CHAPTER 6

Correlation functions

Correlation functions give information about the nature of phases. We collect here some properties of correlation functions for the class of models defined in Eq. (4.5). With $Z_\Lambda = \mathrm{Tr}\, \mathrm{e}^{-\beta H_\Lambda}$ denoting the partition function, the correlation functions at inverse temperature β are given by

$$\langle S_0^i S_x^i \rangle = \frac{1}{Z_\Lambda} \mathrm{Tr}\, S_0^i S_x^i \,\mathrm{e}^{-\beta H_\Lambda} . \tag{6.1}$$

1. Correlation inequalities

It is natural to expect that correlations are stronger among those components of the spins that correspond to stronger coupling parameters in the Hamiltonian. This is the content of the next theorem.

THEOREM 6.1. *Assume that, for all $x, y \in \Lambda$, the couplings satisfy*

$$|J_{xy}^2| \leq J_{xy}^1.$$

Then we have that

$$\bigl|\langle S_0^2 S_x^2 \rangle\bigr| \leq \langle S_0^1 S_x^1 \rangle,$$

for all $x \in \Lambda$. More generally, for all $x_1, \dots x_k \in \Lambda$ and $j_1, \dots, j_k \in \{1,2\}$,

$$\bigl|\langle S_{x_1}^{j_1} \dots S_{x_k}^{j_k} \rangle\bigr| \leq \langle S_{x_1}^1 \dots S_{x_k}^1 \rangle.$$

Further inequalities can be generated using symmetries. Some inequalities hold for the staggered two-point function $(-1)^{|x|} \langle S_0^i S_x^i \rangle$. The proof can be found after that of Trotter's product formula, which is needed.

PROPOSITION 6.2 (Trotter formula). *Let A, B be $N \times N$ matrices. Then*

$$\mathrm{e}^{A+B} = \lim_{n\to\infty} \Bigl(\mathrm{e}^{\frac1n A}\, \mathrm{e}^{\frac1n B} \Bigr)^n = \lim_{n\to\infty} \Bigl[\bigl(1 + \tfrac1n A\bigr)\, \mathrm{e}^{\frac1n B} \Bigr]^n.$$

PROOF. We prove the second formula — the mild changes for the first formula are straightforward. First, we have

$$\mathrm{e}^{A+B} - \bigl(1 + \tfrac1n (A+B)\bigr)^n = \sum_{k=0}^n \frac{1}{k!} (A+B)^k \Bigl(1 - \frac{n(n-1)\dots(n_k+1)}{n^k} \Bigr) + \sum_{k\geq n+1} \frac{1}{k!} (A+B)^k. \tag{6.2}$$

The norm of the right side clearly vanishes as $n \to \infty$. Next, let K_n be the matrix such that

$$\big(1 + \tfrac{1}{n} A\big) \,\mathrm{e}^{\frac{1}{n} B} = 1 + \frac{1}{n}(A+B) + K_n. \tag{6.3}$$

It is clear that $\|K_n\| = O(\frac{1}{n^2})$. We have

$$\begin{aligned}\big((1 + \tfrac{1}{n} A) \,\mathrm{e}^{\frac{1}{n} B}\big)^n - \big(1 + \tfrac{1}{n}(A+B)\big)^n &= \big(1 + \tfrac{1}{n}(A+B) + K_n\big)^n - \big(1 + \tfrac{1}{n}(A+B)\big)^n \\ &= \int_0^1 \mathrm{d}s \frac{\mathrm{d}}{\mathrm{d}s} \big(1 + \tfrac{1}{n}(A+B) + sK_n\big)^n \\ &= \int_0^1 \mathrm{d}s \sum_{k=0}^{n-1} \big(1 + \tfrac{1}{n}(A+B) + sK_n\big)^k K_n \big(1 + \tfrac{1}{n}(A+B) + sK_n\big)^{n-k-1}.\end{aligned} \tag{6.4}$$

Consequently,

$$\Big\| \big((1 + \tfrac{1}{n} A) \,\mathrm{e}^{\frac{1}{n} B}\big)^n - \big(1 + \tfrac{1}{n}(A+B)\big)^n \Big\| \leq \sum_{k=0}^{n-1} \int_0^1 \mathrm{d}s \big\| 1 + \tfrac{1}{n}(A+B) + sK_n \big\|^{n-1} \|K_n\|. \tag{6.5}$$

We have

$$\big\| 1 + \tfrac{1}{n}(A+B) + sK_n \big\|^{n-1} \leq \big(1 + \tfrac{1}{n}\|A+B\| + \|K_n\|\big)^{n-1} \to \mathrm{e}^{\|A+B\|}, \tag{6.6}$$

so that the right side of Eq. (6.5) is less than a constant times $n\|K_n\|$, which vanishes in the limit $n \to \infty$. □

Proof of Theorem 6.1. Let $|a\rangle$, $a \in \{-S, \dots, S\}$ denote the eigenvectors of S^3, and recall the operators $S^\pm$ defined before. The matrix elements of $S^1, S^\pm$ are all nonnegative, and the matrix elements of S^2 are all less than or equal to those of S^1 in absolute values. Using the Trotter formula and multiple resolutions of the identity, we have

$$\begin{aligned}\big|\mathrm{Tr}\, S_0^2 S_x^2 \,\mathrm{e}^{-\beta H_\Lambda}\big| \leq \lim_{N\to\infty} \sum_{\sigma_0,\dots,\sigma_N \in \{-S,\dots,S\}^\Lambda} \Big| &\langle \sigma_0 | S_0^2 S_x^2 | \sigma_1 \rangle \\ &\langle \sigma_1 | \,\mathrm{e}^{\frac{\beta}{N} \sum J_{yz}^3 S_y^3 S_z^3} | \sigma_1 \rangle \langle \sigma_1 | \Big(1 + \frac{\beta}{N} \sum_{y,z \in \Lambda} (J_{yz}^1 S_y^1 S_z^1 + J_{yz}^2 S_y^2 S_z^2)\Big) | \sigma_2 \rangle \\ &\dots \langle \sigma_N | \,\mathrm{e}^{\frac{\beta}{N} \sum J_{yz}^3 S_y^3 S_z^3} | \sigma_N \rangle \langle \sigma_N | \Big(1 + \frac{\beta}{N} \sum_{y,z \in \Lambda} (J_{yz}^1 S_y^1 S_z^1 + J_{yz}^2 S_y^2 S_z^2)\Big) | \sigma_0 \rangle \Big|.\end{aligned} \tag{6.7}$$

Observe that the matrix elements of all operators are nonnegative, except for $S_0^2 S_x^2$. Indeed, this follows from

$$J_{yz}^1 S_y^1 S_z^1 + J_{yz}^2 S_y^2 S_z^2 = \tfrac{1}{4}(J_{yz}^1 - J_{yz}^2)(S_y^+ S_z^+ + S_y^- S_z^-) + \tfrac{1}{4}(J_{yz}^1 + J_{yz}^2)(S_y^+ S_z^- + S_y^- S_z^+). \tag{6.8}$$

We get an upper bound for the right side of (6.7) by replacing $|\langle \sigma_0 | S_0^2 S_x^2 | \sigma_1 \rangle|$ with $\langle \sigma_0 | S_0^1 S_x^1 | \sigma_1 \rangle$. We have obtained

$$\Big| \operatorname{Tr} S_0^2 S_x^2 \, \mathrm{e}^{-\beta H_\Lambda} \Big| \leq \operatorname{Tr} S_0^1 S_x^1 \, \mathrm{e}^{-\beta H_\Lambda} , \tag{6.9}$$

which proves the first claim. The second claim can be proved exactly the same way. □

Corollary 6.3. *Assume that for all $x, y \in \Lambda$, the couplings satisfy*

$$J_{xy}^1 = J_{xy}^2 \geq 0.$$

Then we have for all $x, y, z, u \in \Lambda$

$$\frac{\partial}{\partial J_{xy}^1} \langle S_z^2 S_u^2 \rangle \leq \frac{\partial}{\partial J_{xy}^1} \langle S_0^1 S_x^1 \rangle.$$

Proof. For $i = 1, 2, 3$, we have

$$\frac{1}{\beta} \frac{\partial}{\partial J_{xy}^1} \langle S_z^i S_u^i \rangle = (S_x^1 S_y^1, S_z^i S_u^i) - \langle S_x^1 S_y^1 \rangle \langle S_z^i S_u^i \rangle, \tag{6.10}$$

where (A, B) denotes the Duhamel two-point function,

$$(A, B) = \frac{1}{Z_\Lambda} \int_0^1 \operatorname{Tr} A \, \mathrm{e}^{-s\beta H_\Lambda} \, B \, \mathrm{e}^{-(1-s)\beta H_\Lambda} \, \mathrm{d}s. \tag{6.11}$$

It is not hard to extend the proof of Theorem 6.1 to the Duhamel function, so that

$$\big| (S_x^1 S_y^1, S_z^2 S_u^2) \big| \leq (S_x^1 S_y^1, S_z^1 S_u^1). \tag{6.12}$$

Further, we have $\langle S_z^2 S_u^2 \rangle = \langle S_z^1 S_u^1 \rangle$ by symmetry. The result follows. □

2. Decay of correlations due to symmetries

In this section we prove a variant of the Mermin-Wagner theorem. The result applies to systems that are effectively two-dimensional.

We assume that $J_{xy}^1 = J_{xy}^2$ for all x, y. The decay of correlations is measured by the following expression:

$$\xi_\beta(x) = \sup_{\substack{(\phi_y) \in \mathbb{R}^\Lambda \\ \phi_x = 0}} \Big[\phi_0 - \beta S^2 \sum_{y,z \in \Lambda} |J_{yz}^1| \big(\cosh(\phi_y - \phi_z) - 1 \big) \Big]. \tag{6.13}$$

The solution of this variational problem is essentially a discrete harmonic function. We can estimate it explicitly in the case of "2D-like" graphs with nearest-neighbor couplings. Let Λ denote a graph, i.e. a finite set of vertices and a set of edges, and let $d(x, y)$ denote the graph distance, i.e. the length of the shortest path that connects x and y.

LEMMA 6.4. *Assume that* $J^i_{xy} = 0$ *whenever* $d(x,y) \geq 2$ *and let* $J = \max |J^i_{xy}|$. *Assume in addition that there exists a constant* K *such that, for any* $\ell \in \mathbb{N}$,

$$\#\big\{\{x,y\} \subset \Lambda : d(0,x) = \ell \text{ and } d(0,y) = \ell + 1\big\} \leq K\ell.$$

Then there exists $C = C(\beta, S, J, K)$, *which does not depend on* x, *such that*

$$\xi_\beta(x) \geq \frac{1}{8\beta J S^2 K} \log\big(d(0,x)+1\big) - C.$$

PROOF. With c to be chosen later, let

$$\phi_y = \begin{cases} c \log \frac{d(0,x)+1}{d(0,y)+1} & \text{if } d(0,y) \leq d(0,x), \\ 0 & \text{otherwise.} \end{cases} \tag{6.14}$$

Then

$$\xi_\beta(x) \geq c \log(d(0,x)+1) - 2\beta S^2 J K \sum_{\ell=0}^{d(0,x)-1} \big(\cosh(c \log \tfrac{\ell+2}{\ell+1}) - 1\big)\ell. \tag{6.15}$$

From Taylor expansions of the logarithm and of the hyperbolic cosine, there exist C, C' such that

$$\begin{aligned} \xi_\beta(x) &\geq c \log(d(0,x)+1) - 2\beta S^2 J K c^2 \sum_{\ell=1}^{d(0,x)} \frac{1}{\ell} - C' \\ &\geq \big[c - 2\beta S^2 J K c^2\big] \log(d(0,x)+1) - C. \end{aligned} \tag{6.16}$$

The optimal choice is $c = (4\beta S^2 J K)^{-1}$. □

THEOREM 6.5. *Assume that* $J^1_{xy} = J^2_{xy}$ *for all* $x, y \in \Lambda$. *Then, for* $i = 1, 2$, *we have*

$$\big|\langle S^i_0 S^i_x \rangle\big| \leq 2S^2 \,\mathrm{e}^{-\xi_\beta(x)}.$$

In the case of 2D-like graphs, we can use Lemma 6.4 and we obtain algebraic decay with a power greater than $(8\beta J S^2 K)^{-1}$.

The proof uses the Hölder inequality for traces, which can be proved using chessboard estimates. Recall that the “absolute value” of a matrix is $|A| = (A^* A)^{\frac12}$, where the square root of a nonnegative hermitian matrix can be defined by diagonalising and taking the square root of the eigenvalues. The p-norm of a matrix is then defined as $\|A\|_p = (\operatorname{Tr} |A|^p)^{1/p}$. Notice that $\lim_{p\to\infty} \|A\|_p = \|A\|$.

PROPOSITION 6.6 (Hölder inequality for matrices). *If* $1 \leq p, q, r \leq \infty$ *with* $\frac1p + \frac1q = \frac1r$, *we have*

$$\|AB\|_r \leq \|A\|_p \|B\|_q.$$

It follows from a simple induction that

$$\Big\| \prod_{j=1}^{n} A_j \Big\|_r \leq \prod_{j=1}^{n} \|A_j\|_{p_j} \tag{6.17}$$

whenever $1 \leq r, p_1, \dots, p_n$ with $\sum_{j=1}^{n} \frac{1}{p_j} = \frac{1}{r}$. There are no short proofs of Hölder's inequality for matrices. We prove below Eq. (6.17) for $r = 1$ only. It implies Proposition 6.6 for $r \in \mathbb{N}$ and it is enough for the purpose of proving Theorem 6.5. The proof is due to Fröhlich [1978] and it uses chessboard estimates.

Lemma 6.7 (Chessboard estimate). *For any $n \in \mathbb{N}$ and any matrices $A_1, \dots, A_{2n}$, we have*

$$\big| \mathrm{Tr}\, A_1 \dots A_{2n} \big| \leq \prod_{i=1}^{2n} \Big(\mathrm{Tr}\, (A_i A_i^*)^n \Big)^{1/2n}.$$

Proof. Since $(A, B) \mapsto \mathrm{Tr}\, A^* B$ is an inner product, the following inequality follows from Cauchy-Schwarz:

$$\big| \mathrm{Tr}\, A_1 \dots A_{2n} \big|^2 \leq \mathrm{Tr}\, \big(A_1 \dots A_n A_n^* \dots A_1^* \big)\ \mathrm{Tr}\, \big(A_{2n}^* \dots A_{n+1}^* A_{n+1} \dots A_{2n} \big). \tag{6.18}$$

This allows to use a reflection positivity argument. It is enough to prove the inequality for matrices that satisfy $\mathrm{Tr}\, (A_i A_i^*)^n = 1$; the general result follows from scaling.

Let $A_1, \dots, A_{2n}$ be matrices that maximise $|\mathrm{Tr}\, A_1 \dots A_{2n}|$, with maximum number of matching neighbours $A_{i+1} = A_i^*$. Suppose there exists an index j such that $A_{j+1} \neq A_j^*$. Using cyclicity, we can assume that $j = n$. By the inequality (6.18), $A_1, \dots, A_n, A_n^*, \dots, A_1^*$ and $A_{2n}^*, \dots, A_{n+1}^*, A_{n+1}, \dots, A_{2n}$ are also maximisers. At least one has strictly more matching neighbours, hence a contradiction. The maximum is then $\mathrm{Tr}\, (AA^*)^n$ for some matrix A, which is equal to 1. □

Chessboard estimates allow to prove the case $r = 1$ of Hölder's inequality.

Corollary 6.8. *We have*

$$|\mathrm{Tr}\, A_1 \dots A_n| \leq \prod_{i=1}^{n} \|A_i\|_{m_i}$$

for all n and all rational m_i's such that $\sum_{i=1}^{n} \frac{1}{m_i} = 1$.

Proof. Let ℓ be a positive integer such that $2\ell/m_i$ is integer for all i. Let $A_i = U_i |A_i|$ be the polar decomposition of A_i, and let

$$B_i = |A_i|^{m_i/2\ell}, \qquad \hat{B}_i = U_i |A_i|^{m_i/2\ell}. \tag{6.19}$$

Then $A_i = \hat{B}_i B_i^{(2\ell/m_i)-1}$, and we have

$$\begin{aligned}\big|\mathrm{Tr}\, A_1 \dots A_n\big| &= \big|\mathrm{Tr}\, \hat{B}_1 \underbrace{B_1 \dots B_1}_{(2\ell/m_1)-1} \dots \hat{B}_n \underbrace{B_n \dots B_n}_{(2\ell/m_n)-1}\big| \\ &\leq \prod_{i=1}^{n} (\mathrm{Tr}\, |A_i|^{m_i})^{1/m_i} \\ &= \prod_{i=1}^{n} \|A_i\|_{m_i}.\end{aligned} \tag{6.20}$$

The inequality follows from Lemma 6.7 and from the identities

$$\mathrm{Tr}\,(B_i B_i^*)^\ell = \mathrm{Tr}\,(\hat{B}_i \hat{B}_i^*)^\ell = \mathrm{Tr}\,|A_i|^{m_i}. \tag{6.21}$$

□

LEMMA 6.9. *Let $r, r' \in [1, \infty]$ such that $\frac{1}{r} + \frac{1}{r'} = 1$. Then for any square matrix A, we have*

$$\|A\|_r = \sup_{\|C\|_{r'}=1} \mathrm{Tr}\, C^* A.$$

PROOF. The right side is smaller by Corollary 6.8:

$$\big|\mathrm{Tr}\, C^* A\big| \leq \|C\|_{r'} \|A\|_r = \|A\|_r. \tag{6.22}$$

In order to check that this inequality is saturated, let $A = U|A|$ be the polar decomposition of A, and choose $C = \|A\|_r^{1-r} U |A|^{r-1}$. Then $\|C\|_{r'} = 1$ and $\mathrm{Tr}\, C^* A = \|A\|_r$. □

PROOF OF PROPOSITION 6.6. Starting with Lemma 6.9 and then using Corollary 6.8, we have

$$\begin{aligned}\|AB\|_r &= \sup_{\|C\|_{r'}=1} \mathrm{Tr}\, C^* AB \\ &\leq \sup_{\|C\|_{r'}=1} \|C\|_{r'} \|A\|_p \|B\|_q.\end{aligned} \tag{6.23}$$

□

PROOF OF THEOREM 6.5. We use the method of complex rotations. Let

$$S_y^\pm = S_y^1 \pm \mathrm{i} S_y^2. \tag{6.24}$$

One can check that for any $a \in \mathbb{C}$, we have

$$\mathrm{e}^{a S_y^3}\, S_y^\pm\, \mathrm{e}^{-a S_y^3} \;=\; \mathrm{e}^{\pm a}\, S_y^\pm. \tag{6.25}$$

The Hamiltonian (4.5) can be rewritten as

$$H_\Lambda = -\tfrac{1}{2} \sum_{y,z\in\Lambda} \big(J_{yz}^1 S_y^+ S_z^- + J_{yz}^3 S_y^3 S_z^3 \big) \tag{6.26}$$

Given numbers ϕ_y, let

$$A = \prod_{y\in\Lambda} \mathrm{e}^{\phi_y S_y^3}\,. \tag{6.27}$$

Then

$$\operatorname{Tr} S_0^+ S_x^- \, \mathrm{e}^{-\beta H_\Lambda} = \operatorname{Tr} A S_0^+ S_x^- A^{-1} \, \mathrm{e}^{-\beta A H_\Lambda A^{-1}} . \tag{6.28}$$

We now compute the rotated Hamiltonian.

$$\begin{aligned} A H_\Lambda A^{-1} &= -\tfrac12 \sum_{y,z\in\Lambda} \big(J^1_{yz} \, \mathrm{e}^{\phi_y - \phi_z} \, S_y^+ S_z^- + J^3_{yz} S^3_y S^3_z \big) \\ &= H_\Lambda - \tfrac12 \sum_{y,z\in\Lambda} J^1_{yz} \big(\cosh(\phi_y - \phi_z) - 1 \big) \, S_y^+ S_z^- - \tfrac12 \sum_{y,z\in\Lambda} J^1_{yz} \sinh(\phi_y - \phi_z) \, S_y^+ S_z^- \\ &\equiv H_\Lambda + B + C. \end{aligned} \tag{6.29}$$

Notice that $B^* = B$ and $C^* = -C$. We obtain

$$\operatorname{Tr} S_0^+ S_x^- \, \mathrm{e}^{-\beta H_\Lambda} = \mathrm{e}^{\phi_0 - \phi_x} \operatorname{Tr} S_0^+ S_x^- \, \mathrm{e}^{-\beta H_\Lambda - \beta B - \beta C} . \tag{6.30}$$

We now estimate the trace in the right side using the Trotter product formula and the Hölder inequality for traces. Recall that $\|B\|_s = (\operatorname{Tr} |B|^s)^{1/s}$, with $\|B\|_\infty = \|B\|$ being the usual operator norm.

$$\begin{aligned} \operatorname{Tr} S_0^+ S_x^- \, \mathrm{e}^{-\beta H_\Lambda - \beta B - \beta C} &= \lim_{N\to\infty} \operatorname{Tr} S_0^+ S_x^- \Big(\mathrm{e}^{-\frac{\beta}{N} H_\Lambda} \, \mathrm{e}^{-\frac{\beta}{N} B} \, \mathrm{e}^{-\frac{\beta}{N} C} \Big)^N \\ &\leq \lim_{N\to\infty} \|S_0^+ S_x^-\|_\infty \, \big\| \mathrm{e}^{-\frac{\beta}{N} H_\Lambda} \big\|_N^N \, \big\| \mathrm{e}^{-\frac{\beta}{N} B} \big\|_\infty^N \, \big\| \mathrm{e}^{-\frac{\beta}{N} C} \big\|_\infty^N . \end{aligned} \tag{6.31}$$

Observe now that $\|S_0^+ S_x^-\| = 2S^2$, $\| \mathrm{e}^{-\frac{\beta}{N} H_\Lambda} \|_N^N = Z_\Lambda$, $\| \mathrm{e}^{-\frac{\beta}{N} B} \|^N \leq \mathrm{e}^{\beta \|B\|}$, and $\| \mathrm{e}^{-\frac{\beta}{N} C} \| = 1$. The theorem then follows from

$$\|B\| \leq S^2 \sum_{y,z\in\Lambda} |J^1_{yz}| \big(\cosh(\phi_y - \phi_z) - 1 \big). \tag{6.32}$$

□

Exercise 6.1.

(i) *Show that Eq. (6.17) follows from Proposition 6.6.*

(ii) *Show that Proposition 6.6 for $r \in 2\mathbb{N}$ follows from Eq. (6.17) for $r = 1$.*

Exercise 6.2. *Magnetisation and correlation functions. Let m_Λ denote the magnetisation operator in the 3rd spin direction:*

$$m_\Lambda = \frac{1}{|\Lambda|} \sum_{x\in\Lambda} S^3_x.$$

(i) *Show that $\langle m_\Lambda \rangle = 0$ because of symmetries.*

(ii) *Observe that $\langle m_\Lambda^2 \rangle = \frac{1}{|\Lambda|^2} \sum_{x,y\in\Lambda} \langle S^3_x S^3_y \rangle$. Assume that $|\langle S^3_x S^3_y \rangle| \leq q(\|x-y\|)$ uniformly in Λ, for some function q that satisfies $q(r) \to 0$ as $r \to \infty$. Show that*

$$\limsup_{\Lambda \nearrow \mathbb{Z}^d} \big| \langle m_\Lambda^2 \rangle \big| = 0.$$