CHAPTER 7

Graphical representations

The origin of graphical representations goes back to Feynman's description of the quantum Bose gas in terms of interacting Brownian trajectories (1953). In 1969, Ginibre proved the occurrence of phase transitions in quantum perturbations of classical models, using "space-time configurations" and a Peierls argument. The quantum spin $\frac{1}{2}$ Heisenberg ferromagnet was described using random transpositions by Powers (1976), and independently by Tóth (1993). Tóth used it to derive a bound for the free energy. Another loop representation was proposed by Aizenman and Nachtergaele for the quantum spin $\frac{1}{2}$ Heisenberg antiferromagnet (1994). This allowed them to relate the quantum chain to two classical two-dimensional models, namely the Potts and random cluster models. The synthesis of these representations, that applies to intermediate models such as quantum XY, was only proposed recently (2013).

In this chapter, we restrict our attention to the case $S = \frac{1}{2}$, mainly for pedagogical reasons. The case $S = 1$ is very interesting, though.

We first describe the random loop models; the connection to quantum spins can be found in Section 2.

1. Random loop models

The Poisson point process plays an essential rôle. It can be introduced in many different ways including awfully abstract ones. We choose a pedestrian and intuitive approach, though.

In words, the Poisson point process on the interval $[0, 1]$ describes the occurrence of events that are independent of one another. Let $\lambda > 0$ be the *intensity* of the process. Let us discretise the interval $[0, 1]$ by considering the set $\{\frac{1}{n}, \frac{2}{n}, \dots, 1\}$, with $\lambda/n < 1$. We consider the probability distribution on subsets of $\{\frac{1}{n}, \frac{2}{n}, \dots, 1\}$, where the probability of the subset ω is

$$\mathbb{P}_n(\omega) = \big(\tfrac{\lambda}{n}\big)^{|\omega|} \big(1 - \tfrac{\lambda}{n}\big)^{n-|\omega|}. \tag{7.1}$$

The interpretation is that each point of the form $\frac{i}{n}$, $1 \leq i \leq n$, occurs with probability λ/n. As $n \to \infty$, this process converges to the Poisson point process on $[0, 1]$ with intensity λ.

The total number of events is not fixed; it is a Poisson random variable with parameter λ. Indeed, for $k \in \mathbb{N}_0$, we have

$$\mathbb{P}(|\omega| = k) = \tfrac{\lambda^k}{k!}\, \mathrm{e}^{-\lambda}\,. \tag{7.2}$$

Integration with respect to the Poisson point process can also be defined by a limit. Let $f = (f_k)$ be a collection of smooth functions $f_k : [0,1]^k \to \mathbb{R}$ (with $f_0 = 1$ by definition). Then

$$\begin{aligned}\mathbb{E}(f) &= \lim_{n\to\infty} \sum_{\omega} \big(\tfrac{\lambda}{n}\big)^{|\omega|} \big(1 - \tfrac{\lambda}{n}\big)^{n-|\omega|} f_{|\omega|}(\omega) \\ &= \lim_{n\to\infty} \sum_{k\geq 0} \big(\tfrac{\lambda}{n}\big)^{k} \big(1 - \tfrac{\lambda}{n}\big)^{n-k} \sum_{\omega, |\omega|=k} f_k(\omega) \\ &= \sum_{k\geq 0} \lambda^k \,\mathrm{e}^{-\lambda} \int_{0<t_1<\dots<t_k<1} f_k(t_1,\dots,t_k)\,\mathrm{d}t_1 \dots \mathrm{d}t_k.\end{aligned} \tag{7.3}$$

The exchange of limit and sum can be justified by dominated convergence. The latter expression is concrete and practical.

Next, we describe the model of random loops. Let $(\Lambda, \mathcal{E})$ denote a finite graph, with Λ the set of vertices and $\mathcal{E}$ the set of edges. Let $\beta > 0$ and $u \in [0,1]$ be two parameters. To each edge $\{x,y\} \in \mathcal{E}$ is associated the interval $[0,\beta]$ and a Poisson point process on this interval. There occur two kinds of events:

- "crosses" occur with intensity u;
- "double bars" occur with intensity $1-u$.

(This process can be defined by generalising the Poisson point process above to two kinds of events, or by considering two separate, independent processes.) We let ρ denote the measure of independent Poisson point processes on $\mathcal{E} \times [0,\beta]$. Let ω denote realisations of this probability measure. It contains finitely many objects with probability 1.

To a given realisation ω corresponds a set of loops on $\mathcal{E} \times [0,\beta]$, denoted $\mathcal{L}(\omega)$. The loops consist of vertical lines connected by crosses or bars, with periodic boundary conditions in the continuous direction. This is best understood by looking at Fig. 7.1.

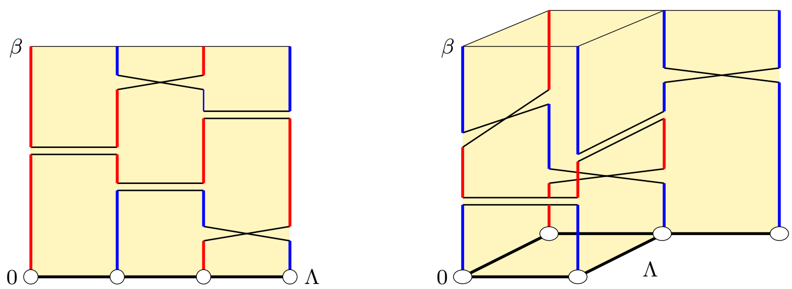

FIGURE 7.1. Graphs and realizations of Poisson point measures, and their loops. In both cases, the number of loops is $|\mathcal{L}(\omega)| = 2$.

The *partition function* $Y(\beta,\Lambda)$ of the model of random loops is defined by

$$Y(\beta,\Lambda) = \int 2^{|\mathcal{L}(\omega)|} \rho(\mathrm{d}\omega). \tag{7.4}$$

The relevant measure for the model of random loops is given by

$$\mu(\mathrm{d}\omega) = \frac{1}{Y(\beta,\Lambda)} 2^{|\mathcal{L}(\omega)|} \rho(\mathrm{d}\omega). \tag{7.5}$$

It can be shown that for β small the loops have small lengths and the probability that two sites belong to the same loop shows exponential decay with respect to the distance between the sites.

The loop correlations are given by the following three events:

- $E^+_{x,y}$ is the set of all realisations ω such that x and y belong to the same loop, and with identical vertical direction at these points.
- $E^-_{x,y}$ is the set of all ω such that x and y belong to the same loop, and with opposite vertical directions at these points.
- $E_{x,y} = E^+_{x,y} \cup E^-_{x,y}$ is the set of all ω such that x and y belong to the same loop.

These events are illustrated in Fig. 7.2. When $u = 1$, that is, when only crosses are present, we have $\mathbb{P}(E^+_{x,y}) = \mathbb{P}(E_{x,y})$. When $u = 0$, and if the graph is bipartite, we have $\mathbb{P}(E^+_{x,y}) = \mathbb{P}(E_{x,y})$ if x, y belong to the same sublattice, and $\mathbb{P}(E^-_{x,y}) = \mathbb{P}(E_{x,y})$ if x, y belong to different sublattices.

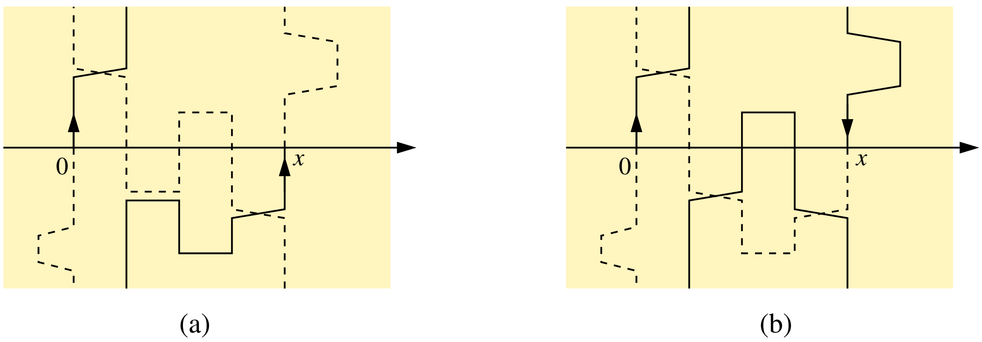

FIGURE 7.2. Illustration for (a) the event $E^+_{0,x}$; (b) the event $E^-_{0,x}$. Here, we shifted the vertical direction and considered $\mathcal{E} \times [-\frac{\beta}{2}, \frac{\beta}{2}]$. Because of periodic boundary conditions in the vertical direction, this is equivalent to $\mathcal{E} \times [0, \beta]$.

2. Relations between random loops and quantum spins

We take $S = \frac{1}{2}$ so the Hilbert space is $\mathcal{H}_\Lambda = \otimes_{x\in\Lambda}\mathbb{C}^2$. We consider the following family of Hamiltonians:

$$H^{(u)}_\Lambda = -2 \sum_{\{x,y\}\in\mathcal{E}} \Big(S^1_x S^1_y + (2u-1) S^2_x S^2_y + S^3_x S^3_y - \tfrac{1}{4} \Big). \tag{7.6}$$

With the parameter u in the interval $[0,1]$, this family interpolates between the Heisenberg ferromagnet ($u=1$), the quantum XY model ($u=\frac{1}{2}$), and the Heisenberg antiferromagnet (more precisely, it is unitarily equivalent to the case $u=0$ if the graph is bipartite).

Here are precise relations between random loops and quantum spins.

THEOREM 7.1.

(i) *The partition functions of both models are identical:*

$$\operatorname{Tr} e^{-\beta H_\Lambda^{(u)}} = \int 2^{|\mathcal{L}(\omega)|} \rho(\mathrm{d}\omega).$$

(ii) *Correlations in the spin directions 1 and 3 are given by loop correlations: For all $x, y \in \Lambda$,*

$$\langle S_x^1 S_y^1 \rangle = \langle S_x^3 S_y^3 \rangle = \tfrac{1}{4} \mathbb{P}(E_{x,y}).$$

(iii) *Correlations in the spin direction 2 are more subtle: For all $x, y \in \Lambda$,*

$$\langle S_x^2 S_y^2 \rangle = \tfrac{1}{4} \big[\mathbb{P}(E_{x,y}^+) - \mathbb{P}(E_{x,y}^-) \big].$$

We prove this theorem by writing the Hamiltonian in terms of suitable interaction operators T_{xy} and Q_{xy}, and by using a "Poisson expansion" of the Gibbs operator $e^{-\beta H_\Lambda^{(u)}}$, see Lemma 7.3.

Let T_{xy} be the transposition operator on $\mathbb{C}^2 \otimes \mathbb{C}^2$:

$$T_{xy} |a,b\rangle = |b,a\rangle, \tag{7.7}$$

for all $a, b = \pm\frac{1}{2}$. And let Q_{xy} be the operator on $\mathbb{C}^2 \otimes \mathbb{C}^2$ with the following matrix elements:

$$\langle a,b | Q_{xy} | c,d \rangle, \tag{7.8}$$

for all $a, b, c, d = \pm\frac{1}{2}$. These operators can be expressed in terms of spin operators.

LEMMA 7.2. *Show that*

(i) $\vec{S}_x \cdot \vec{S}_y = \frac{1}{2} T_{xy} - \frac{1}{4}$.

(ii) $S_x^1 S_y^1 - S_x^2 S_y^2 + S_x^3 S_y^3 = \frac{1}{2} Q_{xy} - \frac{1}{4}$.

The proof of this lemma is done in Exercise 7.5. The Hamiltonian $H_\Lambda^{(u)}$ is a convex combination of these interactions, namely

$$H_\Lambda^{(u)} = - \sum_{\{x,y\} \in \mathcal{E}} \big(u T_{xy} + (1-u) Q_{xy} - 1 \big). \tag{7.9}$$

This is the perfect expression for a Feynman-Kac type of expansion.

Lemma 7.3. *Let $A_1, \dots, A_k$ be bounded operators, and ρ a Poisson point process on $\{1, \dots, k\} \times [0, 1]$ of intensity 1. Then*

$$\exp\Big\{\sum_{j=1}^{k}(A_j - 1)\Big\} = \int \rho(\mathrm{d}\omega) \prod_{(j,t)\in\omega}^{*} A_j.$$

In this lemma, the product $\prod^*$ is over the events of ω in increasing times.

Proof. We start with the Poisson point process. We have

$$\begin{aligned}
\int \rho(\mathrm{d}\omega) \prod_{(j,t)\in\omega}^{*} A_j &= \lim_{n\to\infty} \sum_{m=0}^{n} \sum_{\substack{(j_1,t_1),\dots,(j_m,t_m)\\ t_1<t_2<\dots<t_m}} \big(\tfrac{1}{n}\big)^m \big(1-\tfrac{1}{n}\big)^{kn-m} \prod_{i=1}^{m} A_{j_i} \\
&= \lim_{n\to\infty} \prod_{t=1}^{n}\prod_{j=1}^{k}\big(1 - \tfrac{1}{n} + \tfrac{1}{n}A_j\big) \\
&= \lim_{n\to\infty} \Big(\prod_{j=1}^{k}\big[1 + \tfrac{1}{n}(A_j - 1)\big]\Big)^n \\
&= \lim_{n\to\infty} \Big(1 + \tfrac{1}{n}\sum_{j=1}^{k}(A_j - 1)\Big)^n \\
&= \exp\Big\{\sum_{j=1}^{k}(A_j - 1)\Big\}.
\end{aligned} \tag{7.10}$$

The last identity follows from the Trotter formula, see Proposition 6.2. □

Proof of Theorem 7.1. We start with the equivalence of the partition functions. Given a realisation ω of the Poisson point process on $\mathcal{E} \times [0, \beta]$, let $m = |\omega|$ denote the total number of crosses and double bars. From Lemma 7.3, we have

$$\begin{aligned}
\mathrm{e}^{-\beta H_\Lambda^{(u)}} &= \exp\Big\{\sum_{\{x,y\}\in\mathcal{E}} \big(uT_{xy} + (1-u)Q_{xy} - 1\big)\Big\} \\
&= \int \rho(\mathrm{d}\omega) \prod_{\{x,y,t\}\in\omega}^{*} \left\{ \begin{matrix} T_{xy} \text{ if the event is a cross} \\ \text{or} \\ Q_{xy} \text{ if the event is a double bar} \end{matrix} \right\}.
\end{aligned} \tag{7.11}$$

We actually used a straightforward generalisation of the lemma with two Poisson processes of intensities u and $1 - u$. Using a basis of classical spin configurations, $|\sigma\rangle$ with $\sigma \in \{-\frac{1}{2}, \frac{1}{2}\}^\Lambda$, we get

$$\operatorname{Tr} \mathrm{e}^{-\beta H_\Lambda^{(u)}} = \int \rho(\mathrm{d}\omega) \sum_{\sigma^1,\dots,\sigma^m} \langle\sigma^1|R_{x_1y_1}|\sigma^2\rangle\langle\sigma^2|R_{x_2y_2}|\sigma^3\rangle \dots \langle\sigma^m|R_{x_my_m}|\sigma^1\rangle, \tag{7.12}$$

where $R_{x_iy_i}$ is equal to either $T_{x_iy_i}$ or $Q_{x_iy_i}$, depending on ω. Observe that the product of matrix elements is zero, unless $\sigma^i_z = \sigma^{i+1}_z$ for all $z \neq x_i, y_i$. Further, the spin values at x_i, y_i satisfy

- $\sigma_{x_i}^{i+1} = \sigma_{y_i}^{i}$ and $\sigma_{y_i}^{i+1} = \sigma_{x_i}^{i}$ if $R_{x_i y_i} = T_{x_i y_i}$;
- $\sigma_{x_i}^{i} = \sigma_{y_i}^{i}$ and $\sigma_{x_i}^{i+1} = \sigma_{y_i}^{i+1}$ if $R_{x_i y_i} = Q_{x_i y_i}$.

Let us introduce "space-time spin configurations", which are piecewise constant functions $\sigma : [0,\beta] \to \{-\frac12, \frac12\}^\Lambda$. Given a realisation ω, let $\Sigma(\omega)$ be the set of space-time spin configurations such that σ_{xt} is constant along each loop of ω. This is illustrated in Fig. 7.3.

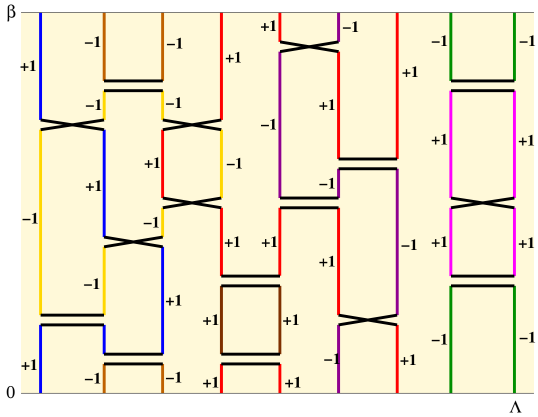

FIGURE 7.3. Illustration for a realisation of the measure ρ and a compatible space-time spin configuration (labels should be $\pm\frac12$ instead of ± 1).

It is possible to rewrite Eq. (7.12) as

$$\mathrm{Tr}\, \mathrm{e}^{-\beta H_\Lambda^{(u)}} = \int \rho(\mathrm{d}\omega) \sum_{\sigma \in \Sigma(\omega)} 1. \tag{7.13}$$

There are two possibilities per loop, so the sum gives $2^{|\mathcal{L}(\omega)|}$. This proves (i).

For (ii), we expand as before and we get

$$\begin{aligned} \mathrm{Tr}\, S_x^3 S_y^3\, \mathrm{e}^{-\beta H_\Lambda^{(u)}} &= \int \rho(\mathrm{d}\omega) \sum_{\sigma\in\Sigma(\omega)} \sigma_{x0}\sigma_{y0} \\ &= \int \rho(\mathrm{d}\omega) 1_{E_{xy}}(\omega) \underbrace{\sum_{\sigma\in\Sigma(\omega)} \sigma_{x0}\sigma_{y0}}_{=\frac14} + \int \rho(\mathrm{d}\omega) 1_{E_{xy}^{\mathrm{c}}}(\omega) \underbrace{\sum_{\sigma\in\Sigma(\omega)} \sigma_{x0}\sigma_{y0}}_{=0} \\ &= \tfrac14 \mathbb{P}(E_{xy}). \end{aligned} \tag{7.14}$$

The proof of (iii) is similar, but the operator $S_x^2 S_y^2$ forces spin flips at $(x,0)$ and $(y,0)$. If ω does not contain a loop that connects x and y, there are no compatible space-time spin configurations and we get 0. If $\omega \in E_{xy}^+$, the factor is

$$\langle \pm\tfrac12 | S_x^2 | \mp \tfrac12 \rangle \langle \mp\tfrac12 | S_y^2 | \pm\tfrac12 \rangle = \tfrac14. \tag{7.15}$$

If $\omega \in E^-_{xy}$, the factor is

$$\langle \pm\tfrac{1}{2}|S^2_x| \mp \tfrac{1}{2}\rangle \, \langle \pm\tfrac{1}{2}|S^2_y| \mp \tfrac{1}{2}\rangle = -\tfrac{1}{4}. \tag{7.16}$$

It follows that $\langle S^2_x S^2_y \rangle$ involves the difference of probabilities of E^+_{xy} and E^-_{xy} and we obtain the identity (iii). □

EXERCISE 7.1.

(i) *Check that the moment generating function (Laplace transform) of a Poisson random variable X is given by*

$$\mathbb{E}(\,\mathrm{e}^{tX}\,) = \,\mathrm{e}^{\lambda(\,\mathrm{e}^{t}\,-1)}\,.$$

(b) *Let ω denote the discrete Poisson process defined in Eq. (7.1). Find the moment generating function of $|\omega|$, and check that*

$$\lim_{n\to\infty} \mathbb{E}_n(\,\mathrm{e}^{t|\omega|}\,) = \,\mathrm{e}^{\lambda(\,\mathrm{e}^{t}\,-1)}\,.$$

(iii) *Find $\mathbb{E}(X)$ and $\mathbb{E}_n(|\omega|)$. (Moment generating functions may help.)*

EXERCISE 7.2. *Let ω denote realisations of crosses and double bars on $\mathcal{E}\times[0,\beta]$. Check that $|\omega|$ is a Poisson random variable with parameter $\beta|\mathcal{E}|$.*

EXERCISE 7.3. *Let $H^{\mathrm{AF}}_{\Lambda} = 2\sum_{\{x,y\}\in\mathcal{E}} \vec{S}_x\cdot\vec{S}_y$. Find the unitary operator U such that*

$$U H^{\mathrm{AF}}_{\Lambda} U^{-1} = H^{(0)}_{\Lambda}.$$

EXERCISE 7.4.

(i) *Show that $T_{xy}\varphi\otimes\psi = \psi\otimes\varphi$ for all $\varphi,\psi\in\mathbb{C}^N$.*
(ii) *Check that T_{xy} is hermitian.*
(iii) *Find the eigenvalues of T_{xy} and their multiplicities. (Hint: Look at T^2_{xy}.)*

EXERCISE 7.5. *Prove Lemma 7.2.*

EXERCISE 7.6. *Consider the transposition operator on $\mathbb{C}^3\otimes\mathbb{C}^3$ and prove that*

$$T_{xy} = -1\!\!1 + \vec{S}_x\cdot\vec{S}_y + (\vec{S}_x\cdot\vec{S}_y)^2.$$

Hint: If an interaction is $SU(2)$-invariant, that is, if $[A, S^i_x + S^i_y] = 0$ for $i = 1,2,3$, then A has the form $\alpha 1\!\!1 + \beta\vec{S}_x\cdot\vec{S}_y + \gamma(\vec{S}_x\cdot\vec{S}_y)^2$ for some $\alpha,\beta,\gamma\in\mathbb{C}$. You can use this fact, and combine it with Lemma 4.1 and Exercise 7.4.

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