## Assignment 5

- 1. Prove that the Fourier transform of a Schwartz function,  $f \in \mathcal{S}(\mathbb{R}^d)$ , is a Schwartz function.
- 2. Area and volume of the unit sphere. You can skip this exercise if you know it already, but it is otherwise a must-do!

Let  $|S_d|$  be the area of the d-dimensional unit sphere (that is, the surface is d-dimensional), and  $|B_d|$  the volume of the d-dimensional ball.

(a) Show that

$$|S_d| = \frac{2\pi^{\frac{d+1}{2}}}{\Gamma(\frac{d+1}{2})},$$

where  $\Gamma(s) = \int_0^\infty e^{-t} t^{s-1} dt$  is the Gamma function.

Hint: Calculate  $\int_{\mathbb{R}^d} e^{-\pi ||x||^2} dx = 1$  using polar coordinates.

(b) Show that  $d|B_d| = |S_{d-1}|$ , hence

$$|B_d| = \frac{\pi^{d/2}}{\Gamma(\frac{d}{2} + 1)}.$$

**3.** Heat equation, formal calculations. Consider the heat equation in  $\mathbb{R}^d$ :

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x_1^2} + \dots + \frac{\partial^2 u}{\partial x_d^2},$$

for t > 0, with initial value u(x, 0) = f(x) and  $f \in \mathcal{S}(\mathbb{R}^d)$ .

Take the Fourier transform of the heat equation, and solve it for any fixed k. Show that the formal solution can be written

$$u(x,t) = \int_{\mathbb{R}^d} \widehat{f}(k) e^{-4\pi^2 t ||k||^2} e^{2\pi i k x} dk.$$

**4.** Heat equation, rigorously. Define the **heat kernel**  $H_t(x)$  by

$$H_t(x) = \frac{1}{(4\pi t)^{d/2}} e^{-\|x\|^2/4t} = \int_{\mathbb{R}^d} e^{-4\pi^2 t \|k\|^2} e^{2\pi i k x} dk.$$

We consider the function  $u(x,t) = (f * H_t)(x)$ .

- (a) Show that u(x,t) is equal to the formal solution of Exercise 3.
- (b) Show that u(x,t) is a Schwartz function in x, for each fixed t > 0.
- (c) Show that u(x,t) solves the heat equation for each t>0.
- (d) Show that  $\lim_{t\to 0+} u(x,t) = f(x)$ , uniformly in x.