbb{Z}^d} \sum_{n\geq 1} \frac{(2\beta)^n}{n!} \sum_{X_1,\dots,X_n : y} \Big(\prod_{k=1}^n \|\Phi_{X_k}\| N^{|X_k|}\Big) \sum_{i=1}^{N-1} \|b_i^{(j_y)}\|_{\mathrm{HS}} \|c_i^{(j_y)}\|_{\mathrm{HS}} \\ &\leq N \sup_{y\in\mathbb{Z}^d} \sum_{n\geq 1} \frac{(2\beta)^n}{n!} \sum_{X_1,\dots,X_n : y} \prod_{k=1}^n \|\Phi_{X_k}\| N^{|X_k|}. \end{aligned} \tag{5.42}$$

We have used Lemma 5.6 to get the last line. The constraint $X_1, \dots, X_n : y$ means that (5.40) must be respected. The final step is to estimate the sum over such subsets. This can be conveniently done with an inductive argument. Namely, let $R_0 = 0$ and, for $m \geq 1$, let

$$R_m = \sup_{y\in\mathbb{Z}^d} \sum_{n=1}^m \frac{(2\beta)^n}{n!} \sum_{X_1,\dots,X_n : y} \prod_{k=1}^n \|\Phi_{X_k}\| N^{|X_k|}. \tag{5.43}$$

Summing first over $X_1 \ni y$, then over sets that intersect sites of X_1, we get

$$\begin{aligned} R_m &\leq 2\beta \sup_y \sum_{X_1 \ni y} \|\Phi_{X_1}\| N^{|X_1|} \prod_{x \in X_1} \Bigl(\sum_{n=1}^m \frac{(2\beta)^{n-1}}{(n-1)!} \sum_{X_2,\dots,X_n : x} \prod_{k=2}^n \|\Phi_{X_k}\| N^{|X_k|} \Bigr) \\ &\leq 2\beta \sup_y \sum_{X_1 \ni y} \|\Phi_{X_1}\| N^{|X_1|} (1 + R_{m-1})^{|X_1|}. \end{aligned} \tag{5.44}$$

It follows easily that $R_m \leq r$ for all m, and all r such that $2\beta \|\Phi\|_{N(1+r)} \leq r$. Then $\|K_\beta\| \leq Nr$, and the assumption of Theorem 5.5 implies the existence of r such that $Nr < 1$. □

4. Lieb-Robinson bounds

Even for small times, the image of a local observable under the evolution map is no longer local; that is, $\alpha_t(a) \notin \mathcal{A}_0$ for all $a \in \mathcal{A}_0$. This follows from the expansion. But one should expect that the evolved observable remains "essentially local". A precise formulation is provided by Lieb-Robinson bounds. Here, we define a new norm on interactions, namely

$$\|\Phi\|_c = \sup_{x \in \Lambda} \sum_{X \ni x} \|\Phi_X\| \, |X| \, \mathrm{e}^{c \, \mathrm{diam} X}. \tag{5.45}$$

The diameter of the finite set X is equal to $\mathrm{diam} X = \max_{x,y \in X} d(x,y)$, where $d(x,y)$ is the graph distance between the sites x and y.

THEOREM 5.7. *Let Λ be a finite graph, and let $a \in \mathcal{A}_X$, and $b \in \mathcal{A}_Y$. Then for every $t \in \mathbb{R}$ and every $c > 0$, we have*

$$\bigl\| [\alpha_t^\Lambda(a), b] \bigr\| \leq \|a\| \, \|b\| \, |X| \, \mathrm{e}^{2\|\Phi\|_c |t| - c\, d(X,Y)}.$$

In order to prove this theorem, we need the following lemma.

LEMMA 5.8. *Let $b : \mathbb{R} \to \mathcal{M}_N$ and $h : \mathbb{R} \times \mathcal{M}_N \to \mathcal{M}_N$ be continuous functions. We assume that $h(t,a)$ is hermitian for all t and all $a \in \mathcal{M}_N$.*

(i) *Let $\gamma_t(a_0)$ be the solution of the equation $\frac{\mathrm{d}}{\mathrm{d}t} a(t) = \mathrm{i}[h(t,a(t)), a(t)]$, $a(0) = a_0$. Then*
$$\|\gamma_t(a_0)\| = \|a_0\|$$
for all $a_0 \in \mathcal{M}_N$.

(ii) *The solution of the equation $\frac{\mathrm{d}}{\mathrm{d}t} a(t) = \mathrm{i}[h(t,a(t)), a(t)] + b(t)$, $a(0) = a_0$, is*
$$a(t) = \gamma_t\Bigl(a_0 + \int_0^t \gamma_{-s}\bigl(b(s)\bigr) \mathrm{d}s \Bigr).$$

(iii) *If $a(t)$ is the above solution, we have the bound*
$$\|a(t) - \gamma_t(a_0)\| \leq \int_0^t \|b(s)\| \mathrm{d}s.$$

Proof. We start with (i); we have

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\|a(t)\| &= \lim_{\varepsilon\to 0}\frac{1}{\varepsilon}\big(\|a(t+\varepsilon)\| - \|a(t)\|\big)\\ &= \lim_{\varepsilon\to 0}\frac{1}{\varepsilon}\Big(\big\|a(t)+\varepsilon\mathrm{i}[h(t,a(t)),a(t)]\big\| - \|a(t)\|\Big)\\ &= \lim_{\varepsilon\to 0}\frac{1}{\varepsilon}\Big(\big\|\,\mathrm{e}^{\mathrm{i}\varepsilon h(t,a(t))}\,a(t)\,\mathrm{e}^{-\mathrm{i}\varepsilon h(t,a(t))}\,\big\| - \|a(t)\|\Big)\\ &= 0.\end{aligned} \tag{5.46}$$

The claim (ii) is straightforwardly verified. We have

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}a(t) &= \lim_{\varepsilon\to 0}\frac{1}{\varepsilon}\Big\{\gamma_{t+\varepsilon}\Big(a_0+\int_0^t\gamma_{-s}\big(b(s)\big)\mathrm{d}s\Big)+\gamma_{t+\varepsilon}\Big(\int_t^{t+\varepsilon}\gamma_{-s}\big(b(s)\big)\mathrm{d}s\Big)-a(t)\Big\}\\ &= \lim_{\varepsilon\to 0}\frac{1}{\varepsilon}\Big\{a(t)+\varepsilon\mathrm{i}\big[h(t,a(t)),a(t)\big]+\varepsilon\gamma_t\big(\gamma_{-t}\big(b(t)\big)\big)-a(t)\Big\}\\ &= \mathrm{i}\big[h(t,a(t)),a(t)\big]+b(t).\end{aligned} \tag{5.47}$$

Finally, the claim (iii) follows from (i) and (ii):

$$\begin{aligned}\|a(t)-\gamma_t(a_0)\| &= \Big\|\gamma_t\Big(\int_0^t\gamma_{-s}\big(b(s)\big)\mathrm{d}s\Big)\Big\|\\ &\le \int_0^t\big\|\gamma_{-s}\big(b(s)\big)\big\|\mathrm{d}s\\ &= \int_0^t\|b(s)\|\mathrm{d}s.\end{aligned} \tag{5.48}$$

□

Proof of Theorem 5.7. Using Jacobi's identity (see Exercise 5.3), we have

$$\frac{\mathrm{d}}{\mathrm{d}t}[\alpha_t^\Lambda(a),b] = \mathrm{i}\Big[\sum_{Z\cap X\neq\emptyset}\alpha_t^\Lambda(\Phi_Z),[\alpha_t^\Lambda(a),b]\Big] - \mathrm{i}\sum_{Z\cap X\neq\emptyset}\big[\alpha_t^\Lambda(a),[\alpha_t^\Lambda(\Phi_Z),b]\big]. \tag{5.49}$$

Here and in the sequel, X denotes the support of a and Y denotes the support of b. By Lemma 5.8, we obtain

$$\big\|[\alpha_t^\Lambda(a),b]\big\| \le \big\|[a,b]\big\| + 2\|a\|\sum_{Z\cap X\neq\emptyset}\int_0^{|t|}\big\|[\alpha_s^\Lambda(\Phi_Z),b]\big\|\mathrm{d}s. \tag{5.50}$$

Let us introduce

$$g(t) = \sup_{a\in\mathcal{A}_0}\frac{\|[\alpha_t^\Lambda(a),b]\|}{\|a\|\,\|b\|\,|X|}\,\mathrm{e}^{c\,d(X,Y)}\,. \tag{5.51}$$

We can insert $g(s)$ in the right side of (5.50) and we get

$$\frac{\|[\alpha_t^\Lambda(a),b]\|}{\|a\|\,\|b\|\,|X|}\,\mathrm{e}^{c\,d(X,Y)} \leq \underbrace{\frac{\|[a,b]\|}{\|a\|\,\|b\|\,|X|}\,\mathrm{e}^{c\,d(X,Y)}}_{\leq 2 \text{ since } [a,b]=0 \text{ when } d(X,Y)\neq 0} + \frac{2}{|X|}\sum_{Z\cap X\neq\emptyset}\|\Phi_Z\|\,|Z|\,\mathrm{e}^{c\,d(X,Y)-c\,d(Z,Y)}\int_0^{|t|} g(s)\mathrm{d}s. \tag{5.52}$$

We now use $d(X,Y)-d(Z,Y)\leq \operatorname{diam} Z$ and we take the supremum over $a\in\mathcal{A}_0$ in the left side. From the definition (5.45) of the norm of Φ, we obtain

$$g(t)\leq 2+2\|\Phi\|_c\int_0^{|t|} g(s)\mathrm{d}s. \tag{5.53}$$

Let h be the solution of the integral equation $h(t)=2+2\|\Phi\|_c\int_0^t h(s)\mathrm{d}s$, $h(0)=2$. One easily finds

$$h(t)=2\,\mathrm{e}^{2\|\Phi\|_c t}. \tag{5.54}$$

Then $g(t)\leq h(|t|)$, which gives the Lieb-Robinson bound. □

5. References for this chapter

The construction of the infinite volume limit of the evolution operator is now textbook material, see Bratteli-Robinson (1981) and Simon (1993); these are good references to learn about the general theory and the background. The present proof of Theorem 5.5 can be found in Fröhlich-Ueltschi (2015). Lieb-Robinson bounds were proposed in 1972; the proof here is a simplification of Nachtergaele-Sims (2010).

Exercise 5.1. *Use a diagonal argument to prove the existence of infinite-volume KMS states.*

Exercise 5.2. *KMS states and symmetries. Assume that there exists a hermitian operator A on $\mathbb{C}^N$ such that for all $\Lambda\subset\subset\mathbb{Z}^d$,*

$$\Big[H_\Lambda,\sum_{x\in\Lambda}A_x\Big]=0.$$

Here, $A_x=A\otimes\mathbb{1}_{\Lambda\setminus\{x\}}$. Let ρ be a KMS state; for $\theta\in\mathbb{R}$, let $U_\Lambda=\mathrm{e}^{\mathrm{i}\theta\sum_{x\in\Lambda}A_x}$ and define

$$\tilde\rho(a)=\rho(U_\Lambda a U_\Lambda^*),$$

for $a\in\mathcal{A}_\Lambda$. Show that $\tilde\rho$ is also a KMS state.

Exercise 5.3. *Check Jacobi's identity for the commutators of matrices:*

$$\big[[a,b],c\big]=\big[a,[b,c]\big]-\big[b,[a,c]\big].$$

CHAPTER 6

Correlation functions

Correlation functions give information about the nature of phases. We collect here some properties of correlation functions for the class of models defined in Eq. (4.5). With $Z_\Lambda = \text{Tr}\, e^{-\beta H_\Lambda}$ denoting the partition function, the correlation functions at inverse temperature β are given by

$$\langle S_0^i S_x^i \rangle = \frac{1}{Z_\Lambda} \text{Tr}\, S_0^i S_x^i \, e^{-\beta H_\Lambda} \, . \tag{6.1}$$

1. Correlation inequalities

It is natural to expect that correlations are stronger among those components of the spins that correspond to stronger coupling parameters in the Hamiltonian. This is the content of the next theorem.

Theorem 6.1. *Assume that, for all $x, y \in \Lambda$, the couplings satisfy*

$$|J_{xy}^2| \leq J_{xy}^1.$$

Then we have that

$$\big|\langle S_0^2 S_x^2 \rangle\big| \leq \langle S_0^1 S_x^1 \rangle,$$

for all $x \in \Lambda$. More generally, for all $x_1, \dots x_k \in \Lambda$ and $j_1, \dots, j_k \in \{1, 2\}$,

$$\big|\langle S_{x_1}^{j_1} \dots S_{x_k}^{j_k} \rangle\big| \leq \langle S_{x_1}^1 \dots S_{x_k}^1 \rangle.$$

Further inequalities can be generated using symmetries. Some inequalities hold for the staggered two-point function $(-1)^{|x|} \langle S_0^i S_x^i \rangle$. The proof can be found after that of Trotter's product formula, which is needed.

Proposition 6.2 (Trotter formula). *Let A, B be $N \times N$ matrices. Then*

$$e^{A+B} = \lim_{n\to\infty} \Big(e^{\frac{1}{n}A} \, e^{\frac{1}{n}B} \Big)^n = \lim_{n\to\infty} \Big[\big(1 + \tfrac{1}{n} A\big) \, e^{\frac{1}{n}B} \Big]^n.$$

Proof. We prove the second formula — the mild changes for the first formula are straightforward. First, we have

$$e^{A+B} - \big(1 + \tfrac{1}{n}(A+B)\big)^n = \sum_{k=0}^{n} \frac{1}{k!} (A+B)^k \Big(1 - \frac{n(n-1)\dots(n-k+1)}{n^k} \Big) + \sum_{k \geq n+1} \frac{1}{k!} (A+B)^k. \tag{6.2}$$

The norm of the right side clearly vanishes as $n \to \infty$. Next, let K_n be the matrix such that

$$\big(1 + \tfrac{1}{n} A\big) \, \mathrm{e}^{\frac{1}{n} B} = 1 + \frac{1}{n}(A+B) + K_n. \tag{6.3}$$

It is clear that $\|K_n\| = O(\frac{1}{n^2})$. We have

$$\begin{aligned}\big((1 + \tfrac{1}{n} A) \, \mathrm{e}^{\frac{1}{n} B}\big)^n - \big(1 + \tfrac{1}{n}(A+B)\big)^n &= \big(1 + \tfrac{1}{n}(A+B) + K_n\big)^n - \big(1 + \tfrac{1}{n}(A+B)\big)^n \\ &= \int_0^1 \mathrm{d}s \frac{\mathrm{d}}{\mathrm{d}s} \big(1 + \tfrac{1}{n}(A+B) + sK_n\big)^n \\ &= \int_0^1 \mathrm{d}s \sum_{k=0}^{n-1} \big(1 + \tfrac{1}{n}(A+B) + sK_n\big)^k K_n \big(1 + \tfrac{1}{n}(A+B) + sK_n\big)^{n-k-1}.\end{aligned} \tag{6.4}$$

Consequently,

$$\Big\| \big((1 + \tfrac{1}{n} A) \, \mathrm{e}^{\frac{1}{n} B}\big)^n - \big(1 + \tfrac{1}{n}(A+B)\big)^n \Big\| \leq \sum_{k=0}^{n-1} \int_0^1 \mathrm{d}s \big\| 1 + \tfrac{1}{n}(A+B) + sK_n \big\|^{n-1} \|K_n\|. \tag{6.5}$$

We have

$$\big\| 1 + \tfrac{1}{n}(A+B) + sK_n \big\|^{n-1} \leq \big(1 + \tfrac{1}{n}\|A+B\| + \|K_n\|\big)^{n-1} \to \mathrm{e}^{\|A+B\|}, \tag{6.6}$$

so that the right side of Eq. (6.5) is less than a constant times $n\|K_n\|$, which vanishes in the limit $n \to \infty$. □

Proof of Theorem 6.1. Let $|a\rangle$, $a \in \{-S, \dots, S\}$ denote the eigenvectors of S^3, and recall the operators $S^\pm$ defined before. The matrix elements of $S^1, S^\pm$ are all nonnegative, and the matrix elements of S^2 are all less than or equal to those of S^1 in absolute values. Using the Trotter formula and multiple resolutions of the identity, we have

$$\begin{aligned}\big|\mathrm{Tr}\, S_0^2 S_x^2 \, \mathrm{e}^{-\beta H_\Lambda}\big| \leq \lim_{N\to\infty} \sum_{\sigma_0,\dots,\sigma_N \in \{-S,\dots,S\}^\Lambda} \Big| &\langle \sigma_0 | S_0^2 S_x^2 | \sigma_1 \rangle \\ &\langle \sigma_1 | \, \mathrm{e}^{\frac{\beta}{N} \sum J^3_{yz} S^3_y S^3_z} | \sigma_1 \rangle \langle \sigma_1 | \Big(1 + \frac{\beta}{N} \sum_{y,z\in\Lambda} (J^1_{yz} S^1_y S^1_z + J^2_{yz} S^2_y S^2_z) \Big) | \sigma_2 \rangle \\ &\dots \langle \sigma_N | \, \mathrm{e}^{\frac{\beta}{N} \sum J^3_{yz} S^3_y S^3_z} | \sigma_N \rangle \langle \sigma_N | \Big(1 + \frac{\beta}{N} \sum_{y,z\in\Lambda} (J^1_{yz} S^1_y S^1_z + J^2_{yz} S^2_y S^2_z) \Big) | \sigma_0 \rangle \Big|.\end{aligned} \tag{6.7}$$

Observe that the matrix elements of all operators are nonnegative, except for $S_0^2 S_x^2$. Indeed, this follows from

$$J^1_{yz} S^1_y S^1_z + J^2_{yz} S^2_y S^2_z = \tfrac{1}{4}(J^1_{yz} - J^2_{yz})(S^+_y S^+_z + S^-_y S^-_z) + \tfrac{1}{4}(J^1_{yz} + J^2_{yz})(S^+_y S^-_z + S^-_y S^+_z). \tag{6.8}$$

We get an upper bound for the right side of (6.7) by replacing $|\langle\sigma_0|S_0^2 S_x^2|\sigma_1\rangle|$ with $\langle\sigma_0|S_0^1 S_x^1|\sigma_1\rangle$. We have obtained

$$\Big|\mathrm{Tr}\, S_0^2 S_x^2\, \mathrm{e}^{-\beta H_\Lambda}\,\Big| \leq \mathrm{Tr}\, S_0^1 S_x^1\, \mathrm{e}^{-\beta H_\Lambda}\,, \tag{6.9}$$

which proves the first claim. The second claim can be proved exactly the same way. □

Corollary 6.3. *Assume that for all $x, y \in \Lambda$, the couplings satisfy*

$$J_{xy}^1 = J_{xy}^2 \geq 0.$$

Then we have for all $x, y, z, u \in \Lambda$

$$\frac{\partial}{\partial J_{xy}^1} \langle S_z^2 S_u^2 \rangle \leq \frac{\partial}{\partial J_{xy}^1} \langle S_0^1 S_x^1 \rangle.$$

Proof. For $i = 1, 2, 3$, we have

$$\frac{1}{\beta}\frac{\partial}{\partial J_{xy}^1} \langle S_z^i S_u^i \rangle = (S_x^1 S_y^1, S_z^i S_u^i) - \langle S_x^1 S_y^1 \rangle \langle S_z^i S_u^i \rangle, \tag{6.10}$$

where (A, B) denotes the Duhamel two-point function,

$$(A, B) = \frac{1}{Z_\Lambda} \int_0^1 \mathrm{Tr}\, A\, \mathrm{e}^{-s\beta H_\Lambda}\, B\, \mathrm{e}^{-(1-s)\beta H_\Lambda}\, \mathrm{d}s. \tag{6.11}$$

It is not hard to extend the proof of Theorem 6.1 to the Duhamel function, so that

$$\big|(S_x^1 S_y^1, S_z^2 S_u^2)\big| \leq (S_x^1 S_y^1, S_z^1 S_u^1). \tag{6.12}$$

Further, we have $\langle S_z^2 S_u^2 \rangle = \langle S_z^1 S_u^1 \rangle$ by symmetry. The result follows. □

2. Decay of correlations due to symmetries

In this section we prove a variant of the Mermin-Wagner theorem. The result applies to systems that are effectively two-dimensional.

We assume that $J_{xy}^1 = J_{xy}^2$ for all x, y. The decay of correlations is measured by the following expression:

$$\xi_\beta(x) = \sup_{\substack{(\phi_y) \in \mathbb{R}^\Lambda \\ \phi_x = 0}} \Big[\phi_0 - \beta S^2 \sum_{y,z \in \Lambda} |J_{yz}^1| \big(\cosh(\phi_y - \phi_z) - 1\big) \Big]. \tag{6.13}$$

The solution of this variational problem is essentially a discrete harmonic function. We can estimate it explicitly in the case of "2D-like" graphs with nearest-neighbor couplings. Let Λ denote a graph, i.e. a finite set of vertices and a set of edges, and let $d(x, y)$ denote the graph distance, i.e. the length of the shortest path that connects x and y.

LEMMA 6.4. *Assume that* $J^i_{xy} = 0$ *whenever* $d(x,y) \geq 2$ *and let* $J = \max |J^i_{xy}|$. *Assume in addition that there exists a constant* K *such that, for any* $\ell \in \mathbb{N}$,

$$\#\big\{\{x,y\} \subset \Lambda : d(0,x) = \ell \text{ and } d(0,y) = \ell+1\big\} \leq K\ell.$$

Then there exists $C = C(\beta, S, J, K)$, *which does not depend on* x, *such that*

$$\xi_\beta(x) \geq \frac{1}{8\beta J S^2 K} \log\big(d(0,x)+1\big) - C.$$

PROOF. With c to be chosen later, let

$$\phi_y = \begin{cases} c \log \frac{d(0,x)+1}{d(0,y)+1} & \text{if } d(0,y) \leq d(0,x), \\ 0 & \text{otherwise.} \end{cases} \tag{6.14}$$

Then

$$\xi_\beta(x) \geq c \log(d(0,x)+1) - 2\beta S^2 JK \sum_{\ell=0}^{d(0,x)-1} \big(\cosh(c \log \tfrac{\ell+2}{\ell+1}) - 1\big)\ell. \tag{6.15}$$

From Taylor expansions of the logarithm and of the hyperbolic cosine, there exist C, C' such that

$$\begin{aligned} \xi_\beta(x) &\geq c \log(d(0,x)+1) - 2\beta S^2 JKc^2 \sum_{\ell=1}^{d(0,x)} \frac{1}{\ell} - C' \\ &\geq \big[c - 2\beta S^2 JKc^2\big] \log(d(0,x)+1) - C. \end{aligned} \tag{6.16}$$

The optimal choice is $c = (4\beta S^2 JK)^{-1}$. □

THEOREM 6.5. *Assume that* $J^1_{xy} = J^2_{xy}$ *for all* $x,y \in \Lambda$. *Then, for* $i = 1,2$, *we have*

$$\big|\langle S^i_0 S^i_x \rangle\big| \leq 2S^2\, \mathrm{e}^{-\xi_\beta(x)}.$$

In the case of 2D-like graphs, we can use Lemma 6.4 and we obtain algebraic decay with a power greater than $(8\beta JS^2K)^{-1}$.

The proof uses the Hölder inequality for traces, which can be proved using chessboard estimates. Recall that the "absolute value" of a matrix is $|A| = (A^*A)^{\frac12}$, where the square root of a nonnegative hermitian matrix can be defined by diagonalising and taking the square root of the eigenvalues. The p-norm of a matrix is then defined as $\|A\|_p = (\mathrm{Tr}\,|A|^p)^{1/p}$. Notice that $\lim_{p\to\infty} \|A\|_p = \|A\|$.

PROPOSITION 6.6 (Hölder inequality for matrices). *If* $1 \leq p,q,r \leq \infty$ *with* $\frac1p + \frac1q = \frac1r$, *we have*

$$\|AB\|_r \leq \|A\|_p \|B\|_q.$$

It follows from a simple induction that

$$\Big\| \prod_{j=1}^{n} A_j \Big\|_r \leq \prod_{j=1}^{n} \|A_j\|_{p_j} \tag{6.17}$$

whenever $1 \leq r, p_1, \dots, p_n$ with $\sum_{j=1}^{n} \frac{1}{p_j} = \frac{1}{r}$. There are no short proofs of Hölder's inequality for matrices. We prove below Eq. (6.17) for $r = 1$ only. It implies Proposition 6.6 for $r \in \mathbb{N}$ and it is enough for the purpose of proving Theorem 6.5. The proof is due to Fröhlich [1978] and it uses chessboard estimates.

Lemma 6.7 (Chessboard estimate). *For any $n \in \mathbb{N}$ and any matrices $A_1, \dots, A_{2n}$, we have*

$$\big|\mathrm{Tr}\, A_1 \dots A_{2n}\big| \leq \prod_{i=1}^{2n} \Big(\mathrm{Tr}\, (A_i A_i^*)^n \Big)^{1/2n}.$$

Proof. Since $(A, B) \mapsto \mathrm{Tr}\, A^* B$ is an inner product, the following inequality follows from Cauchy-Schwarz:

$$\big|\mathrm{Tr}\, A_1 \dots A_{2n}\big|^2 \leq \mathrm{Tr}\, \big(A_1 \dots A_n A_n^* \dots A_1^*\big)\ \mathrm{Tr}\, \big(A_{2n}^* \dots A_{n+1}^* A_{n+1} \dots A_{2n}\big). \tag{6.18}$$

This allows to use a reflection positivity argument. It is enough to prove the inequality for matrices that satisfy $\mathrm{Tr}\, (A_i A_i^*)^n = 1$; the general result follows from scaling.

Let $A_1, \dots, A_{2n}$ be matrices that maximise $|\mathrm{Tr}\, A_1 \dots A_{2n}|$, with maximum number of matching neighbours $A_{i+1} = A_i^*$. Suppose there exists an index j such that $A_{j+1} \neq A_j^*$. Using cyclicity, we can assume that $j = n$. By the inequality (6.18), $A_1, \dots, A_n, A_n^*, \dots, A_1^*$ and $A_{2n}^*, \dots, A_{n+1}^*, A_{n+1}, \dots, A_{2n}$ are also maximisers. At least one has strictly more matching neighbours, hence a contradiction. The maximum is then $\mathrm{Tr}\, (AA^*)^n$ for some matrix A, which is equal to 1. ☐

Chessboard estimates allow to prove the case $r = 1$ of Hölder's inequality.

Corollary 6.8. *We have*

$$|\mathrm{Tr}\, A_1 \dots A_n| \leq \prod_{i=1}^{n} \|A_i\|_{m_i}$$

for all n and all rational m_i's such that $\sum_{i=1}^{n} \frac{1}{m_i} = 1$.

Proof. Let ℓ be a positive integer such that $2\ell/m_i$ is integer for all i. Let $A_i = U_i |A_i|$ be the polar decomposition of A_i, and let

$$B_i = |A_i|^{m_i/2\ell}, \qquad \hat{B}_i = U_i |A_i|^{m_i/2\ell}. \tag{6.19}$$

Then $A_i = \hat{B}_i B_i^{(2\ell/m_i)-1}$, and we have

$$\begin{aligned}\bigl|\mathrm{Tr}\, A_1 \dots A_n\bigr| &= \bigl|\mathrm{Tr}\, \hat{B}_1 \underbrace{B_1 \dots B_1}_{(2\ell/m_1)-1} \dots \hat{B}_n \underbrace{B_n \dots B_n}_{(2\ell/m_n)-1}\bigr| \\ &\leq \prod_{i=1}^{n} (\mathrm{Tr}\, |A_i|^{m_i})^{1/m_i} \\ &= \prod_{i=1}^{n} \|A_i\|_{m_i}.\end{aligned} \tag{6.20}$$

The inequality follows from Lemma 6.7 and from the identities

$$\mathrm{Tr}\, (B_i B_i^*)^\ell = \mathrm{Tr}\, (\hat{B}_i \hat{B}_i^*)^\ell = \mathrm{Tr}\, |A_i|^{m_i}. \tag{6.21}$$

□

LEMMA 6.9. *Let $r, r' \in [1, \infty]$ such that $\frac{1}{r} + \frac{1}{r'} = 1$. Then for any square matrix A, we have*

$$\|A\|_r = \sup_{\|C\|_{r'}=1} \mathrm{Tr}\, C^* A.$$

PROOF. The right side is smaller by Corollary 6.8:

$$\bigl|\mathrm{Tr}\, C^* A\bigr| \leq \|C\|_{r'} \|A\|_r = \|A\|_r. \tag{6.22}$$

In order to check that this inequality is saturated, let $A = U|A|$ be the polar decomposition of A, and choose $C = \|A\|_r^{1-r} U |A|^{r-1}$. Then $\|C\|_{r'} = 1$ and $\mathrm{Tr}\, C^* A = \|A\|_r$. □

PROOF OF PROPOSITION 6.6. Starting with Lemma 6.9 and then using Corollary 6.8, we have

$$\begin{aligned}\|AB\|_r &= \sup_{\|C\|_{r'}=1} \mathrm{Tr}\, C^* AB \\ &\leq \sup_{\|C\|_{r'}=1} \|C\|_{r'} \|A\|_p \|B\|_q.\end{aligned} \tag{6.23}$$

□

PROOF OF THEOREM 6.5. We use the method of complex rotations. Let

$$S_y^\pm = S_y^1 \pm \mathrm{i} S_y^2. \tag{6.24}$$

One can check that for any $a \in \mathbb{C}$, we have

$$\mathrm{e}^{a S_y^3}\, S_y^\pm\, \mathrm{e}^{-a S_y^3} \;=\; \mathrm{e}^{\pm a}\, S_y^\pm. \tag{6.25}$$

The Hamiltonian (4.5) can be rewritten as

$$H_\Lambda = -\tfrac{1}{2} \sum_{y,z \in \Lambda} \bigl(J_{yz}^1 S_y^+ S_z^- + J_{yz}^3 S_y^3 S_z^3 \bigr) \tag{6.26}$$

Given numbers ϕ_y, let

$$A = \prod_{y \in \Lambda} \mathrm{e}^{\phi_y S_y^3}\,. \tag{6.27}$$

Then

$$\operatorname{Tr} S_0^+ S_x^- \,\mathrm{e}^{-\beta H_\Lambda} = \operatorname{Tr} A S_0^+ S_x^- A^{-1} \,\mathrm{e}^{-\beta A H_\Lambda A^{-1}} . \tag{6.28}$$

We now compute the rotated Hamiltonian.

$$\begin{aligned} A H_\Lambda A^{-1} &= -\tfrac12 \sum_{y,z\in\Lambda} \big(J^1_{yz} \,\mathrm{e}^{\phi_y - \phi_z} \, S_y^+ S_z^- + J^3_{yz} S^3_y S^3_z \big) \\ &= H_\Lambda - \tfrac12 \sum_{y,z\in\Lambda} J^1_{yz} \big(\cosh(\phi_y - \phi_z) - 1 \big) \; S_y^+ S_z^- - \tfrac12 \sum_{y,z\in\Lambda} J^1_{yz} \sinh(\phi_y - \phi_z) \; S_y^+ S_z^- \\ &\equiv H_\Lambda + B + C. \end{aligned} \tag{6.29}$$

Notice that $B^* = B$ and $C^* = -C$. We obtain

$$\operatorname{Tr} S_0^+ S_x^- \,\mathrm{e}^{-\beta H_\Lambda} = \mathrm{e}^{\phi_0 - \phi_x} \operatorname{Tr} S_0^+ S_x^- \,\mathrm{e}^{-\beta H_\Lambda - \beta B - \beta C} . \tag{6.30}$$

We now estimate the trace in the right side using the Trotter product formula and the Hölder inequality for traces. Recall that $\|B\|_s = (\operatorname{Tr} |B|^s)^{1/s}$, with $\|B\|_\infty = \|B\|$ being the usual operator norm.

$$\begin{aligned} \operatorname{Tr} S_0^+ S_x^- \,\mathrm{e}^{-\beta H_\Lambda - \beta B - \beta C} &= \lim_{N\to\infty} \operatorname{Tr} S_0^+ S_x^- \Big(\mathrm{e}^{-\frac{\beta}{N} H_\Lambda} \, \mathrm{e}^{-\frac{\beta}{N} B} \, \mathrm{e}^{-\frac{\beta}{N} C} \Big)^N \\ &\leq \lim_{N\to\infty} \| S_0^+ S_x^- \|_\infty \, \big\| \mathrm{e}^{-\frac{\beta}{N} H_\Lambda} \big\|_N^N \, \big\| \mathrm{e}^{-\frac{\beta}{N} B} \big\|_\infty^N \, \big\| \mathrm{e}^{-\frac{\beta}{N} C} \big\|_\infty^N . \end{aligned} \tag{6.31}$$

Observe now that $\| \mathrm{e}^{-\frac{\beta}{N} H_\Lambda} \|_N^N = Z_\Lambda$, $\| \mathrm{e}^{-\frac{\beta}{N} B} \|^N \leq \mathrm{e}^{\beta \|B\|}$, and $\| \mathrm{e}^{-\frac{\beta}{N} C} \| = 1$. We also have $\|S_0^+ S_x^-\| = 2S^2$ by Exercise 3.4 and Eq. (3.16). The theorem then follows from

$$\|B\| \leq S^2 \sum_{y,z\in\Lambda} |J^1_{yz}| \big(\cosh(\phi_y - \phi_z) - 1 \big). \tag{6.32}$$

□

3. References for this chapter

The correlation inequalities described in this chapter were proposed in Fröhlich-Ueltschi (2015). The decay of correlation due to symmetries is usually referred to as a "Mermin-Wagner theorem". Decay of correlations was first proposed by Fisher-Jasnow (1971), Bonato-Fernando Perez-Klein (1982), and Ito (1982); these results use the Fourier transform and they only apply to the regular lattice $\mathbb{Z}^2$. A more general result, similar to Theorem 6.5, was proposed in Koma-Tasaki (1992). The present proof is a bit simpler and can be found in Fröhlich-Ueltschi (2015).

Exercise 6.1.

(i) *Show that Eq. (6.17) follows from Proposition 6.6.*
(ii) *Show that Proposition 6.6 for $r \in 2\mathbb{N}$ follows from Eq. (6.17) for $r = 1$.*

EXERCISE 6.2. *Magnetisation and correlation functions. Let m_Λ denote the magnetisation operator in the 3rd spin direction:*

$$m_\Lambda = \frac{1}{|\Lambda|} \sum_{x\in\Lambda} S_x^3.$$

(i) *Show that $\langle m_\Lambda \rangle = 0$ because of symmetries.*
(ii) *Observe that $\langle m_\Lambda^2 \rangle = \frac{1}{|\Lambda|^2} \sum_{x,y\in\Lambda} \langle S_x^3 S_y^3 \rangle$. Assume that $|\langle S_x^3 S_y^3 \rangle| \leq q(\|x-y\|)$ uniformly in Λ, for some function q that satisfies $q(r) \to 0$ as $r \to \infty$. Show that*

$$\limsup_{\Lambda \nearrow \mathbb{Z}^d} \big| \langle m_\Lambda^2 \rangle \big| = 0.$$

CHAPTER 7

Graphical representations

The origin of graphical representations goes back to Feynman's description of the quantum Bose gas in terms of interacting Brownian trajectories (1953). In 1969, Ginibre proved the occurrence of phase transitions in quantum perturbations of classical models, using "space-time configurations" and a Peierls argument. The quantum spin $\frac{1}{2}$ Heisenberg ferromagnet was described using random transpositions by Powers (1976), and independently by Tóth (1993). Tóth used it to derive a bound for the free energy. Another loop representation was proposed by Aizenman and Nachtergaele for the quantum spin $\frac{1}{2}$ Heisenberg antiferromagnet (1994). This allowed them to relate the quantum chain to two classical two-dimensional models, namely the Potts and random cluster models. The synthesis of these representations, that applies to intermediate models such as quantum XY, was only proposed recently (2013).

In this chapter, we restrict our attention to the case $S = \frac{1}{2}$, mainly for pedagogical reasons. The case $S = 1$ is very interesting, though.

We first describe the random loop models; the connection to quantum spins can be found in Section 2.

1. Random loop models

The Poisson point process plays an essential rôle. It can be introduced in many different ways including awfully abstract ones. We choose a pedestrian and intuitive approach, though.

In words, the Poisson point process on the interval $[0, 1]$ describes the occurrence of events that are independent of one another. Let $\lambda > 0$ be the *intensity* of the process. Let us discretise the interval $[0, 1]$ by considering the set $\{\frac{1}{n}, \frac{2}{n}, \dots, 1\}$, with $\lambda/n < 1$. We consider the probability distribution on subsets of $\{\frac{1}{n}, \frac{2}{n}, \dots, 1\}$, where the probability of the subset ω is

$$\mathbb{P}_n(\omega) = \big(\tfrac{\lambda}{n}\big)^{|\omega|} \big(1 - \tfrac{\lambda}{n}\big)^{n-|\omega|}. \tag{7.1}$$

The interpretation is that each point of the form $\frac{i}{n}$, $1 \leq i \leq n$, occurs with probability λ/n. As $n \to \infty$, this process converges to the Poisson point process on $[0, 1]$ with intensity λ.

The total number of events is not fixed; it is a Poisson random variable with parameter λ. Indeed, for $k \in \mathbb{N}_0$, we have

$$\mathbb{P}(|\omega| = k) = \tfrac{\lambda^k}{k!} \, \mathrm{e}^{-\lambda} \, . \tag{7.2}$$

Integration with respect to the Poisson point process can also be defined by a limit. Let $f = (f_k)$ be a collection of smooth functions $f_k : [0,1]^k \to \mathbb{R}$ (with $f_0 = 1$ by definition). Then

$$\begin{aligned}\mathbb{E}(f) &= \lim_{n\to\infty} \sum_{\omega} \big(\tfrac{\lambda}{n}\big)^{|\omega|} \big(1 - \tfrac{\lambda}{n}\big)^{n-|\omega|} f_{|\omega|}(\omega) \\ &= \lim_{n\to\infty} \sum_{k\geq 0} \big(\tfrac{\lambda}{n}\big)^{k} \big(1 - \tfrac{\lambda}{n}\big)^{n-k} \sum_{\omega, |\omega|=k} f_k(\omega) \\ &= \sum_{k\geq 0} \lambda^k \,\mathrm{e}^{-\lambda} \int_{0<t_1<\dots<t_k<1} f_k(t_1,\dots,t_k)\,\mathrm{d}t_1 \dots \mathrm{d}t_k.\end{aligned} \tag{7.3}$$

The exchange of limit and sum can be justified by dominated convergence. The latter expression is concrete and practical.

Next, we describe the model of random loops. Let $(\Lambda, \mathcal{E})$ denote a finite graph, with Λ the set of vertices and $\mathcal{E}$ the set of edges. Let $\beta > 0$ and $u \in [0,1]$ be two parameters. To each edge $\{x,y\} \in \mathcal{E}$ is associated the interval $[0,\beta]$ and a Poisson point process on this interval. There occur two kinds of events:

- "crosses" occur with intensity u;
- "double bars" occur with intensity $1-u$.

(This process can be defined by generalising the Poisson point process above to two kinds of events, or by considering two separate, independent processes.) We let ρ denote the measure of independent Poisson point processes on $\mathcal{E} \times [0,\beta]$. Let ω denote realisations of this probability measure. It contains finitely many objects with probability 1.

To a given realisation ω corresponds a set of loops on $\mathcal{E} \times [0,\beta]$, denoted $\mathcal{L}(\omega)$. The loops consist of vertical lines connected by crosses or bars, with periodic boundary conditions in the continuous direction. This is best understood by looking at Fig. 7.1.

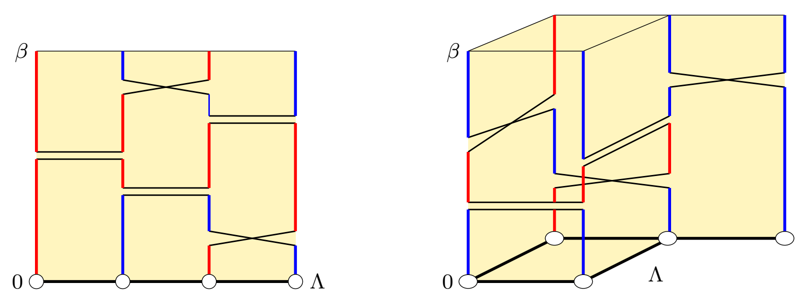

FIGURE 7.1. Graphs and realizations of Poisson point measures, and their loops. In both cases, the number of loops is $|\mathcal{L}(\omega)| = 2$.

The *partition function* $Y(\beta,\Lambda)$ of the model of random loops is defined by

$$Y(\beta,\Lambda) = \int 2^{|\mathcal{L}(\omega)|} \rho(\mathrm{d}\omega). \tag{7.4}$$

The relevant measure for the model of random loops is given by

$$\mu(\mathrm{d}\omega) = \frac{1}{Y(\beta,\Lambda)} 2^{|\mathcal{L}(\omega)|} \rho(\mathrm{d}\omega). \tag{7.5}$$

It can be shown that for β small the loops have small lengths and the probability that two sites belong to the same loop shows exponential decay with respect to the distance between the sites.

The loop correlations are given by the following three events:

- $E^+_{x,y}$ is the set of all realisations ω such that x and y belong to the same loop, and with identical vertical direction at these points.
- $E^-_{x,y}$ is the set of all ω such that x and y belong to the same loop, and with opposite vertical directions at these points.
- $E_{x,y} = E^+_{x,y} \cup E^-_{x,y}$ is the set of all ω such that x and y belong to the same loop.

These events are illustrated in Fig. 7.2. When $u = 1$, that is, when only crosses are present, we have $\mathbb{P}(E^+_{x,y}) = \mathbb{P}(E_{x,y})$. When $u = 0$, and if the graph is bipartite, we have $\mathbb{P}(E^+_{x,y}) = \mathbb{P}(E_{x,y})$ if x, y belong to the same sublattice, and $\mathbb{P}(E^-_{x,y}) = \mathbb{P}(E_{x,y})$ if x, y belong to different sublattices.

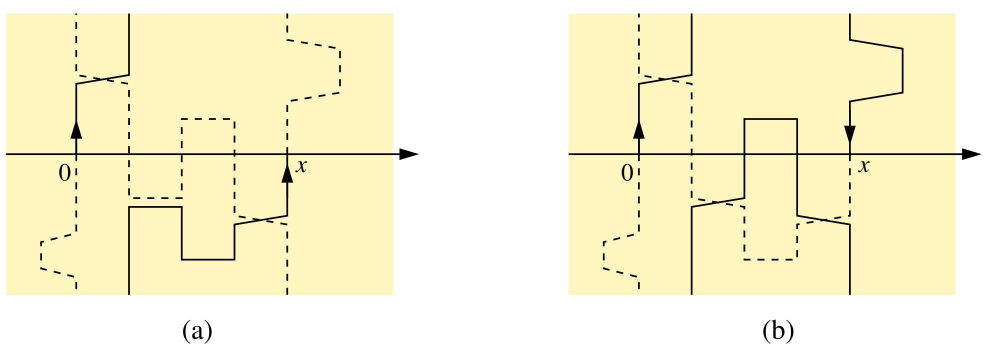

FIGURE 7.2. Illustration for (a) the event $E^+_{0,x}$; (b) the event $E^-_{0,x}$. Here, we shifted the vertical direction and considered $\mathcal{E} \times [-\frac{\beta}{2}, \frac{\beta}{2}]$. Because of periodic boundary conditions in the vertical direction, this is equivalent to $\mathcal{E} \times [0, \beta]$.

2. Relations between random loops and quantum spins

We take $S = \frac{1}{2}$ so the Hilbert space is $\mathcal{H}_\Lambda = \otimes_{x\in\Lambda} \mathbb{C}^2$. We consider the following family of Hamiltonians:

$$H^{(u)}_\Lambda = -2 \sum_{\{x,y\}\in\mathcal{E}} \Big(S^1_x S^1_y + (2u-1) S^2_x S^2_y + S^3_x S^3_y - \tfrac{1}{4} \Big). \tag{7.6}$$

With the parameter u in the interval $[0,1]$, this family interpolates between the Heisenberg ferromagnet ($u=1$), the quantum XY model ($u=\frac12$), and the Heisenberg antiferromagnet (more precisely, it is unitarily equivalent to the case $u=0$ if the graph is bipartite).

Here are precise relations between random loops and quantum spins.

THEOREM 7.1.

(i) *The partition functions of both models are identical:*

$$\mathrm{Tr}\, \mathrm{e}^{-\beta H_\Lambda^{(u)}} = \int 2^{|\mathcal{L}(\omega)|} \rho(\mathrm{d}\omega).$$

(ii) *Correlations in the spin directions 1 and 3 are given by loop correlations: For all $x,y \in \Lambda$,*

$$\langle S_x^1 S_y^1 \rangle = \langle S_x^3 S_y^3 \rangle = \tfrac14 \mathbb{P}(E_{x,y}).$$

(iii) *Correlations in the spin direction 2 are more subtle: For all $x,y \in \Lambda$,*

$$\langle S_x^2 S_y^2 \rangle = \tfrac14 \big[\mathbb{P}(E_{x,y}^+) - \mathbb{P}(E_{x,y}^-) \big].$$

We prove this theorem by writing the Hamiltonian in terms of suitable interaction operators T_{xy} and Q_{xy}, and by using a "Poisson expansion" of the Gibbs operator $\mathrm{e}^{-\beta H_\Lambda^{(u)}}$, see Lemma 7.3.

Let T_{xy} be the transposition operator on $\mathbb{C}^2 \otimes \mathbb{C}^2$:

$$T_{xy}|a,b\rangle = |b,a\rangle, \tag{7.7}$$

for all $a,b=\pm\frac12$. And let Q_{xy} be the operator on $\mathbb{C}^2 \otimes \mathbb{C}^2$ with the following matrix elements:

$$\langle a,b|Q_{xy}|c,d\rangle, \tag{7.8}$$

for all $a,b,c,d = \pm\frac12$. These operators can be expressed in terms of spin operators.

LEMMA 7.2. *Show that*

(i) $\vec{S}_x \cdot \vec{S}_y = \frac12 T_{xy} - \frac14$.

(ii) $S_x^1 S_y^1 - S_x^2 S_y^2 + S_x^3 S_y^3 = \frac12 Q_{xy} - \frac14$.

The proof of this lemma is done in Exercise 7.5. The Hamiltonian $H_\Lambda^{(u)}$ is a convex combination of these interactions, namely

$$H_\Lambda^{(u)} = - \sum_{\{x,y\}\in\mathcal{E}} \big(uT_{xy} + (1-u)Q_{xy} - 1 \big). \tag{7.9}$$

This is the perfect expression for a Feynman-Kac type of expansion.

LEMMA 7.3. *Let $A_1, \dots, A_k$ be bounded operators, and ρ a Poisson point process on $\{1, \dots, k\} \times [0,1]$ of intensity 1. Then*

$$\exp\Big\{\sum_{j=1}^k (A_j - 1)\Big\} = \int \rho(\mathrm{d}\omega) \prod_{(j,t)\in\omega}^{*} A_j.$$

In this lemma, the product $\prod^*$ is over the events of ω in increasing times.

PROOF. We start with the Poisson point process. We have

$$\begin{aligned}
\int \rho(\mathrm{d}\omega) \prod_{(j,t)\in\omega}^{*} A_j &= \lim_{n\to\infty} \sum_{m=0}^{n} \sum_{\substack{(j_1,t_1),\dots,(j_m,t_m) \\ t_1<t_2<\dots<t_m}} \big(\tfrac1n\big)^m \big(1-\tfrac1n\big)^{kn-m} \prod_{i=1}^m A_{j_i} \\
&= \lim_{n\to\infty} \prod_{t=1}^{n} \prod_{j=1}^{k} \big(1 - \tfrac1n + \tfrac1n A_j\big) \\
&= \lim_{n\to\infty} \Big(\prod_{j=1}^{k} \big[1 + \tfrac1n (A_j - 1)\big]\Big)^n \\
&= \lim_{n\to\infty} \Big(1 + \tfrac1n \sum_{j=1}^{k} (A_j - 1)\Big)^n \\
&= \exp\Big\{\sum_{j=1}^k (A_j - 1)\Big\}.
\end{aligned} \tag{7.10}$$

The last identity follows from the Trotter formula, see Proposition 6.2. □

PROOF OF THEOREM 7.1. We start with the equivalence of the partition functions. Given a realisation ω of the Poisson point process on $\mathcal{E} \times [0,\beta]$, let $m = |\omega|$ denote the total number of crosses and double bars. From Lemma 7.3, we have

$$\begin{aligned}
\mathrm{e}^{-\beta H_\Lambda^{(u)}} &= \exp\Big\{\beta \sum_{\{x,y\}\in\mathcal{E}} \big(uT_{xy} + (1-u)Q_{xy} - 1\big)\Big\} \\
&= \int \rho(\mathrm{d}\omega) \prod_{\{x,y,t\}\in\omega}^{*} \left\{ \begin{array}{c} T_{xy} \text{ if the event is a cross} \\ \text{or} \\ Q_{xy} \text{ if the event is a double bar} \end{array} \right\}.
\end{aligned} \tag{7.11}$$

We actually used a straightforward generalisation of the lemma with two Poisson processes of intensities u and $1-u$, in the interval $[0,\beta]$ rather than $[0,1]$. Using a basis of classical spin configurations, $|\sigma\rangle$ with $\sigma \in \{-\frac12, \frac12\}^\Lambda$, we get

$$\mathrm{Tr}\, \mathrm{e}^{-\beta H_\Lambda^{(u)}} = \int \rho(\mathrm{d}\omega) \sum_{\sigma^1,\dots,\sigma^m} \langle\sigma^1|R_{x_1y_1}|\sigma^2\rangle\langle\sigma^2|R_{x_2y_2}|\sigma^3\rangle \dots \langle\sigma^m|R_{x_my_m}|\sigma^1\rangle, \tag{7.12}$$

where $R_{x_iy_i}$ is equal to either $T_{x_iy_i}$ or $Q_{x_iy_i}$, depending on ω. Observe that the product of matrix elements is zero, unless $\sigma_z^i = \sigma_z^{i+1}$ for all $z \neq x_i, y_i$. Further, the spin values at x_i, y_i satisfy

- $\sigma^{i+1}_{x_i} = \sigma^i_{y_i}$ and $\sigma^{i+1}_{y_i} = \sigma^i_{x_i}$ if $R_{x_i y_i} = T_{x_i y_i}$;
- $\sigma^i_{x_i} = \sigma^i_{y_i}$ and $\sigma^{i+1}_{x_i} = \sigma^{i+1}_{y_i}$ if $R_{x_i y_i} = Q_{x_i y_i}$.

Let us introduce "space-time spin configurations", which are piecewise constant functions $\sigma : [0, \beta] \to \{-\frac12, \frac12\}^\Lambda$. Given a realisation ω, let $\Sigma(\omega)$ be the set of space-time spin configurations such that σ_{xt} is constant along each loop of ω. This is illustrated in Fig. 7.3.

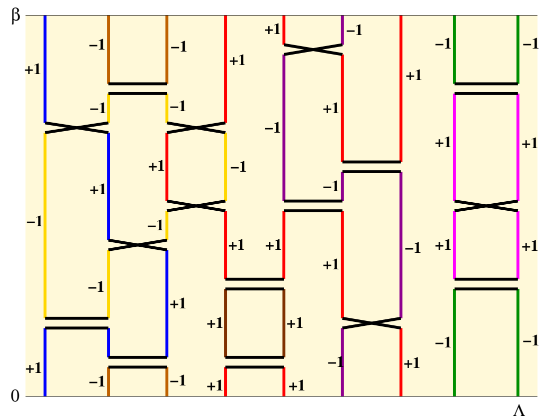

FIGURE 7.3. Illustration for a realisation of the measure ρ and a compatible space-time spin configuration (labels should be $\pm\frac12$ instead of ± 1).

It is possible to rewrite Eq. (7.12) as

$$\operatorname{Tr} \mathrm{e}^{-\beta H^{(u)}_\Lambda} = \int \rho(\mathrm{d}\omega) \sum_{\sigma \in \Sigma(\omega)} 1. \tag{7.13}$$

There are two possibilities per loop, so the sum gives $2^{|\mathcal{L}(\omega)|}$. This proves (i).

For (ii), we expand as before and we get

$$\begin{aligned} \operatorname{Tr} S^3_x S^3_y \mathrm{e}^{-\beta H^{(u)}_\Lambda} &= \int \rho(\mathrm{d}\omega) \sum_{\sigma \in \Sigma(\omega)} \sigma_{x0}\sigma_{y0} \\ &= \int \rho(\mathrm{d}\omega) 1_{E_{xy}}(\omega) \underbrace{\sum_{\sigma \in \Sigma(\omega)} \sigma_{x0}\sigma_{y0}}_{=\frac14} + \int \rho(\mathrm{d}\omega) 1_{E^{\mathrm{c}}_{xy}}(\omega) \underbrace{\sum_{\sigma \in \Sigma(\omega)} \sigma_{x0}\sigma_{y0}}_{=0} \\ &= \tfrac14 \mathbb{P}(E_{xy}). \end{aligned} \tag{7.14}$$

The proof of (iii) is similar, but the operator $S^2_x S^2_y$ forces spin flips at $(x, 0)$ and $(y, 0)$. If ω does not contain a loop that connects x and y, there are no compatible space-time spin configurations and we get 0. If $\omega \in E^+_{xy}$, the factor is

$$\langle \pm\tfrac12 | S^2_x | \mp \tfrac12 \rangle \, \langle \mp\tfrac12 | S^2_y | \pm \tfrac12 \rangle = \tfrac14. \tag{7.15}$$

If $\omega \in E_{xy}^-$, the factor is

$$\langle \pm\tfrac{1}{2}|S_x^2|\mp\tfrac{1}{2}\rangle\,\langle \pm\tfrac{1}{2}|S_y^2|\mp\tfrac{1}{2}\rangle = -\tfrac{1}{4}. \tag{7.16}$$

It follows that $\langle S_x^2 S_y^2\rangle$ involves the difference of probabilities of E_{xy}^+ and E_{xy}^- and we obtain the identity (iii). □

3. References for this chapter

Probabilistic representations of the Gibbs operator $\mathrm{e}^{-\beta H}$ go back to Feynman (1953) for interacting bosons, and to Ginibre (1969) and Conlon-Solovej (1991) for quantum spin systems. The representation described in this chapter is due to Tóth (1993) in the case $u = 1$ (see also Powers, 1976); to Aizenman-Nachtergaele (1994) in the case $u = 0$; and to Ueltschi (2013) for $u \in (0, 1)$.

EXERCISE 7.1.

(i) *Check that the moment generating function (Laplace transform) of a Poisson random variable X is given by*

$$\mathbb{E}(\,\mathrm{e}^{tX}\,) = \mathrm{e}^{\lambda(\,\mathrm{e}^t-1)}\,.$$

(b) *Let ω denote the discrete Poisson process defined in Eq. (7.1). Find the moment generating function of $|\omega|$, and check that*

$$\lim_{n\to\infty}\mathbb{E}_n(\,\mathrm{e}^{t|\omega|}\,) = \mathrm{e}^{\lambda(\,\mathrm{e}^t-1)}\,.$$

(iii) *Find $\mathbb{E}(X)$ and $\mathbb{E}_n(|\omega|)$. (Moment generating functions may help.)*

EXERCISE 7.2. *Let ω denote realisations of crosses and double bars on $\mathcal{E}\times[0,\beta]$. Check that $|\omega|$ is a Poisson random variable with parameter $\beta|\mathcal{E}|$.*

EXERCISE 7.3. *Let $H_\Lambda^{\mathrm{AF}} = 2\sum_{\{x,y\}\in\mathcal{E}}\vec{S}_x\cdot\vec{S}_y$. Find the unitary operator U such that*

$$UH_\Lambda^{\mathrm{AF}}U^{-1} = H_\Lambda^{(0)}.$$

EXERCISE 7.4.

(i) *Show that $T_{xy}\varphi\otimes\psi = \psi\otimes\varphi$ for all $\varphi,\psi\in\mathbb{C}^N$.*

(ii) *Check that T_{xy} is hermitian.*

(iii) *Find the eigenvalues of T_{xy} and their multiplicities. (Hint: Look at T_{xy}^2.)*

EXERCISE 7.5. *Prove Lemma 7.2.*

EXERCISE 7.6. *Consider the transposition operator on $\mathbb{C}^3\otimes\mathbb{C}^3$ and prove that*

$$T_{xy} = -\mathbb{1} + \vec{S}_x\cdot\vec{S}_y + (\vec{S}_x\cdot\vec{S}_y)^2.$$

Hint: If an interaction is $SU(2)$-invariant, that is, if $[A, S_x^i + S_y^i] = 0$ for $i = 1,2,3$, then A has the form $\alpha\mathbb{1} + \beta\vec{S}_x\cdot\vec{S}_y + \gamma(\vec{S}_x\cdot\vec{S}_y)^2$ for some $\alpha,\beta,\gamma\in\mathbb{C}$. You can use this fact, and combine it with Lemma 4.1 and Exercise 7.4.

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