15% of the credit for this module will come from your work on eight assignments. Each assignment will be marked out of 25 for answers to one randomly chosen 'A' and one 'B' questions. Working through all questions is vital for understanding lecture material and success at the exam. 'A' questions will constitute a base for the first exam problem worth 40% of the final mark, the rest of the exam will be based on 'B' questions.

The answers to **all questions** are to be submitted by the deadline of **3pm on Monday 13 October 2014**. Your work should be stapled together, and you should state legibly at the top your name, your department and the name of your teaching assistant. Your work should be deposited in the dropbox labelled with your teaching assistant's name, opposite the Maths Undergraduate Office.

Step functions. Integration of step functions.

1. A. Function $\psi : [a, b] \to \mathbb{R}$ is called a squelch function if for **each** partition $a = p_0 < p_1 < \ldots < p_{k-1} < p_k = b$, ψ is constant on each open interval (p_{j-1}, p_j), $1 \leq j \leq k$.
(i) What is the maximal possible cardinality of the set of values of ψ? (ii) Is ψ is a step function? (iii) True or false: any step function on $[a, b]$ is also a squelch function. Justify your answers.
2. B. (i) Prove that an arbitrary linear combination of step functions $\phi_1, \phi_2, \ldots \phi_n \in S[a, b]$ is a step function. (ii) Let $\{\phi_k\}_{k=1}^{\infty}$ be an infinite sequence of step functions on $[a, b]$. Suppose that the series $\sum_{k=1}^{\infty} \phi_k(x)$ converges at every $x \in [a, b]$. Define $\phi(x) := \sum_{k=1}^{\infty} \phi_n(x)$. True or false: $\phi \in S[a, b]$.
3. A. Find an example of a function $\phi : [0, 1] \to \mathbb{R}$ such that: ϕ is not a step function, but $|\phi|$ is a step function.
4. A. Suppose h is a step function on $[a, b]$ and that $P = \{z_0 < z_1 < \ldots < z_n\}$ is a partition of $[a, b]$ for which h is constant on each subinterval (z_{i-1}, z_i). (i) Prove that
$$\int_a^b h = \sum_{i=1}^{n} h(w_i)(z_i - z_{i-1}),$$
where, for each $1 \leq i \leq n$, w_i is any point in (z_{i-1}, z_i). (If you are familiar with the Riemann theory of integration, you will notice that the integral of a step function takes the form of a Riemann sum.) (ii) Show that $\int h$ is independent of the values of h at the points $\{z_i\}$ of the partition P.
5. B. Let h_1 and h_2 be two step functions on $[a, b]$. (i) Suppose that $h_1(x) = h_2(x)$ for all $x \in [a; b]$ except for one point c. Prove that
$$\int_a^b h_1 = \int_a^b h_2.$$
(ii) Suppose $h_1(x) = h_2(x)$ for all but a finite number of points $c_1, \ldots, c_N \in [a, b]$. Prove that
$$\int_a^b h_1 = \int_a^b h_2.$$
6. A. Let $h_1, h_2 \in S[a, b]$. Prove that if $h_1 \geq h_2$, then $\int_a^b h_1 \geq \int_a^b h_2$.
7. A. Let $\varphi, \psi : [a, b] \to \mathbb{R}$ be step functions.
(i) Prove that $|\varphi|$ is a step function and that $|\int_a^b \varphi| \leq \int_a^b |\varphi|$.
(ii) Prove that $\int_a^b |\varphi + \psi| \leq \int_a^b |\varphi| + \int_a^b |\psi|$.
8. A. Define the step function $\varphi : [-5, 5] \to \mathbb{R}$ by $\varphi(x) = -1$, for $-5 \leq x \leq 0$, and $\varphi(x) = +1$, for $0 < x \leq 5$. What is $\Phi : [-5, 5] \to \mathbb{R}$, $\Phi(x) := \int_{-5}^{x} \varphi$? Show that Φ is differentiable, except at 0, with derivative φ.
9. B. (Poor man's fundamental theorem of calculus.) Let $\phi \in S[a, b]$. Let P be a partition of $[a, b]$, $a = p_0 < p_1 < p_2 < \ldots < p_N = b$, such that $\phi \mid_{(p_{i-1}, p_i)} = \phi_i = const$, $i = 1, 2, \ldots N$. Denote $\phi(a) = \phi_0$. Assume that ϕ is continuous at $x = b$, i. e. $\phi(b) = \phi_N$. Let us define the function $D\phi$ on $[a, b]$:
$$D\phi \mid_{(p_{i-1}, p_i)} = \frac{\phi_i - \phi_{i-1}}{p_i - p_{i-1}}, \; i = 1, 2 \ldots, N.$$
Assign any real values you like to $D\phi(p_i)$. Clearly, $D\phi$ is a step function. (Notice however that its construction does depend on P.) Calculate $\int_a^b D\phi(x)dx$.
10. B. Let $\varphi, \psi : [a, b] \to \mathbb{R}$ be step functions.
(a) Prove that their product $\varphi\psi : [a, b] \to \mathbb{R}$ is a step function.
(b) Prove that $(\int_a^b \varphi\psi)^2 \leq (\int_a^b \varphi^2)(\int_a^b \psi^2)$.
[This is called the Cauchy-Schwarz inequality for the integral of step functions.
Suggestion: consider the quadratic function of t defined by $\int_a^b (t\varphi + \psi)^2$.]

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