MA258 Analysis III **Assignment 7.**

15% of the credit for this module will come from your work on eight assignments. Each assignment will be marked out of 10 for answers to one randomly chosen 'A' and one 'B' questions. Working through all questions is vital for understanding lecture material and success at the exam. 'A' questions will constitute a base for the first exam problem worth 40% of the final mark, the rest of the exam will be based on 'B' questions.

The answers to **all questions** are to be submitted by the deadline of **3pm on Monday 24 November 2014**. Your work should be stapled together, and you should state legibly at the top your name, your department and the name of your teaching assistant. Your work should be deposited in the dropbox labelled with your teaching assistant's name, opposite the Maths Undergraduate Office.

Norms

1. A. In our course we only study norms on real vector spaces. Let V be a **complex** vector space. A function $\|\cdot\| : V \to \mathbb{R}$ is called a norm if:

   - $\|\mathbf{z}\| \geq \mathbf{0}$ for all $\mathbf{z} \in \mathbf{V}$
   - If $\|\mathbf{z}\| = 0$, then $\mathbf{z} = \mathbf{0}$
   - If $\lambda \in \mathbb{C}$ and $\mathbf{z} \in V$, then $\|\lambda \mathbf{z}\| = |\lambda| \cdot \|\mathbf{z}\|$ (Here $|\lambda|$ is the modulus of $\lambda \in \mathbb{C}$)
   - If $\mathbf{z}, \mathbf{w} \in V$, then $\|\mathbf{z} + \mathbf{w}\| \leq \|\mathbf{z}\| + \|\mathbf{w}\|$.

   Which of the following functions are norms on $\mathbb{C}^n$?

   (a) $N_1(\mathbf{z}) = \sum_{k=1}^{n} (z_i + \bar{z}_i)$;

   (b) $N_2(\mathbf{z}) = \left(\sum_{k=1}^{n} \bar{z}_i z_i\right)^{1/2}$.

   (Here $\mathbf{z} = \{z_k = x_k + iy_k\}_{k=1}^n, \bar{\mathbf{z}} = \{x_k - iy_k\}_{k=1}^n \in \mathbb{C}^n$). Justify your answers

2. A. A pair (X, d) is called a metric space if X is a set and $d : X \times X \to \mathbb{R}$ is a function with the following properties: for any $x, y, z \in X$,

   - $d(x, y) \geq 0$ (non-negativity)
   - $d(x, y) = 0$ iff $x = y$ (separation of points)
   - $d(x, y) = d(y, x)$ (symmetry)
   - $d(x, z) \leq d(x, y) + d(y, z)$ (triangle inequality)

   Let $(V, \|\cdot\|)$ be a normed vector space. Define $d(x, y) := \|x - y\|,\ x, y \in V$. Prove that (V, d) is a metric space.

3. A. Let $a_j \in \mathbb{R}$ for $1 \leq j \leq n$ and write

   $$\|x\|_a = \sum_{j=1}^{n} a_j |x_j|.$$

   State and prove necessary and sufficient conditions for $\|\cdot\|_a$ to be a norm on $\mathbb{R}^n$.

4. B. Consider S_F, the space of real sequences $\mathbf{a} = (a_n)_{n=1}^{\infty}$, such that **all but finitely many** of the a_n's are zero. (In other words, each sequence $\mathbf{a} \in S_F$ is eventually zero.)

   (a) Show that if we use the natural definition of addition and scalar multiplication

   $$(a_n) + (b_n) = (a_n + b_n),\ \ \lambda(a_n) = (\lambda a_n),\ \ \lambda \in \mathbb{R},$$

   then S_F is a vector space.

   (b) Show that the following definitions all give norms on S_F,

   $$\|\mathbf{a}\|_\infty = \max_{n\geq 1} |a_n|, \tag{1}$$

   $$\|\mathbf{a}\|_w = \max_{n\geq 1} |n a_n|, \tag{2}$$

   $$\|\mathbf{a}\|_1 = \sum_{n=1}^{\infty} |a_n|, \tag{3}$$

   $$\|\mathbf{a}\|_2 = \left(\sum_{n=1}^{\infty} |a_n|^2\right)^{1/2}, \tag{4}$$

   $$\|\mathbf{a}\|_u = \sum_{n=1}^{\infty} n|a_n|. \tag{5}$$

   (c) Show that norms (2), (3), (4), (5) are NOT equivalent to norm (1). (In fact no two of norms from the above list are equivalent. Verify this if you accidentally got a free Saturday night. At least, check the most interesting case of (2) and (3).)

Completeness.

5. A. Let S_F be the space defined in Question 4. Prove that the normed vector space $(S_F, \|\cdot\|_1)$ is not Banach. Here $\|\cdot\|_1$ is the norm defined in (3).

6. B. Let l_1 be the set of real sequences $\mathbf{a}$ with $\sum_{j=1}^{\infty} |a_j|$ convergent.

   (a) Show that l_1 is a vector space given the natural definitions of addition and multiplications:

   $$(a_n) + (b_n) = (a_n + b_n),\ \ \lambda(a_n) = (\lambda a_n),\ \ \lambda \in \mathbb{R}.$$

   (b) Prove that $(l_1, \|\cdot\|_1)$ is a complete normed vector space (Banach space). Here $\|\cdot\|_1$ is the function defined in (3).You may assume without a proof that $\|\cdot\|_1$ is a norm on l_1.

17th of November 2014 — Daniel Ueltschi and Oleg Zaboronski