Theorem For a function $g : \mathbb{R}^3 \to [0,\infty)$ we have

$$27\left(\frac{\pi}{2}\right)^4 \int_{\mathbb{R}^3} g^3 \geq \left[\int_{\mathbb{R}^3} \left(\frac{1}{|x|} - \frac{1}{3}\right)\cdot g(x)\, dx\right]^3.$$

Corollary $\inf\left\{ 3\left(\frac{\pi}{2}\right)^{4/3} \|\psi\|_6^2 - \int_{\mathbb{R}^3} \frac{1}{|x|} |\psi(x)|^2\, dx \ ; \ \|\psi\|_{L_2(\mathbb{R}^3)} = 1 \right\}$
$= -\frac{1}{3}$

Proof Put $g = |\psi|^2$. $\square$

Proof of the theorem Using $\frac{1}{|x|} - \frac{1}{3} < 0$ for $|x| > 3$, and Hölder's inequality on the set $\{|x| < 3\}$ we obtain

$$\left[\int_{\mathbb{R}^3} \left(\frac{1}{|x|} - \frac{1}{3}\right) g(x)\, dx\right]^3 \leq \left[\int_{|x|<3} \left(\frac{1}{|x|} - \frac{1}{3}\right) g(x)\, dx\right]^3$$

Hölder's: $\frac{1}{3} + \frac{1}{3/2} = 1$

$$\leq \left(\int_{|x|<3} \left(\frac{1}{|x|} - \frac{1}{3}\right)^{3/2} dx\right)^2 \left(\int_{|x|<3} g^3\right)$$

$$\leq 27\left(\frac{\pi}{2}\right)^4 \cdot \int_{\mathbb{R}^3} g^3. \ \square$$

⚠ The inequality is sharp – take $g(x) = \text{const}\cdot\left(\frac{1}{|x|} - \frac{1}{3}\right)^{1/2} \mathbb{1}_{\{|x|<}$

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