CHAPTER 5

Gibbs states

The most striking phenomenon of equilibrium statistical mechanics is that of a *phase transition.* We are all familiar with transitions from ice to water and water to vapour; more generally, from solid to liquid to gas. Here, we are concerned with magnetic properties of solids. Ferromagnetism is the ability for a material to remain magnetised, after the external magnetic field has been removed. This actually take place only if the temperature is below the *Curie temperature.* The statistical mechanics explanation is that low temperature ferromagnets can be in several different *Gibbs states* that are characterised by the direction of their magnetisation. Since only one Gibbs state exists at high temperatures, their number must change as the temperature is lowered, which corresponds to the occurrence of a phase transition.

1. Evolution operator

Let $N = 2S + 1$. We also use the notation $\Lambda \subset\subset \mathbb{Z}^d$ when Λ is a finite subset of $\mathbb{Z}^d$. Let $\mathcal{H}_\Lambda = \otimes_{x\in\Lambda}\mathbb{C}^N$ and $\mathcal{A}_\Lambda = \mathcal{B}(\mathcal{H}_\Lambda)$ the algebra of linear operators on $\mathcal{H}_\Lambda$. It is a C^* algebra, namely, it possesses of hermitian conjugation and a norm. If $\Lambda \subset \Lambda'$, we view $\mathcal{A}_\Lambda$ as a subalgebra of $\mathcal{A}_{\Lambda'}$ by identifying $A \in \mathcal{A}_\Lambda$ with $A \otimes \mathbb{1}_{\Lambda'\setminus\Lambda} \in \mathcal{A}_{\Lambda'}$.

Let $(\Phi_X)_{X\subset\mathbb{Z}^d}$ denote an "interaction", that is, a collection of operators $\Phi_X \in \mathcal{A}_X$, for any finite subset X of $\mathbb{Z}^d$. The norm of an interaction is defined by

$$\|\Phi\|_r = \sup_{x\in\mathbb{Z}^d} \sum_{X\ni x} \|\Phi_X\| \, \mathrm{e}^{r|X|} . \tag{5.1}$$

Here, $\|\Phi_X\|$ denotes the usual operator norm in $\mathcal{A}_X$, and $r \geq 0$ is a parameter. The Hamiltonian associated with a finite domain $\Lambda \subset \mathbb{Z}^d$ is given by

$$H_\Lambda = \sum_{X\subset\Lambda} \Phi_X. \tag{5.2}$$

It is time to give a precise meaning to the notion of infinite-volume limit. The next definition will apply to several different objects such as vectors, operators, states, etc...

DEFINITION 5.1. *Let $(\mathcal{X}, d)$ be a metric space and let (x_Λ) be a family of elements of $\mathcal{X}$ indexed by finite subsets $\Lambda \subset\subset \mathbb{Z}^d$. We say that (x_Λ) converges to $x \in \mathcal{X}$ as $\Lambda \nearrow \mathbb{Z}^d$,*

$$\lim_{\Lambda\nearrow\mathbb{Z}^d} x_\Lambda = x,$$

if for every $\varepsilon > 0$, there exists $\Lambda_\varepsilon \subset\subset \mathbb{Z}^d$ such that $d(x_\Lambda, x) < \varepsilon$ for all finite $\Lambda \supset \Lambda_\varepsilon$.

In order to describe infinite systems, we consider the C^*-algebra $\mathcal{A}$ of quasi-local observables, which is the norm-completion of the usual algebra of local observables

$$\mathcal{A} = \overline{\mathcal{A}_0}, \qquad \text{where} \qquad \mathcal{A}_0 = \bigvee_{\Lambda \nearrow \mathbb{Z}^d} \mathcal{A}_\Lambda. \tag{5.3}$$

For $t \in \mathbb{C}$, let α_t^Λ be the linear automorphism of $\mathcal{A}_\Lambda$ that describes the time evolution of operators ("observables") in $\mathcal{A}_\Lambda$, namely

$$\alpha_t^\Lambda(a) = \mathrm{e}^{\mathrm{i}tH_\Lambda}\, a\, \mathrm{e}^{-\mathrm{i}tH_\Lambda}\,. \tag{5.4}$$

By tensoring with identities, α_t^Λ can be extended as a bounded operator $\mathcal{A} \to \mathcal{A}$. Its norm depends a priori on Λ.

We first address the question of the existence of the infinite-volume limit of α_t^Λ. In view of the discussion of KMS states below, we need to consider complex times as well. It turns out that α_t^Λ converges uniformly to a bounded operator when $t \in \mathbb{R}$; it converges pointwise when $|\mathrm{Im}\, t|$ is small; it is not known otherwise.

Proposition 5.1 (Infinite-volume limit of the evolution operator). *Assume that $\|\Phi\|_r < \infty$ for some $r > 0$. Then*

(a) *If $t \in \mathbb{C}$ and $|\mathrm{Im}\, t| < \frac{r}{2\|\Phi\|_r}$, there exists an automorphism $\alpha_t : \mathcal{A}_0 \to \mathcal{A}$ such that*

$$\lim_{\Lambda \nearrow \mathbb{Z}^d} \|\alpha_t^\Lambda(a) - \alpha_t(a)\| = 0$$

for all $a \in \mathcal{A}_0$. Further, we have

$$\|\alpha_t^\Lambda(a)\| \leq \|a\|\, \mathrm{e}^{r|X|} \Big(1 - |t| \frac{2\|\Phi\|_r}{r}\Big)^{-1}$$

whenever $a \in \mathcal{A}_X$.

(b) *For $t \in \mathbb{R}$, α_t is a *isomorphism with $\|\alpha_t\| = 1$ and it satisfies the group property*

$$\alpha_{s+t}(a) = \alpha_s\big(\alpha_t(a)\big).$$

In case clarification is needed, α_t is an automorphism in the sense that $\alpha_t(ab) = \alpha_t(a)\alpha_t(b)$. We also have $\alpha_t(a)^* = \alpha_{\bar{t}}(a^*)$, so that α_t is a *automorphism for real t.

The proof consists of the following steps.

(i) If $|t| < \frac{r}{2\|\Phi\|_r}$, $(\alpha_t^\Lambda)_{\Lambda \subset\subset \mathbb{Z}^d}$ is Cauchy for each fixed $a \in \mathcal{A}_0$. We denote the limit $\alpha_t(a)$.
(ii) For $t \in \mathbb{R}$, we have $\|\alpha_t^\Lambda(a)\| = \|a\|$ for all Λ, so $\|\alpha_t\| = 1$.
(ii) We use the group property to extend α_t it to the whole real line, then to the infinite strip.

For the first step, we need the multicommutator expansion. Let $\mathrm{ad}_A(B) = [A, B]$ denote the "adjoint endomorphism".

Lemma 5.2 (Multicommutator expansion). *Let A and B be two operators on the same finite-dimensional Hilbert space. Then*

$$\mathrm{e}^{A} B \, \mathrm{e}^{-A} = \sum_{n \geq 0} \frac{1}{n!} \mathrm{ad}_A^n(B).$$

Proof. We show that $\mathrm{e}^{sA} B \, \mathrm{e}^{-sA}$ and $\sum_n \frac{s^n}{n!} \mathrm{ad}_A^n(B)$ satisfy the same differential equation. First,

$$\frac{\mathrm{d}}{\mathrm{d}s} \mathrm{e}^{sA} B \, \mathrm{e}^{-sA} = \big[A, \mathrm{e}^{sA} B \, \mathrm{e}^{-sA}\big]. \tag{5.5}$$

Second,

$$\frac{\mathrm{d}}{\mathrm{d}s} \sum_{n \geq 0} \frac{s^n}{n!} \mathrm{ad}_A^n(B) = \sum_{n \geq 1} \frac{s^{n-1}}{(n-1)!} \mathrm{ad}_A\Big(\mathrm{ad}_A^{n-1}(B)\Big) = \Big[A, \sum_{n \geq 0} \frac{s^n}{n!} \mathrm{ad}_A^n(B)\Big]. \tag{5.6}$$

□

Proof of Proposition 5.1. Let $a \in \mathcal{A}_Y$ for some $Y \subset\subset \mathbb{Z}^d$. We show that $(\alpha_t^\Lambda(a))_{\Lambda \subset\subset \mathbb{Z}^d}$ is Cauchy. By Lemma 5.2, we have

$$\begin{aligned} \alpha_t^\Lambda(a) &= \sum_{n \geq 0} \frac{(\mathrm{i}t)^n}{n!} \mathrm{ad}_{H_\Lambda}^n(a) \\ &= \sum_{n \geq 0} \frac{(\mathrm{i}t)^n}{n!} \sum_{X_1, \dots, X_n \subset \Lambda} [\Phi_{X_n}, [\Phi_{X_{n-1}}, \dots [\Phi_{X_1}, a] \dots]. \end{aligned} \tag{5.7}$$

We show that this series converges absolutely for small $|t|$. In order for the commutators to differ from zero, the sets must satisfy

$$\begin{aligned} &X_1 \cap Y \neq \emptyset, \\ &X_2 \cap (X_1 \cup Y) \neq \emptyset, \\ &\quad \vdots \\ &X_n \cap (X_1 \cup \dots \cup X_{n-1} \cup Y) \neq \emptyset. \end{aligned} \tag{5.8}$$

The sum over such sets can be realised by first summing over sets that contain the origin, then by summing over translations so that (5.8) is satisfied. There are no more than

$$\begin{aligned} |Y| &\text{ translations for } X_1, \\ |X_1| + |Y| &\text{ translations for } X_2, \\ &\vdots \\ |X_1| + \dots + |X_{n-1}| + |Y| &\text{ translations for } X_n. \end{aligned} \tag{5.9}$$

We get

$$\Big\| \sum_{X_1,\dots,X_n} [\Phi_{X_n}, \dots [\Phi_{X_1}, a] \dots] \Big\| \leq \|a\| 2^n \sum_{X_1,\dots,X_n \ni 0} \big(|X_1| + \dots + |X_n| + |Y| \big)^n \prod_{i=1}^n \|\Phi_{X_i}\| \\ \leq \|a\| \, \mathrm{e}^{r|Y|} \, n! \Big(\frac{2\|\Phi\|_r}{r} \Big)^n. \tag{5.10}$$

We used $c^n \leq n! r^{-n} \mathrm{e}^{rc}$, which is obvious from the Taylor series of e^{rc}. It follows that $\alpha_t^\Lambda(a)$ is absolutely convergent whenever $|t| < \frac{r}{2\|\Phi\|_r}$ for some $r > 0$. Notice the bound

$$\|\alpha_t^\Lambda(a)\| \leq \|a\| \, \mathrm{e}^{r|Y|} \Big(1 - |t| \frac{2\|\Phi\|_r}{r} \Big)^{-1}. \tag{5.11}$$

for all $a \in \mathcal{A}_Y$. It is uniform in Λ but not in Y.

If $\Lambda' \supset \Lambda$, we have

$$\alpha_t^{\Lambda'}(a) - \alpha_t^\Lambda(a) = \sum_{n \geq 0} \frac{(\mathrm{i}t)^n}{n!} \sum_{\substack{X_1,\dots,X_n : Y \\ \cup X_i \not\subset \Lambda}} [\Phi_{X_n}, [\Phi_{X_{n-1}}, \dots [\Phi_{X_1}, a] \dots]. \tag{5.12}$$

The second sum is over sets in Λ' that satisfy the constraint (5.8) and whose union is not contained in Λ. For small $|t|$, it follows from the absolute convergence of the series that (5.12) is as small as we want by taking Λ large enough. Hence $(\alpha_t^\Lambda(a))_\Lambda$ is Cauchy, and it converges since $\mathcal{A}$ is complete. We define $\alpha_t(a)$ to be equal to the limit.

The map α_t is clearly linear and it satisfies $\alpha_t(a^*) = \alpha_t(a)^*$.

If $\|\alpha_s^\Lambda - \alpha_s\| \to 0$ and $\|\alpha_t^\Lambda - \alpha_t\| \to 0$, we can define $\alpha_{s+t} = \alpha_s \circ \alpha_t$ and we have, for all $\|a\| = 1$,

$$\|\alpha_{s+t}^\Lambda(a) - \alpha_{s+t}(a)\| \leq \|\alpha_s^\Lambda\big(\alpha_t^\Lambda(a) - \alpha_t(a)\big)\| + \|(\alpha_s^\Lambda - \alpha_s)(\alpha_t(a))\| \\ \leq \|\alpha_t^\Lambda - \alpha_t\| + \|\alpha_s^\Lambda - \alpha_s\|, \tag{5.13}$$

which goes to 0. This allows to extend α_t to the whole real line; the group property is indeed satisfied. Finally, if $z = t + \mathrm{i}\beta$ with $|\beta| < \frac{r}{2\|\Phi\|_r}$, we have

$$\alpha_{t+\mathrm{i}\beta}^\Lambda(a) = \alpha_t^\Lambda\big(\alpha_{\mathrm{i}\beta}^\Lambda(a)\big) \to \alpha_t\big(\alpha_{\mathrm{i}\beta}(a)\big). \tag{5.14}$$

This allows to define $\alpha_{t+\mathrm{i}\beta} = \alpha_t \circ \alpha_{\mathrm{i}\beta}$. □

2. KMS states

A "state" is a bounded, positive, normalised linear functional on $\mathcal{A}$. That is, it satisfies

$$\rho(\mathbb{1}) = 1, \\ \rho(a^* a) \geq 0, \tag{5.15}$$

for all $a \in \mathcal{A}$. Notice that $\rho(a) \in \mathbb{R}$ when a is hermitian, and $\rho(a^*) = \overline{\rho(a)}$ in general, since any operator can be written as the sum of hermitian and anti-hermitian operators, $a = \frac{1}{2}(a + a^*) + \frac{1}{2}(a - a^*)$.

Lemma 5.3. *States are bounded linear functionals with norm 1.*

Proof. Since $\rho(\mathbb{1}) = 1$, it is clear that $\|\rho\| \geq 1$. We show that $|\rho(a)| \leq 1$ for all $\|a\| = 1$. If $a = a^*$, we have $1 \pm a \geq 0$ and $\rho(1 \pm a) = 1 \pm \rho(a) \geq 0$, so that $|\rho(a)| \leq 1$.

For general a, we have

$$\rho\big((a^* - \overline{\rho(a)})(a - \rho(a))\big) \geq 0 \tag{5.16}$$

and linearity implies that $|\rho(a)|^2 \leq \rho(a^* a) \leq 1$. □

For finite volumes, equilibrium states of quantum statistical mechanics are given by the Gibbs states

$$\rho_\Lambda(a) = \frac{1}{\operatorname{Tr} \mathrm{e}^{-\beta H_\Lambda}} \operatorname{Tr} a \, \mathrm{e}^{-\beta H_\Lambda} \,, \tag{5.17}$$

where the parameter β represents the inverse temperature of the system. Using a compactness argument, one obtains the existence of cluster points in the infinite-volume limit. But this limit is rather delicate.An alternative approach is to seek a property that directly deals with states on $\mathcal{A}$ and that characterises equilibrium. This is the motivation for the *KMS condition*, named after Kubo, Martin, and Schwinger. It is not straightforward and it requires a discussion.

The starting point is the following identity for finite volumes, that follows from cyclicity of the trace:

$$\begin{aligned} \rho_\Lambda\big(ab\big) &= \frac{1}{\operatorname{Tr} \mathrm{e}^{-\beta H_\Lambda}} \operatorname{Tr} ab \, \mathrm{e}^{-\beta H_\Lambda} \\ &= \frac{1}{\operatorname{Tr} \mathrm{e}^{-\beta H_\Lambda}} \operatorname{Tr} b \, \mathrm{e}^{\mathrm{i}(\mathrm{i}\beta) H_\Lambda} \, a \, \mathrm{e}^{-\mathrm{i}(\mathrm{i}\beta) H_\Lambda} \; \mathrm{e}^{-\beta H_\Lambda} \\ &= \rho_\Lambda\big(b \, \alpha^\Lambda_{\mathrm{i}\beta}(a)\big). \end{aligned} \tag{5.18}$$

When the infinite volume limit of $\alpha^\Lambda_{\mathrm{i}\beta}$ exists, which the case for β small, we get the following rather simple condition.

Definition 5.2. *Assume that the inverse temperature satisfies* $0 \leq \beta \leq \frac{r}{2\|\Phi\|_r}$. *A state* ρ *satisfies the KMS condition if*

$$\rho\big(ab\big) = \rho\big(b \, \alpha_{\mathrm{i}\beta}(a)\big), \tag{5.19}$$

for all a, b *in* $\mathcal{A}_0$.

Condensed matter physics is much more interesting at low temperatures, and it is worth extending this definition to large β. A key ingredient is the Three-Line Lemma of complex analysis; recall that if $f(z)$ is analytic in the strip $0 \leq \operatorname{Im} z \leq \beta$, the lemma states that for all $0 \leq c \leq 1$, we have

$$\sup_{t \in \mathbb{R}} |f(t + \mathrm{i}c\beta)| \leq \big(\sup_{t \in \mathbb{R}} |f(t)|\big)^{1-c} \big(\sup_{t \in \mathbb{R}} |f(t + \mathrm{i}\beta)|\big)^c. \tag{5.20}$$

For finite volume, we have

$$\rho_\Lambda\big(b \, \alpha^\Lambda_{t+\mathrm{i}\beta}(a)\big) = \rho_\Lambda\big(\alpha^\Lambda_t(a) \, b\big). \tag{5.21}$$

Since $\|\alpha_t^\Lambda\| = 1$ and $\|\rho\| = 1$, we have

$$\big|\rho_\Lambda\big(b\alpha_t^\Lambda(a)\big)\big|,\ \big|\rho_\Lambda\big(b\alpha_{t+\mathrm{i}\beta}^\Lambda(a)\big)\big| \leq \|a\|\,\|b\|. \tag{5.22}$$

It follows from the Three-Line Lemma that $|\rho_\Lambda(\alpha_z^\Lambda(a)\,b)| \leq \|a\|\,\|b\|$ for z in the strip $0 \leq \operatorname{Im} z \leq \beta$. This holds uniformly in Λ. Further, the sequence $\rho_\Lambda(\alpha_z^\Lambda(a)\,b)$ converges uniformly as $\Lambda \nearrow \mathbb{Z}^d$, so the limit is analytic by Weierstrass convergence theorem. These properties suggest the following generalisation of the KMS condition

Definition 5.3. *The state ρ satisfies the KMS condition at inverse temperature β if for any $a, b \in \mathcal{A}$, there exists an analytic function $f(z)$ in the strip $0 \leq \operatorname{Im} z \leq \beta$ such that*

$$f(t) = \rho(b\,\alpha_t(a)) \quad \textit{and} \quad f(t+\mathrm{i}\beta) = \rho(\alpha_t(a)\,b). \tag{5.23}$$

If KMS states represent equilibrium, they should be invariant under time evolution. This is easy to check in finite volume Gibbs states, but it remains true in general.

Proposition 5.4. *If ρ is a KMS state, we have $\rho(\alpha_t(a)) = \rho(a)$ for all $a \in \mathcal{A}$ and all $t \in \mathbb{R}$.*

Proof. Choose $b = \mathbb{1}$ in the KMS definition. Then $f(t) = f(t+\mathrm{i}\beta)$, so $f(z)$ can be extended to an analytic function in the whole of $\mathbb{C}$. It is periodic in the imaginary time direction with period β. Since it is bounded in the strip $0 \leq \operatorname{Im} z \leq \beta$, it is bounded everywhere, and Liouville theorem implies that it is constant. □

3. Uniqueness theorem

Theorem 5.5. *Assume that*

$$\beta\,\|\Phi\|_{N+1} < (2N)^{-1}.$$

Then there exists a unique KMS state at inverse temperature β.

We actually prove the theorem under the more general condition that there exists $s < 1/N$ such that $2\beta\|\Phi\|_{N(1+s)} < s$. As mentioned above, the strategy of our proof is to reformulate the KMS condition as an equation for the equilibrium state that has a unique solution when β is small enough. In order to derive this equation, we express observables as commutators of operators. The proof of Theorem 5.5 will be given after the one of Lemma 5.6, which we state next.

Here and in the sequel, $\|\cdot\|_{\mathrm{HS}}$ denotes the normalized Hilbert-Schmidt norm

$$\|A\|_{\mathrm{HS}}^2 = \frac{1}{\dim \mathcal{H}_\Lambda} \operatorname{Tr} A^* A. \tag{5.24}$$

Notice that

$$\frac{1}{\sqrt{\dim \mathcal{H}_\Lambda}} \|A\| \leq \|A\|_{\mathrm{HS}} \leq \|A\| \tag{5.25}$$

for all $A \in \mathcal{A}_\Lambda$.

Lemma 5.6. *Let A be a hermitian $N \times N$ matrix with the property that $\operatorname{Tr} A = 0$. Then there exist hermitian $N \times N$ matrices $B_1, \dots, B_{N-1}$ and $C_1, \dots, C_{N-1}$ such that*

$$A = \sum_{i=1}^{N-1} [B_i, C_i],$$
$$\sum_{i=1}^{N-1} \|B_i\|_{\rm HS} \, \|C_i\|_{\rm HS} \leq \sqrt{N} \, \|A\|_{\rm HS}.$$

Proof. Let $a_1, \dots, a_N$ be the eigenvalues of A (repeated according to their multiplicity). We have that

$$\sum_{i=1}^N a_i = 0, \qquad \sum_{i=1}^N |a_i|^2 = N \, \|A\|_{\rm HS}^2. \tag{5.26}$$

In particular, each $|a_i|$ is bounded above by $\sqrt{N}\|A\|_{\rm HS}$. Let us order the eigenvalues so that

$$\Bigl| \sum_{i=1}^k a_i \Bigr| \leq \sqrt{N} \, \|A\|_{\rm HS} \tag{5.27}$$

for all $1 \leq k \leq N-1$. This is indeed possible, as can be seen by induction using $\sum a_i = 0$: If $0 \leq \sum^k a_i \leq \sqrt{N}\|A\|_{\rm HS}$, we can find $a_{k+1} \leq 0$ among the remaining eigenvalues such that $|\sum^{k+1} a_i| \leq \sqrt{N}\|A\|_{\rm HS}$. And if the partial sum is negative, we can find $a_{k+1} \geq 0$ among the remaining eigenvalues, with the same conclusion.

We work in a basis such that A is diagonal and its eigenvalues are ordered so they satisfy the properties above. Let $\tilde a_k = \sum_{i=1}^k a_i$, and let $\sigma^1_{j,j+1}, \sigma^2_{j,j+1}, \sigma^3_{j,j+1}$ be $N \times N$ matrices that are equal to Pauli matrices on the 2×2 block that contains (j,j) and $(j+1,j+1)$, and that are equal to zero everywhere else. It is not hard to check that

$$A = \sum_{j=1}^{N-1} \tilde a_j \, \sigma^3_{j,j+1}. \tag{5.28}$$

We therefore have that

$$A = \tfrac12 \sum_{j=1}^{N-1} \tilde a_j \, [\sigma^1_{j,j+1}, \sigma^2_{j,j+1}], \tag{5.29}$$

which proves the first claim. The bound follows from $|\tilde a_j| \leq \sqrt{N}\|A\|_{\rm HS}$ and $\|\sigma^i_{j,j+1}\|^2_{\rm HS} = 2/N$. □

Proof of Theorem 5.5. Let $(e_i)_{i=0}^{N^2-1}$ be a hermitian basis of $\mathcal{M}_N(\mathbb{C})$, with $e_0 = \mathbb{1}$, $\operatorname{Tr} e_i = 0$ if $1 \neq 0$, and $\|e_i\| = 1$, for all i. Let J be the set of multi-indices $j = (j_x)_{x \in \mathbb{Z}^d}$, $0 \leq j_x \leq N^2 - 1$, with finite support

$$\operatorname{supp} j = \{x \in \mathbb{Z}^d | j_x \neq 0\}. \tag{5.30}$$

Given $j \in J$, let $e_j = \otimes_{x \in \operatorname{supp} j} e_{j_x} \in \mathcal{A}_{\operatorname{supp} j}$. The linear span of $\{e_j\}_{j \in J}$ is dense in $\mathcal{A}$.

Let tr denote the normalized trace on $\mathcal{A}$; it is equal to $\frac{1}{\dim \mathcal{H}_\Lambda} \operatorname{Tr}$ on $\mathcal{A}_\Lambda$ and it can be extended to $\mathcal{A}$ by continuity. The state ρ can be written as $\rho = \operatorname{tr} + \varepsilon$ where $\varepsilon(\mathbb{1}) = 0$. We actually have that

$$\varepsilon(e_j) = \begin{cases} \rho(e_j) & \text{if } j \not\equiv 0, \\ 0 & \text{if } j \equiv 0. \end{cases} \tag{5.31}$$

Using Lemma 5.6, we have that

$$e_j = \frac{1}{|\operatorname{supp} j|} \sum_{y \in \operatorname{supp} j} \sum_{i=1}^{N-1} \big[\otimes_{x \neq y} e_{j_x} \otimes b_i^{(j_y)}, \otimes_{x \neq y} \mathbb{1} \otimes c_i^{(j_y)} \big], \tag{5.32}$$

for $j \not\equiv 0$. Here, $b_i^{(k)}, c_i^{(k)}$ are the matrices B_i, C_i of Lemma 5.6 in the case where the matrix A is e_k.

We now use this decomposition and the KMS condition, Definition 5.2, in order to get an equation for ε. For $j \not\equiv 0$,

$$\begin{aligned} \varepsilon(e_j) &= \frac{1}{|\operatorname{supp} j|} \sum_{y \in \operatorname{supp} j} \sum_{i=1}^{N-1} \rho\Big(\big[\otimes_{x \neq y} e_{j_x} \otimes b_i^{(j_y)}, \otimes_{x \neq y} \mathbb{1} \otimes c_i^{(j_y)} \big] \Big) \\ &= \frac{1}{|\operatorname{supp} j|} \sum_{y \in \operatorname{supp} j} \sum_{i=1}^{N-1} \rho\Big(\otimes_{x \neq y} e_{j_x} \otimes b_i^{(j_y)} \cdot (\mathbb{1} - \alpha_{\mathrm{i}\beta}) \otimes_{x \neq y} \mathbb{1} \otimes c_i^{(j_y)} \Big) \\ &= \delta(e_j) + K_\beta \varepsilon(e_j). \end{aligned} \tag{5.33}$$

In the above equation, we set

$$\delta(e_j) = \frac{1}{|\operatorname{supp} j|} \sum_{y \in \operatorname{supp} j} \sum_{i=1}^{N-1} \operatorname{tr} \Big(\otimes_{x \neq y} e_{j_x} \otimes b_i^{(j_y)} \cdot (\mathbb{1} - \alpha_{\mathrm{i}\beta}) \otimes_{x \neq y} \mathbb{1} \otimes c_i^{(j_y)} \Big), \tag{5.34}$$

and the operator K_β is defined by

$$(K_\beta \phi)(e_j) = \frac{1}{|\operatorname{supp} j|} \sum_{y \in \operatorname{supp} j} \sum_{i=1}^{N-1} \phi\Big(\otimes_{x \neq y} e_{j_x} \otimes b_i^{(j_y)} \cdot (\mathbb{1} - \alpha_{\mathrm{i}\beta}) \otimes_{x \neq y} \mathbb{1} \otimes c_i^{(j_y)} \Big). \tag{5.35}$$

Notice that K_β is a linear operator on the Banach space $\mathcal{L}(\mathcal{A})$ of linear functionals on $\mathcal{A}$. Equation (5.33) can be written as

$$(\mathbb{1} - K_\beta)\varepsilon = \delta. \tag{5.36}$$

Let us introduce the following norm on $\mathcal{L}(\mathcal{A})$:

$$|||\phi||| = \sup_{j \in J} |\phi(e_j)|. \tag{5.37}$$

Because $\|e_j\| = 1$ for all j, we have $|||\phi||| \leq \|\phi\|$ and $(\mathcal{L}(\mathcal{A}), |||\cdot|||)$ is a normed vector space. We consider K_β as an operator on $(\mathcal{L}(\mathcal{A}), |||\cdot|||)$ and we show that its norm is strictly less than 1; the solution of (5.36) is then unique. The norm of K_β is equal to

$$\|K_\beta\| = \sup_{\|\phi\|=1} \sup_{j\in J} |K_\beta \phi(e_j)|. \tag{5.38}$$

Recall that $\alpha_{\mathrm{i}\beta} = \lim_\Lambda \alpha^\Lambda_{\mathrm{i}\beta}$ (with convergence in the operator norm) and that $\alpha^\Lambda_{\mathrm{i}\beta}(A)$, $A \in \mathcal{A}$, has an expansion in multiple commutators. From (5.35), we get

$$\begin{aligned}\big|K_\beta\phi(e_j)\big| \leq \frac{1}{|\operatorname{supp} j|} \sum_{y\in\operatorname{supp} j} \sum_{i=1}^{N-1} \sum_{n\geq 1} \frac{\beta^n}{n!} \sup_{\Lambda\subset\mathbb{Z}^d} \sum_{X_1,\dots,X_n\subset\Lambda} \\ \Big|\phi\Big(\otimes_{x\neq y} e_{j_x} \otimes b_i^{(j_y)} \big[\Phi_{X_n}, \dots, \big[\Phi_{X_1}, \otimes_{x\neq y}\mathbb{1}\otimes c_i^{(j_y)}\big]\cdots\big]\Big)\Big|.\end{aligned} \tag{5.39}$$

Because of the commutators, the sum over the X_k's is restricted to subsets that satisfy the following constraints, as in (5.8):

$$\begin{aligned} &X_1 \ni y, \\ &X_2 \cap X_1 \neq \emptyset, \\ &\quad\vdots \\ &X_n \cap (X_1 \cup \dots \cup X_{n-1}) \neq \emptyset. \end{aligned} \tag{5.40}$$

Let $A = \sum_{(j'_x)_{x\in X}} a_{j'} e_{j'}$ be an operator in $\mathcal{A}_X$. For any $(j_x)_{x\notin X}$, we have

$$\begin{aligned}\big|\phi\big(\otimes_{x\notin X} e_{j_x} \otimes A\big)\big| &= \Big| \sum_{(j'_x)_{x\in X}} a_{j'}\ \phi\big(\otimes_{x\notin X} e_{j_x} \otimes_{x\in X} e_{j'_x}\big)\Big| \\ &\leq \|\phi\| \sum_{(j'_x)_{x\in X}} |a_{j'}| \\ &\leq \|\phi\| \|A\|_{\mathrm{HS}} N^{|X|}.\end{aligned} \tag{5.41}$$

Using Eq. (5.41) with $\|\phi\| = 1$, $\|AB\|_{\mathrm{HS}} \leq \|A\|\,\|B\|_{\mathrm{HS}}$, and $\|c_i^{(j_y)}\| \leq \sqrt{N}\|c_i^{(j_y)}\|_{\mathrm{HS}}$, we get

$$\begin{aligned}\big|K_\beta\phi(e_j)\big| &\leq \sqrt{N} \sup_{y\in\mathbb{Z}^d} \sum_{n\geq 1} \frac{(2\beta)^n}{n!} \sum_{X_1,\dots,X_n:y} \Big(\prod_{k=1}^n \|\Phi_{X_k}\| N^{|X_k|}\Big) \sum_{i=1}^{N-1} \|b_i^{(j_y)}\|_{\mathrm{HS}} \|c_i^{(j_y)}\|_{\mathrm{HS}} \\ &\leq N \sup_{y\in\mathbb{Z}^d} \sum_{n\geq 1} \frac{(2\beta)^n}{n!} \sum_{X_1,\dots,X_n:y} \prod_{k=1}^n \|\Phi_{X_k}\| N^{|X_k|}.\end{aligned} \tag{5.42}$$

We have used Lemma 5.6 to get the last line. The constraint $X_1, \dots, X_n : y$ means that (5.40) must be respected. The final step is to estimate the sum over such subsets. This can be conveniently done with an inductive argument. Namely, let $R_0 = 0$ and, for $m \geq 1$, let

$$R_m = \sup_{y\in\mathbb{Z}^d} \sum_{n=1}^m \frac{(2\beta)^n}{n!} \sum_{X_1,\dots,X_n:y} \prod_{k=1}^n \|\Phi_{X_k}\| N^{|X_k|}. \tag{5.43}$$

Summing first over $X_1 \ni y$, then over sets that intersect sites of X_1, we get

$$\begin{aligned} R_m &\leq 2\beta \sup_y \sum_{X_1 \ni y} \|\Phi_{X_1}\| N^{|X_1|} \prod_{x \in X_1} \Big(\sum_{n=1}^{m} \frac{(2\beta)^{n-1}}{(n-1)!} \sum_{X_2,\dots,X_n : x} \prod_{k=2}^{n} \|\Phi_{X_k}\| N^{|X_k|} \Big) \\ &\leq 2\beta \sup_y \sum_{X_1 \ni y} \|\Phi_{X_1}\| N^{|X_1|} (1 + R_{m-1})^{|X_1|}. \end{aligned} \tag{5.44}$$

It follows easily that $R_m \leq r$ for all m, and all r such that $2\beta \|\Phi\|_{N(1+r)} \leq r$. Then $\|K_\beta\| \leq Nr$, and the assumption of Theorem 5.5 implies the existence of r such that $Nr < 1$. □

4. Lieb-Robinson bounds

Even for small times, the image of a local observable under the evolution map is no longer local; that is, $\alpha_t(a) \notin \mathcal{A}_0$ for all $a \in \mathcal{A}_0$. This follows from the expansion. But one should expect that the evolved observable remains "essentially local". A precise formulation is provided by Lieb-Robinson bounds. Here, we define a new norm on interactions, namely

$$\|\Phi\|_c = \sup_{x \in \Lambda} \sum_{X \ni x} \|\Phi_X\| \, |X| \, \mathrm{e}^{c \, \mathrm{diam} X} . \tag{5.45}$$

The diameter of the finite set X is equal to $\mathrm{diam} X = \max_{x,y \in X} d(x,y)$, where $d(x,y)$ is the graph distance between the sites x and y.

Theorem 5.7. *Let Λ be a finite graph, and let $a \in \mathcal{A}_X$, and $b \in \mathcal{A}_Y$. Then for every $t \in \mathbb{R}$ and every $c > 0$, we have*

$$\big\| [\alpha_t^\Lambda(a), b] \big\| \leq \|a\| \, \|b\| \, |X| \, \mathrm{e}^{2\|\Phi\|_c |t| - c \, d(X,Y)} .$$

In order to prove this theorem, we need the following lemma.

Lemma 5.8. *Let $b : \mathbb{R} \to \mathcal{M}_N$ and $h : \mathbb{R} \times \mathcal{M}_N \to \mathcal{M}_N$ be continuous functions. We assume that $h(t,a)$ is hermitian for all t and all $a \in \mathcal{M}_N$.*

(i) *Let $\gamma_t(a_0)$ be the solution of the equation $\frac{\mathrm{d}}{\mathrm{d}t} a(t) = \mathrm{i}[h(t, a(t)), a(t)]$, $a(0) = a_0$. Then*

$$\|\gamma_t(a_0)\| = \|a_0\|$$

for all $a_0 \in \mathcal{M}_N$.

(ii) *The solution of the equation $\frac{\mathrm{d}}{\mathrm{d}t} a(t) = \mathrm{i}[h(t, a(t)), a(t)] + b(t)$, $a(0) = a_0$, is*

$$a(t) = \gamma_t \Big(a_0 + \int_0^t \gamma_{-s}\big(b(s)\big) \mathrm{d}s \Big).$$

(iii) *If $a(t)$ is the above solution, we have the bound*

$$\|a(t) - \gamma_t(a_0)\| \leq \int_0^t \|b(s)\| \mathrm{d}s.$$

PROOF. We start with (i); we have

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\|a(t)\| &= \lim_{\varepsilon\to0}\frac{1}{\varepsilon}\big(\|a(t+\varepsilon)\| - \|a(t)\|\big)\\ &= \lim_{\varepsilon\to0}\frac{1}{\varepsilon}\Big(\big\|a(t)+\varepsilon\mathrm{i}[h(t,a(t)),a(t)]\big\| - \|a(t)\|\Big)\\ &= \lim_{\varepsilon\to0}\frac{1}{\varepsilon}\Big(\big\|\,\mathrm{e}^{\mathrm{i}\varepsilon h(t,a(t))}\,a(t)\,\mathrm{e}^{-\mathrm{i}\varepsilon h(t,a(t))}\,\big\| - \|a(t)\|\Big)\\ &= 0.\end{aligned} \tag{5.46}$$

The claim (ii) is straightforwardly verified. We have

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}a(t) &= \lim_{\varepsilon\to0}\frac{1}{\varepsilon}\Big\{\gamma_{t+\varepsilon}\Big(a_0+\int_0^t\gamma_{-s}\big(b(s)\big)\mathrm{d}s\Big)+\gamma_{t+\varepsilon}\Big(\int_t^{t+\varepsilon}\gamma_{-s}\big(b(s)\big)\mathrm{d}s\Big)-a(t)\Big\}\\ &= \lim_{\varepsilon\to0}\frac{1}{\varepsilon}\Big\{a(t)+\varepsilon\mathrm{i}\big[h(t,a(t)),a(t)\big]+\varepsilon\gamma_t\big(\gamma_{-t}\big(b(t)\big)\big)-a(t)\Big\}\\ &= \mathrm{i}\big[h(t,a(t)),a(t)\big]+b(t).\end{aligned} \tag{5.47}$$

Finally, the claim (iii) follows from (i) and (ii):

$$\begin{aligned}\|a(t)-\gamma_t(a_0)\| &= \Big\|\gamma_t\Big(\int_0^t\gamma_{-s}\big(b(s)\big)\mathrm{d}s\Big)\Big\|\\ &\leq \int_0^t\big\|\gamma_{-s}\big(b(s)\big)\big\|\mathrm{d}s\\ &= \int_0^t\|b(s)\|\mathrm{d}s.\end{aligned} \tag{5.48}$$

□

PROOF OF THEOREM 5.7. Using Jacobi's identity (see Exercise 5.3), we have

$$\frac{\mathrm{d}}{\mathrm{d}t}[\alpha_t^\Lambda(a),b] = \mathrm{i}\Big[\sum_{Z\cap X\neq\emptyset}\alpha_t^\Lambda(\Phi_Z),[\alpha_t^\Lambda(a),b]\Big]-\mathrm{i}\sum_{Z\cap X\neq\emptyset}\big[\alpha_t^\Lambda(a),[\alpha_t^\Lambda(\Phi_Z),b]\big]. \tag{5.49}$$

Here and in the sequel, X denotes the support of a and Y denotes the support of b. By Lemma 5.8, we obtain

$$\big\|[\alpha_t^\Lambda(a),b]\big\| \leq \big\|[a,b]\big\|+2\|a\|\sum_{Z\cap X\neq\emptyset}\int_0^{|t|}\big\|[\alpha_s^\Lambda(\Phi_Z),b]\big\|\mathrm{d}s. \tag{5.50}$$

Let us introduce

$$g(t) = \sup_{a\in\mathcal{A}_0}\frac{\|[\alpha_t^\Lambda(a),b]\|}{\|a\|\,\|b\|\,|X|}\,\mathrm{e}^{c\,d(X,Y)}\,. \tag{5.51}$$

We can insert $g(s)$ in the right side of (5.50) and we get

$$\frac{\|[\alpha_t^\Lambda(a),b]\|}{\|a\|\,\|b\|\,|X|}\,\mathrm{e}^{c\,d(X,Y)} \leq \underbrace{\frac{\|[a,b]\|}{\|a\|\,\|b\|\,|X|}\,\mathrm{e}^{c\,d(X,Y)}}_{\leq 2 \text{ since } [a,b]=0 \text{ when } d(X,Y)\neq 0} + \frac{2}{|X|}\sum_{Z\cap X\neq\emptyset}\|\Phi_Z\|\,|Z|\,\mathrm{e}^{c\,d(X,Y)-c\,d(Z,Y)}\int_0^{|t|} g(s)\mathrm{d}s. \tag{5.52}$$

We now use $d(X,Y)-d(Z,Y)\leq \operatorname{diam} Z$ and we take the supremum over $a\in\mathcal{A}_0$ in the left side. From the definition (5.45) of the norm of Φ, we obtain

$$g(t)\leq 2+2\|\Phi\|_c\int_0^{|t|} g(s)\mathrm{d}s. \tag{5.53}$$

Let h be the solution of the integral equation $h(t)=2+2\|\Phi\|_c\int_0^t h(s)\mathrm{d}s$, $h(0)=2$. One easily finds

$$h(t)=2\,\mathrm{e}^{2\|\Phi\|_c t}. \tag{5.54}$$

Then $g(t)\leq h(|t|)$, which gives the Lieb-Robinson bound. □

EXERCISE 5.1. *Use a diagonal argument to prove the existence of infinite-volume KMS states.*

EXERCISE 5.2. *KMS states and symmetries. Assume that there exists a hermitian operator A on $\mathbb{C}^N$ such that for all $\Lambda \subset\subset \mathbb{Z}^d$,*

$$\big[H_\Lambda,\sum_{x\in\Lambda}A_x\big]=0.$$

Here, $A_x = A\otimes \mathbb{1}_{\Lambda\setminus\{x\}}$. Let ρ be a KMS state; for $\theta\in\mathbb{R}$, let $U_\Lambda=\mathrm{e}^{\mathrm{i}\theta\sum_{x\in\Lambda}A_x}$ and define

$$\tilde\rho(a)=\rho(U_\Lambda a U_\Lambda^*),$$

for $a\in\mathcal{A}_\Lambda$. Show that $\tilde\rho$ is also a KMS state.

EXERCISE 5.3. *Check Jacobi's identity for the commutators of matrices:*

$$\big[[a,b],c\big]=\big[a,[b,c]\big]-\big[b,[a,c]\big].$$