15% of the credit for this module will come from your work on eight assignments. Each assignment will be marked out of 10 for answers to one randomly chosen 'A' and one 'B' questions. Working through all questions is vital for understanding lecture material and success at the exam. 'A' questions will constitute a base for the first exam problem worth 40% of the final mark, the rest of the exam will be based on 'B' questions.

The answers to **all questions** are to be submitted by the deadline of **3pm on Monday 1 December 2014**. Your work should be stapled together, and you should state legibly at the top your name, your department and the name of your teaching assistant. Your work should be deposited in the dropbox labelled with your teaching assistant's name, opposite the Maths Undergraduate Office.

Completeness.

1. A. A subset X of a normed vector space $(V, \|\cdot\|)$ is said to possess the Bolzano-Weierstrass property if every sequence $(x_n)_{n=1}^{\infty} \subset X$ has a convergent subsequence $x_{n_k} \stackrel{k\to\infty}{\longrightarrow} x \in X$. Prove that X is complete in the sense that every Cauchy sequence in X converges to a point in X.

2. A. This question is a preparation for the discussion of the Banach contraction mapping theorem. Let $(X, \|\cdot\|)$ be a normed vector space and $T : X \to X$ a mapping. We say that $w \in X$ is a fixed point of T if $Tw = w$. We say that T is a contraction mapping if there exists a positive number $K < 1$ with $\|Tx - Ty\| \leq K\|x - y\|$ for all $x, y \in X$. Suppose that $x_0 \in X$ and define x_n inductively by $x_n = Tx_{n-1}$, $n > 0$. Show that the sequence $(x_n)_{n\geq 1}$ converges to w for any $x_0 \in X$.

Closed and open sets, continuity

3. B. Consider the real normed vector space l^∞ of bounded real sequences $\mathbf{a} = (a_1, a_2, \dots)$ with norm $\|a\|_\infty = sup_{n\geq 1}|a_n|$. The linear structure is given by component-wise addition and scalar multiplication defined in Question 4(a).

   (a) Show that the set
   $$E = \{\mathbf{a} \in l^\infty : \exists N(\mathbf{a}) : \ \forall n > N(\mathbf{a}), a_n = 0\}.$$
   is a linear subspace of l^∞, but not a closed subset.

   (b) Show that the set
   $$F = \{\mathbf{a} \in l^\infty : a_{2n} = 0 \ \forall n = 1, 2, 3, \dots\}.$$
   is a closed infinite-dimensional linear subspace of l^∞.

   (c) Show that any finite-dimensional linear subspace of any normed vector space is closed.

4. B. Consider the vector space S_F defined in Question 4.

   (a) Show that the map $T : S_F \to \mathbb{R}$ defined by $T\mathbf{a} = \sum_{j=1}^{\infty} a_j$ is linear.

   (b) B. If we equip $\mathbb{R}$ with the usual Euclidean norm $|\cdot|$ and S_F with one of the five norms (1)-(5), state, with reasons, whether T is continuous.

5. C. Let $U = V = S_F$, where S_F a normed space defined in Question 4. Let $\|\mathbf{a}\|_V = \|\mathbf{a}\|_U = \sum_{n=1}^{\infty} |a_n|$.

   (a) Show that if $T : U \to V$ is defined by $T\mathbf{a} = \mathbf{b}$ with $b_j = (1 - j^{-1})a_j$, $j = 1, 2, 3, \ldots$, then T is a continuous linear map. However, prove that there does not exist an $\mathbf{a} \in U$ with $a \neq 0$ such that $\|T\mathbf{a}\|_V = \|T\|\|\mathbf{a}\|_U$.

   (b) If U and V are two finite-dimensional normed vector spaces and $T : U \to V$ is linear, can we always find an $\mathbf{a} \in U$ with $\mathbf{a} \neq 0$ such that $\|T\mathbf{a}\|_V = \|T\|\|\mathbf{a}\|_U$? Give reasons.

17th of November 2014 Daniel Ueltschi and Oleg Zaboronski