**MA258 Analysis III** **Assignment 6.**

*15% of the credit for this module will come from your work on eight assignments. Each assignment will be marked out of 10 for answers to one randomly chosen 'A' and one 'B' questions. Working through all questions is vital for understanding lecture material and success at the exam. 'A' questions will constitute a base for the first exam problem worth 40% of the final mark, the rest of the exam will be based on 'B' questions.*

The answers to **all questions** are to be submitted by the deadline of **3pm on Monday 17 November 2014**. Your work should be stapled together, and you should state legibly at the top your name, your department and the name of your teaching assistant. Your work should be deposited in the dropbox labelled with your teaching assistant's name, opposite the Maths Undergraduate Office.

# Further exercises in integration

1. B. Use definite integrals to find the limits of the following sums:
   (a) $\lim_{n\to\infty}\left(\frac{1}{n^2}+\frac{2}{n^2}+\dots\frac{n-1}{n^2}\right)$
   (b) $\lim_{n\to\infty}\left(\frac{1}{n+1}+\frac{1}{n+2}+\dots\frac{1}{n+n}\right)$
   (c) $\lim_{n\to\infty}\frac{1^p+2^p+\dots n^p}{n^{p+1}}$, $p>0$
2. A. Determine the sign of the following integral:
$$\int_0^{2\pi} f(x)dx,$$
   where $f(0)=1$, $f(x)=\frac{sin(x)}{x}$ for $x>0$.
3. A. Determine (without evaluating) which of the following integrals is greater:
   - $\int_0^1\sqrt{1+x^2}dx$ or $\int_0^1 1dx$
   - $\int_0^1 x^2\sin^2(x)dx$ or $\int_0^1 xsin^2(x)dx$
   - $\int_1^2 e^{x^2}dx$ or $\int_1^2 e^x dx$

# Uniform convergence.

4. B. Find $f'(x)$ if
$$f(x)=\int_x^{10} e^{-xy^2}dy,\ 0\le x\le 10.$$
   Justify all steps of the calculation.
   **Hint.** Let $I(a,b,c)=\int_a^b dxf(x,c)$. You need to show that $f$ satisfies all conditions for necessary for FTC1 and the theorem on the differentiation of integrals depending on a parameter to hold. Then
$$I'(a(x),b(x),c(x))=\frac{\partial I}{\partial a}(a(x),b(x),c(x))a'(x)$$
$$+\frac{\partial I}{\partial b}(a(x),b(x),c(x))b'(x)+\frac{\partial I}{\partial c}(a(x),b(x),c(x))c'(x),$$
   and each of the partial derivatives can be evaluated using either FTC1 or the theorem on the differentiation of integrals depending on a parameter.

5. B. Applying differentiation with respect to a parameter $\alpha$, evaluate the following integral

$$\int_0^\infty \frac{e^{-\alpha x} - e^{-\beta x}}{x} dx,\ \alpha, \beta > 0.$$

**Hint.** You may exchange the order of integration and differentiation without a justification. The exchange is justified due to the uniform convergence of the above improper integral depending on parameters. The theory of improper integrals depending on parameters is not currently covered by our course.

6. B. The Laplace transform of $f : [0, \infty) \to \mathbb{R}$ is a function $F : (0, \infty) \to \mathbb{R}$ defined by the formula

$$F(p) = \int_0^\infty e^{-pt} f(t) dt,\ p > 0.$$

Find the Laplace transform of the following functions:

(a) $f(t) = 1$

(b) $f(t) = e^{-\alpha t}$, $\alpha > 0$

(c) $f(t) = cos(\beta t),\ \beta \in \mathbb{R}$

Notice that the Laplace transform of a bounded function is not necessarily bounded.

7. B. Prove the uniform convergence of the functional series in the indicated intervals:

(a) $\sum_{n=1}^\infty \frac{x^n}{n^2},\ x \in [-1, 1]$

(b) $\sum_{n=1}^\infty \frac{\sin(nx)}{2^n}$, $x \in \mathbb{R}$

8. B. Applying term-wise differentiation and integration, find the sums of the series in the indicated intervals:

(a) $\sum_{k=1}^\infty \frac{x^k}{k}$, $x \in [a, b]$, $-1 < a < 0 < b < 1$

(b) $\sum_{k=1}^\infty (-1)^{k-1} \frac{x^k}{k}$, $x \in [a, b]$, $-1 < a < 0 < b < 1$

(c) $\sum_{k=1}^\infty \frac{x^{2k-1}}{2k-1}$, $x \in [a, b]$, $-1 < a < 0 < b < 1$.

Justify all steps of your calculations.

3rd of November 2014 Daniel Ueltschi and Oleg Zaboronski