MA258 Analysis III **Assignment 5.**

15% of the credit for this module will come from your work on eight assignments. Each assignment will be marked out of 10 for answers to one randomly chosen 'A' and one 'B' questions. Working through all questions is vital for understanding lecture material and success at the exam. 'A' questions will constitute a base for the first exam problem worth 40% of the final mark, the rest of the exam will be based on 'B' questions.

The answers to **all questions** are to be submitted by the deadline of **3pm on Monday 10 November 2014**. Your work should be stapled together, and you should state legibly at the top your name, your department and the name of your teaching assistant. Your work should be deposited in the dropbox labelled with your teaching assistant's name, opposite the Maths Undergraduate Office.

Further exercises in integration

1. A. Test the convergence of the probability integral

$$\int_0^\infty e^{-x^2} dx$$

Hint. Use the following comparison principle: if $\int_a^\infty F$ converges and $|f| < F$, then $\int_a^\infty f$ converges.

2. B. Prove that the Euler integrals of the first kind (beta-function) converges when $p > 0$ and $q > 0$:

$$B(p, q) = \int_0^1 x^{p-1}(1-x)^{q-1} dx.$$

3. B. Prove that the Euler integral of the second kind (gamma-function) converges for $p > 0$:

$$\Gamma(p) = \int_0^\infty x^{p-1} e^{-x} dx.$$

4. B. Prove the following generalization of the fundamental theorem of calculus: Let $F : [a, b] \to \mathbb{R}$ be continuous and piecewise continuously differentiable. Then

$$\int_a^b F'(x)dx = F(b) - F(a).$$

Remark. The definition of $\int_a^b F'$: F' is not defined at a finite set of points of $[a, b]$ where it has jump discontinuities. However, F' has one sided-limits at every point of $[a, b]$. Let us assign any values we like to F' at the points of discontinuity. We get a regulated function on $[a, b]$ which can be integrated. The value of the integral is denoted by $\int_a^b F'$. Notice that arbitrary choices we made to extend F' to the whole of $[a, b]$ do not influence the value of the integral.

5. A. Use the theorem of question 4 to calculate

$$\int_0^{\pi n} sign(\sin(x))dx,$$

for $n = 1, 2, 3, \ldots$. Here $sign : \mathbb{R} \to \mathbb{R}$ is a sign function: $sign(x) = 1$ for $x > 0$, $sign(x) = -1$, $sign(0) = 0$.
Hint. Sketch the graphs of $sin(x), sign(sin(x))$ - this will help you to guess the shape of F: $F'(x) = sign(sin(x))$ for $x \neq \pi k$, $k \in \mathbb{Z}$.

6. A. Let $f : [-a, a] \to \mathbb{R}$ be continuous, $a > 0$. Prove or disprove the following two statements:

   (a) If f is an even function ($f(x) = f(-x)$),

$$\int_{-a}^{a} f(x)dx = 2\int_{0}^{a} f(x)dx.$$

   (b) If f is an odd function ($f(x) = -f(-x)$),

$$\int_{-a}^{a} f(x)dx = 0.$$

7. B. Let n be a natural number. Use integration by parts to show that for the integral

$$I_n = \int_{0}^{\pi/2} \cos^n(x)dx$$

   the following reduction formula holds true:

$$I_n = \frac{n-1}{n} I_{n-2}, n > 1.$$

   Use the reduction formula to find I_n, then calculate I_9, I_{10}.

3rd of November 2014 Daniel Ueltschi and Oleg Zaboronski