15% of the credit for this module will come from your work on eight assignments. Each assignment will be marked out of 25 for answers to one randomly chosen 'A' and one 'B' questions. Working through all questions is vital for understanding lecture material and success at the exam. 'A' questions will constitute a base for the first exam problem worth 40% of the final mark, the rest of the exam will be based on 'B' questions.

The answers to **all questions** are to be submitted by the deadline of **3pm on Monday 20 October 2014**. Your work should be stapled together, and you should state legibly at the top your name, your department and the name of your teaching assistant. Your work should be deposited in the dropbox labelled with your teaching assistant's name, opposite the Maths Undergraduate Office.

Regulated functions. Integration of regulated functions

1. A. Let $f : [0,1] \to \mathbb{R}$ be given by $f(0) = 0$, $f(x) = x \cdot sin(1/x)$ for $x > 0$. Prove that f is regulated.
2. A. Let $f : [0,1] \to \mathbb{R}$ be given by $f(0) = 0$, $f(x) = 1/x$ for $x > 0$. Prove that f is not regulated. .
3. B. Let $f : [a,b] \to \mathbb{R}$ be regulated and non-negative. Prove that $g : [a,b] \to \mathbb{R}$ defined by $g(x) = \sqrt{f(x)}$ is regulated.
4. A. Let $a > 0$. By using a suitable sequence of step functions on $[0,a]$, show directly from the definition that $\int_0^a dxx = a^2/2$.
5. A. By using a suitable sequence of step functions ϕ_n for the partition $P = \{0, \frac{1}{n}, \frac{2}{n}, \dots, \frac{n-1}{n}, 1\}$, show directly from the definition that $\int_0^1 e^x \, dx = e - 1$.
6. B. Let $f : [0,1] \to \mathbb{R}$ be regulated. Let $g : [0,1] \to \mathbb{R}$ be equal to f at every point of $[0,1]$ except for $x_1, x_2, \dots, x_k \in [0,1]$ where it is equal to $g_1, g_2, \dots, g_k$ correspondingly. Prove that g is regulated and that $\int_0^1 f = \int_0^1 g$.
7. A. If the regulated function $f : [-\pi, \pi] \to \mathbb{R}$ is even (meaning that $\forall x \in [-\pi, \pi]$ $f(x) = f(-x)$) show that $\int_{-\pi}^{\pi} f = 2 \int_0^{\pi} f$.
8. B. For $p < q$ in $\mathbb{R}$ show that (i) $\int_p^q \sin(tx)\, dx \to 0$ as $t \to \infty$. (ii) For $\varphi \in S[a,b]$ show that $\int_a^b \varphi(x).\sin(tx)\, dx \to 0$ as $t \to \infty$. [Suggestion: treat separately each of the intervals on which φ is constant.] (iii) For $f \in R[a,b]$ show that $\int_a^b f(x).\sin(tx)\, dx \to 0$ as $t \to \infty$.
9. B. Let $\mathbb{R}[x]$ denote the real vector space of polynomials in x with real coefficients, and let $D, I : \mathbb{R}[x] \to \mathbb{R}[x]$ be given by $D(p) := p'$ and $I(p) := q$ where $q(x) := \int_0^x p$ (using the integration you have studied before). Show that D, I are linear maps. Are they surjective or injective; what is the kernel? Is it true that $D \circ I = I \circ D = id_{\mathbb{R}[x]}$?
10. A. Suppose $f : [a,b] \to \mathbb{R}$ is continuous. Show that there exists $c \in (a,b)$ so that

$$\frac{1}{b-a} \int_a^b f(x)\, dx = f(c).$$

[Hint: one method is to use the Mean Value Theorem.]

9 October 2014

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