CHAPTER 3

Mathematical setting

1. Tensor products (of Hilbert spaces)

Let $\mathcal{H}$ be a separable Hilbert space and $\{e_i\}_{i\geq 1}$ be a finite or countable orthonormal basis. Recall that span$\{e_i\}$ denotes the space of finite linear combinations of $\{e_i\}$, and that its completion is isomorphic to $\mathcal{H}$.

Definition 3.1 (Tensor product). *Let $\mathcal{H}_1, \mathcal{H}_2$ be two Hilbert spaces with respective bases $\{e_i^1\}, \{e_i^2\}$. The* **tensor product** *of $\mathcal{H}_1$ and $\mathcal{H}_2$, denoted $\mathcal{H}_1 \otimes \mathcal{H}_2$, is the completion of the linear span of $\{(e_i^1, e_j^2)\}_{i,j\geq 1}$.*

The dimension of the tensor product space satisfies

$$\dim \mathcal{H}_1 \otimes \mathcal{H}_2 = \dim \mathcal{H}_1 \cdot \dim \mathcal{H}_2. \tag{3.1}$$

Given two vectors $\varphi_1 \in \mathcal{H}_1$ and $\varphi_2 \in \mathcal{H}_2$, we can construct the element $\varphi_1 \otimes \varphi_2$ as follows. Let $\sum_i a_i e_i^1$ and $\sum_j b_j e_j^2$ be the decompositions of φ_1, φ_2 in the bases $\{e_i^1\}$ and $\{e_j^2$, respectively. Then

$$\varphi_1 \otimes \varphi_2 = \sum_{i,j\geq 1} a_i b_j e_i^1 \otimes e_j^2. \tag{3.2}$$

Notice that $\varphi_1 \otimes 2\varphi_2 = 2\varphi_1 \otimes \varphi_2 = 2(\varphi_1 \otimes \varphi_2)$. Not all elements of $\mathcal{H}_1 \otimes \mathcal{H}_2$ can be written as tensor product vectors (see Exercise 3.3).

The inner product on $\mathcal{H}_1 \otimes \mathcal{H}_2$ is

$$\langle \varphi_1 \otimes \varphi_2, \psi_1 \otimes \psi_2 \rangle = \langle \varphi_1, \psi_1 \rangle \cdot \langle \varphi_2, \psi_2 \rangle, \tag{3.3}$$

where the inner products in the right side are in $\mathcal{H}_1$ and $\mathcal{H}_2$, respectively. This extends to general elements of $\mathcal{H}_1 \otimes \mathcal{H}_2$ by linearity.

Let $A_1 \in \mathcal{B}(\mathcal{H}_1)$ and $A_2 \in \mathcal{B}(\mathcal{H}_2)$ be two bounded operators. The tensor product operator $A_1 \otimes A_2$ is an operator acting on $\mathcal{H}_1 \otimes \mathcal{H}_2$ and its action on tensor product vectors is

$$(A_1 \otimes A_2)(\varphi_1 \otimes \varphi_2) = A_1\varphi_1 \otimes A_2\varphi_2. \tag{3.4}$$

Its action on general vectors is obtained by linearity.

This construction is easily generalised to more than two Hilbert spaces. The tensor product space $\mathcal{H}_1 \otimes \cdots \otimes \mathcal{H}_n$ is the completion of the linear span of $\{(e_{j_1}^1, \dots, e_{j_n}^n)\}_{j_1,\dots,j_n\geq 1}$, where $\{e_j^i\}_{j\geq 1}$ is an orthonormal basis of $\mathcal{H}_i$.

2. Direct sums

DEFINITION 3.2. *The* **direct sum** *of two Hilbert spaces* $\mathcal{H}_1$ *and* $\mathcal{H}_2$*, denoted* $\mathcal{H}_1 \oplus \mathcal{H}_2$*, is the space of pairs* (φ_1, φ_2) *with* $\varphi_1 \in \mathcal{H}_1$*,* $\varphi_2 \in \mathcal{H}_2$*, with operations*

$$\alpha(\varphi_1, \varphi_2) + \beta(\psi_1, \psi_2) = (\alpha\varphi_1 + \beta\psi_1, \alpha\varphi_2 + \beta\psi_2), \qquad \alpha, \beta \in \mathbb{C}.$$

If $\{e_i^1\}$ and $\{e_j^2\}$ are bases of $\mathcal{H}_1$ and $\mathcal{H}_2$, then $\{(e_i^1, 0)\} \cup \{(0, e_j^2)\}$ is a basis of $\mathcal{H}_1 \oplus \mathcal{H}_2$. It follows that

$$\dim \mathcal{H}_1 \oplus \mathcal{H}_2 = \dim \mathcal{H}_1 + \dim \mathcal{H}_2. \tag{3.5}$$

3. Spin operators

Let $S \in \frac{1}{2}\mathbb{N}$. On $\mathbb{C}^{2S+1}$, let S^1, S^2, S^3 be hermitian matrices that satisfy the following properties:

$$[S^1, S^2] = \mathrm{i}S^3, \quad [S^2, S^3] = \mathrm{i}S^1, \quad [S^3, S^1] = \mathrm{i}S^2, \tag{3.6}$$

$$[S^1]^2 + [S^2]^2 + [S^3]^2 = S(S+1)\mathtt{Id}. \tag{3.7}$$

The existence of such matrices follows by construction: Let $|a\rangle$, $a \in \{-S, -S+1, \dots, S\}$ denote an orthonormal basis of $\mathbb{C}^{2S+1}$, and define $S^3|a\rangle = a|a\rangle$. Next, let S^+, S^- be defined by

$$S^+|a\rangle = \sqrt{S(S+1) - a(a+1)}\;|a+1\rangle, \quad S^-|a\rangle = \sqrt{S(S+1) - (a-1)a}\;|a-1\rangle. \tag{3.8}$$

Then we set $S^1 = \frac{1}{2}(S^+ + S^-)$ and $S^2 = \frac{1}{2\mathrm{i}}(S^+ - S^-)$.

LEMMA 3.1. *The operators* S^1, S^2, S^3 *constructed above satisfy the relations (3.6) and (3.7).*

PROOF. One can check the following commutation relations:

$$[S^3, S^+] = S^+, \qquad [S^3, S^-] = -S^-, \qquad [S^+, S^-] = 2S^3. \tag{3.9}$$

The relations (3.6) follow. Finally,

$$[S^1]^2 + [S^2]^2 + [S^3]^2 = S^+S^- + [S^3]^2 - S^3 = S(S+1)\mathtt{Id}. \tag{3.10}$$

□

For $S = \frac{1}{2}$, the choice above gives the Pauli matrices

$$S^1 = \tfrac{1}{2}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \qquad S^2 = \tfrac{1}{2}\begin{pmatrix} 0 & -\mathrm{i} \\ \mathrm{i} & 0 \end{pmatrix}, \qquad S^3 = \tfrac{1}{2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \tag{3.11}$$

For $S = 1$, we get

$$S^1 = \frac{1}{\sqrt{2}}\begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}, \qquad S^2 = \frac{1}{\sqrt{2}}\begin{pmatrix} 0 & -\mathrm{i} & 0 \\ \mathrm{i} & 0 & -\mathrm{i} \\ 0 & \mathrm{i} & 0 \end{pmatrix}, \qquad S^3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix}. \tag{3.12}$$

Notice that, for $S > 1$, the matrix of S^1 is not proportional to $\delta_{|i-j|,1}$. Spin operators are not unique, but their spectrum is uniquely determined by the commutation relations.

Lemma 3.2. *Assume that S^1, S^2, S^3 are hermitian matrices in $\mathbb{C}^{2S+1}$ that satisfy the relations (3.6) and (3.7). Then each S^i has eigenvalues $\{-S, -S+1, \dots, S\}$.*

Proof. It is enough to prove the claim for S^3. Define $S^+ = S^1 + \mathrm{i}S^2$ and $S^- = S^1 - \mathrm{i}S^2$. One can check that

$$\begin{aligned} S^+S^- &= S(S+1)\mathtt{Id} - [S^3]^2 + S^3, \\ S^-S^+ &= S(S+1)\mathtt{Id} - [S^3]^2 - S^3. \end{aligned} \tag{3.13}$$

Let $|a\rangle$ be an eigenvector of S^3 with eigenvalue a. It follows from Eq. (3.13) that

$$\begin{aligned} \big\|S^+|a\rangle\big\|^2 &= \langle a|S^-S^+|a\rangle = S(S+1) - a^2 - a \geq 0, \\ \big\|S^-|a\rangle\big\|^2 &= \langle a|S^+S^-|a\rangle = S(S+1) - a^2 + a \geq 0. \end{aligned} \tag{3.14}$$

Then $|a| \leq S$, and $S^+|a\rangle \neq 0$ if $a \neq S$. Next, observe that $[S^3, S^+] = S^+$. Then

$$S^3S^+|a\rangle = (a+1)S^+|a\rangle. \tag{3.15}$$

Then if $a \neq S$ is an eigenvalue, $a+1$ is also an eigenvalue. There are similar relations with S^-, so that if $a \neq -S$ is an eigenvalue, $a-1$ is also an eigenvalue. It follows that $\{-S, -S+1, \dots S\}$ is the set of eigenvalues. □

Notice that the relations (3.8) always hold; this follows from (3.15) and (3.14). It follows from the parallelogram identity that $\|S^\pm\| = \sqrt{2}S$:

$$\begin{aligned} \|S^+\|^2 &= \tfrac{1}{4}(2\|S^+\|^2 + 2\|S^-\|^2) = \tfrac{1}{4}(\|S^+ + S^-\|^2 + \|S^+ - S^-\|^2) \\ &= \tfrac{1}{4}(4\|S^1\|^2 + 4\|S^2\|^2) = 2S^2. \end{aligned} \tag{3.16}$$

Spin operators are related to rotations in $\mathbb{R}^3$. Let $\vec{S} = (S^1, S^2, S^3)$. Given $\vec{a} \in \mathbb{R}^3$, let

$$S^{\vec{a}} = \vec{a} \cdot \vec{S} = a_1S^1 + a_2S^2 + a_3S^3. \tag{3.17}$$

By linearity, the commutation relations (3.6) generalize as

$$[S^{\vec{a}}, S^{\vec{b}}] = \mathrm{i}S^{\vec{a}\times\vec{b}}. \tag{3.18}$$

Finally, let $R_{\vec{a}}\vec{b}$ denote the vector $\vec{b}$ rotated around $\vec{a}$ by the angle $\|\vec{a}\|$.

Lemma 3.3.

$$\mathrm{e}^{-\mathrm{i}S^{\vec{a}}}\, S^{\vec{b}}\, \mathrm{e}^{\mathrm{i}S^{\vec{a}}} = S^{R_{\vec{a}}\vec{b}}.$$

Proof. We replace $\vec{a}$ by $s\vec{a}$, and we check that both sides of the identity satisfy the same differential equation. We find

$$\frac{\mathrm{d}}{\mathrm{d}s}\, \mathrm{e}^{-\mathrm{i}S^{s\vec{a}}}\, S^{\vec{b}}\, \mathrm{e}^{\mathrm{i}S^{s\vec{a}}} = -\mathrm{i}[S^{\vec{a}},\, \mathrm{e}^{-\mathrm{i}S^{s\vec{a}}}\, S^{\vec{b}}\, \mathrm{e}^{\mathrm{i}S^{s\vec{a}}}\,], \tag{3.19}$$

and

$$\frac{\mathrm{d}}{\mathrm{d}s} S^{R_{s\vec{a}}\vec{b}} = \Big(\frac{\mathrm{d}}{\mathrm{d}s} R_{s\vec{a}}\vec{b}\Big) \cdot \vec{S} = \Big(\vec{a} \times R_{s\vec{a}}\vec{b}\Big) \cdot \vec{S} = -\mathrm{i}[S^{\vec{a}}, S^{R_{s\vec{a}}\vec{b}}]. \tag{3.20}$$

We used (3.18) for the last identity. □

It also follows from Lemmas 3.2 and 3.3 that any matrix $S^{\vec{a}}$, $\vec{a} \in \mathbb{R}^3$ with $\|\vec{a}\| = 1$, has eigenvalues $\{-S, -S+1, \dots, S\}$.

Corollary 3.4. *Let $\psi_{\vec{b},c}$ be the eigenvector of $S^{\vec{b}}$ with eigenvalue c. Then $\mathrm{e}^{-\mathrm{i}S^{\vec{a}}}\,\psi_{\vec{b},c}$ is eigenvector of $S^{R_{\vec{a}}\vec{b}}$ with eigenvalue c.*

Proof. Using Lemma 3.3,

$$S^{R_{\vec{a}}\vec{b}}\,\mathrm{e}^{-\mathrm{i}S^{\vec{a}}}\,\psi_{\vec{b},c} = \mathrm{e}^{-\mathrm{i}S^{\vec{a}}}\,S^{\vec{b}}\psi_{\vec{b},c} = c\,\mathrm{e}^{-\mathrm{i}S^{\vec{a}}}\,\psi_{\vec{b},c}. \tag{3.21}$$

□

Finally, let us note the following useful relations:

$$\begin{aligned} \mathrm{e}^{-\mathrm{i}aS^3}\,S^+\,\mathrm{e}^{\mathrm{i}aS^3} &= \mathrm{e}^{-\mathrm{i}a}\,S^+, \\ \mathrm{e}^{-\mathrm{i}aS^3}\,S^-\,\mathrm{e}^{\mathrm{i}aS^3} &= \mathrm{e}^{\mathrm{i}a}\,S^-. \end{aligned} \tag{3.22}$$

Exercise 3.1. *For $S = 1$, check that the following matrices satisfy the spin relations.*

$$S^1 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -\mathrm{i} \\ 0 & \mathrm{i} & 0 \end{pmatrix}, \qquad S^2 = \begin{pmatrix} 0 & 0 & -\mathrm{i} \\ 0 & 0 & 0 \\ \mathrm{i} & 0 & 0 \end{pmatrix}, \qquad S^3 = \begin{pmatrix} 0 & -\mathrm{i} & 0 \\ \mathrm{i} & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

Exercise 3.2. *For $S = 1$, check that the following matrices do not satisfy the spin relations.*

$$S^1 = \tfrac{1}{2}\begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \qquad S^2 = \tfrac{1}{2}\begin{pmatrix} 0 & -\mathrm{i} & 0 \\ \mathrm{i} & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \qquad S^3 = \tfrac{1}{2}\begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

Exercise 3.3. *Show that there exist no $\varphi_1 \in \mathcal{H}_1$, $\varphi_2 \in \mathcal{H}_2$ such that*

$$e_1^1 \otimes e_1^2 + e_2^1 \otimes e_2^2 = \varphi_1 \otimes \varphi_2.$$

Exercise 3.4. *Show that*

(a) $\|\varphi_1 \otimes \varphi_2\| = \|\varphi_1\| \cdot \|\varphi_2\|$ *for all* $\varphi_1 \in \mathcal{H}_1$, $\varphi_2 \in \mathcal{H}_2$.
(b) $\|A_1 \otimes A_2\| = \|A_1\| \cdot \|A_2\|$ *for all* $A_1 \in \mathcal{B}(\mathcal{H}_1)$, $A_2 \in \mathcal{B}(\mathcal{H}_2)$.

Exercise 3.5. *Show that*

$$\underbrace{\mathcal{H} \oplus \cdots \oplus \mathcal{H}}_{n \text{ times}} \simeq \mathcal{H} \otimes \mathbb{C}^n.$$

CHAPTER 4

Models of quantum spins

1. Origin and motivation

The electron is a particle that possesses a mass m, a charge $-e$, and also a spin. In quantum mechanics, the state space for an electron in domain Ω is the Hilbert space $\mathcal{H}_1 = L^2(\Omega) \otimes \mathbb{C}^2$. The description of an atom with Z protons and N electrons turns out to be very complicated (except for $N = 1$). The Hilbert space is the antisymmetric subspace of $\mathcal{H}_1^{\otimes N}$ and the Hamiltonian is the operator

$$H = -\tfrac{\hbar^2}{2m} \sum_{i=1}^N \Delta_i - Ze^2 \sum_{i=1}^N \frac{1}{\|X_i\|} + e^2 \sum_{1 \leq i < j \leq N} \frac{1}{\|X_i - X_j\|}. \tag{4.1}$$

Here, Δ_i is the Laplacian for the ith particle, that is

$$\Delta_i = \Bigl(1\!\!1_{L^2(\Omega)} \otimes 1\!\!1_{\mathbb{C}^2} \Bigr) \otimes \dots \otimes \Bigl(\Delta_{L^2(\Omega)} \otimes 1\!\!1_{\mathbb{C}^2} \Bigr) \otimes \dots \otimes \Bigl(1\!\!1_{L^2(\Omega)} \otimes 1\!\!1_{\mathbb{C}^2} \Bigr). \tag{4.2}$$

The position operator of the ith particle, $X_i = (X_i^1, X_i^2, X_i^3)$, is defined similarly. We assumed that the nucleus is located at the origin. A system of condensed matter is even more complicated, as it consists of many atoms and many electrons. Evidence shows that, in many cases, atoms arrange themselves in periodic lattices. This is ill-understood but we accept it, so we assume that the positions of the atoms are given by the vertices of a regular lattice. Here, "lattice" means a graph with a periodic structure.

Our goal is to understand the behaviour of the electrons, that is, to understand the electronic properties of the system. The evolution of the system is formidably complex. However, a large system at equilibrium is described by statistical mechanics. The expectation of the observable A is given by

$$\langle A \rangle = \frac{\operatorname{Tr} A \, \mathrm{e}^{-\beta H}}{\operatorname{Tr} \mathrm{e}^{-\beta H}}. \tag{4.3}$$

Here, β is a parameter that is equal to the inverse temperature of the system. This linear functional is called a *finite-volume Gibbs state*. Its justification is physically and mathematically delicate, but we accept it.

Eq. (4.3) is still intractable and we are led to the notion of *models*. Models are grossly simplified systems that nevertheless capture several relevant mechanisms at work in the original systems. The main approach of theoretical condensed matter physics consists in introducing interesting models, to work out their properties, and to link them with actual physical systems.

2. Models of quantum spin systems

We obtain an important class of models by assuming that only one electron per atom is relevant, and by restricting our attention to its spin. We assume that the total Hamiltonian is the sum of two-body interactions. If $S = \frac{1}{2}$ and if we also assume that the interaction is rotation invariant, it necessarily is of the form $\pm\vec{S}_x \cdot \vec{S}_y$. We actually consider the following more general class of models.

Let Λ be the (finite) set of vertices. The Hilbert space is

$$\mathcal{H}_\Lambda = \otimes_{x\in\Lambda}\mathbb{C}^{2S+1}, \tag{4.4}$$

where $S \in \frac{1}{2}\mathbb{N}$ is a fixed parameter. The Hamiltonian is

$$H_\Lambda = -\tfrac{1}{2}\sum_{x,y\in\Lambda}\Big(J^1_{xy}S^1_xS^1_y + J^2_{xy}S^2_xS^2_y + J^3_{xy}S^3_xS^3_y\Big). \tag{4.5}$$

Here, $J^i_{xy} = J^i_{yx}$ are real parameters. The spin operator S^i_x is equal to

$$S^i_x = S^i \otimes \mathbb{1}_{\Lambda\setminus\{x\}} \tag{4.6}$$

where $\mathbb{1}_{\Lambda\setminus\{x\}}$ is the identity in $\otimes_{y\in\Lambda\setminus\{x\}}\mathbb{C}^{2S+1}$.

It is natural to choose Λ to be a box in $\mathbb{Z}^d$ and to set $J^i_{xy} = 0$ unless x, y are nearest-neighbours. Several famous models belong to the general class:

- The case $J^3_{xy} = J$ for all neighbours x, y, and $J^i_{xy} = 0$ for $i = 1, 2$ or x, y not neighbours. This is the Ising model, invented by Lenz in 1920. All relevant operators commute with one another and the quantum setting is superfluous.
- The case $J^1_{xy} = J^2_{xy} = J$ for all neighbours x, y, and $J^i_{xy} = 0$ for $i = 3$ or x, y not neighbours. This model is known as the quantum XY model, or model of quantum rotators.
- The case $J^1_{xy} = J^2_{xy} = J^3_{xy} = J$ for all neighbours x, y, and $J^i_{xy} = 0$ for x, y not neighbours. This is the ferromagnetic Heisenberg model when $J > 0$ and the antiferromagnetic Heisenberg model when $J < 0$.

3. System of two spins

Systems of two spins are relevant for interaction operators. The Hilbert space is $\mathbb{C}^{2S+1} \otimes \mathbb{C}^{2S+1}$, and the spin operators are $S^i_1 = S^i \otimes \mathbb{1}$, $S^i_2 = \mathbb{1} \otimes S^i$, $i = 1, 2, 3$.

LEMMA 4.1.

- *The eigenvalues of* $(\vec{S}_1+\vec{S}_2)^2$ *are* $J(J+1)$, *with* $J = 0, 1, \ldots, 2S$. *The degeneracy of* J *is* $2J + 1$.
- $[S^i_1 + S^i_2, (\vec{S}_1 + \vec{S}_2)^2] = 0$, $i = 1, 2, 3$, *and the eigenvalues of* $S^i_1 + S^i_2$ *in the sector* J *are* $-J, -J+1, \ldots, J$.

In the sequel we call J an eigenvalue of $(\vec{S}_1 + \vec{S}_2)^2$, even though the actual eigenvalue is $J(J + 1)$.

PROOF. It is enough to consider the case $i = 3$. We already have a basis of eigenvectors of $S_1^3 + S_2^3$, namely $|a\rangle \otimes |b\rangle$ with $a, b \in \{-S, -S+1, \dots, S\}$, so that the eigenvalues of $S_1^3 + S_2^3$ are $m = -2S, -2S+1, \dots, 2S$, with degeneracy $2S+1-|m|$. The following relations are easily checked:

$$\begin{aligned}(\vec{S}_1 + \vec{S}_2)^2 &= 2S(S+1) + 2\vec{S}_1 \cdot \vec{S}_2 \\ &= \tfrac{1}{2}(S_1^+ + S_2^+)(S_1^- + S_2^-) + \tfrac{1}{2}(S_1^- + S_2^-)(S_1^+ + S_2^+) + (S_1^3 + S_2^3)^2,\end{aligned} \tag{4.7}$$

and

$$2\vec{S}_1 \cdot \vec{S}_2 = S_1^+ S_2^- + S_1^- S_2^+ + 2S_1^3 S_2^3. \tag{4.8}$$

The commutation relation of the lemma follows. Furthermore,

$$[S_1^+ + S_2^+, \vec{S}_1 \cdot \vec{S}_2] = [S_1^- + S_2^-, \vec{S}_1 \cdot \vec{S}_2] = 0. \tag{4.9}$$

The idea of the proof is similar to that of Lemma 3.2. Let ψ be an eigenvector of $(\vec{S}_1 + \vec{S}_2)^2$ and $S_1^3 + S_2^3$ with eigenvalues (J, m), and consider $(S_1^{(\pm)} + S_2^{(\pm)})\psi$. Its norm is nonnegative, so that $|m| \leq J$. It differs from 0 if $m \neq \pm J$ and it is eigenvector with eigenvalues $(J, m \pm 1)$. Then J must be an integer smaller or equal to $2S$ — otherwise, it is possible to construct eigenvectors of $S_1^3 + S_2^3$ with eigenvalues satisfying $|m| > 2S$.

Let $D_{J,m}$ denote the degeneracy of (J, m). If ψ and ψ' are two orthogonal eigenvectors with eigenvalues (J, m), we can check that $(S_1^+ + S_2^+)\psi$ and $(S_1^+ + S_2^+)\psi'$ are also orthogonal — this can be seen with the help of

$$(S_1^- + S_2^-)(S_1^+ + S_2^+) = 2S(S+1)\mathtt{Id} + 2\vec{S}_1 \cdot \vec{S}_2 - (S_1^3 + S_2^3)^2 - (S_1^3 + S_2^3). \tag{4.10}$$

It follows that $D_{J,m+1} \geq D_{J,m}$. A similar argument with $(S_1^- + S_2^-)$ implies that $D_{J,m-1} \geq D_{J,m}$. Then $D_{J,m}$ does not depend on m; call it $\bar{D}_J$.

The degeneracy D_m of m can be written as

$$D_m = \sum_{J=|m|}^{2S} \bar{D}_J. \tag{4.11}$$

Then $\bar{D}_J = D_J - D_{J+1}$. Since $D_m = 2S+1-|m|$, we obtain $\bar{D}_J = 1$ and the lemma follows. □

EXERCISE 4.1. *In each case, show that the following Hamiltonians are related by unitary transformations. Write the unitary matrices explicitly.*

(a) $H_1 = -\frac{1}{2}\sum_{x,y} J_{xy} S_x^1 S_y^1$, $H_2 = -\frac{1}{2}\sum_{x,y} J_{xy} S_x^2 S_y^2$, $H_3 = -\frac{1}{2}\sum_{x,y} J_{xy} S_x^3 S_y^3$.

(b) $H_1 = -\frac{1}{2}\sum_{x,y} J_{xy}\Big(S_x^2 S_y^2 + S_x^3 S_y^3\Big)$, $H_2 = -\frac{1}{2}\sum_{x,y} J_{xy}\Big(S_x^1 S_y^1 + S_x^3 S_y^3\Big)$, $H_3 = -\frac{1}{2}\sum_{x,y} J_{xy}\Big(S_x^1 S_y^1 + S_x^2 S_y^2\Big)$.

(c) *Assume that the graph is bipartite, that is,* $\Lambda = \Lambda_{\mathrm{A}} \cup \Lambda_{\mathrm{B}}$ *and* $J_{xy} = 0$ *if* $x, y \in \Lambda_{\mathrm{A}}$ *or* $x, y \in \Lambda_{\mathrm{B}}$. *Then let* $H_1 = -\frac{1}{2} \sum_{x,y} J_{xy} \Big(S_x^1 S_y^1 + S_x^2 S_y^2 \Big)$, $H_2 = -\frac{1}{2} \sum_{x,y} J_{xy} \Big(S_x^1 S_y^1 - S_x^2 S_y^2 \Big)$, $H_3 = +\frac{1}{2} \sum_{x,y} J_{xy} \Big(S_x^1 S_y^1 + S_x^2 S_y^2 \Big)$.

EXERCISE 4.2. *Consider the Hilbert space* $\mathbb{C}^2 \otimes \mathbb{C}^2$ *(case* $S = \frac{1}{2}$*) and the basis* $|a\rangle \otimes |b\rangle$, $a, b = \pm\frac{1}{2}$. *Write down the eigenvector of* $(\vec{S}_1 + \vec{S}_2)^2$ *with eigenvalue* $J = 0$.