6. DISTRIBUTIONS

6.1 Context & examples

Engineers and physicists as far back as Heaviside in the 1890s have used objects that "look like rough functions". The chief example is Dirac's "function" on $\mathbb{R}$:

$$\delta(x) = \begin{cases} \infty & \text{if } x = 0 \\ 0 & \text{if } x \neq 0 \end{cases}$$

in such a way that $\int_{-\infty}^{\infty} \delta(x)\,dx = 1$, and $\int_{-\infty}^{\infty} f(x)\,\delta(x)\,dx = f(0)$.
These objects have proved useful, they allow for short calculations that often lead to the correct outcome. They were nonetheless received with disdain and perplexity by mathematicians. It was only in 1950-51 that Laurent Schwartz introduced the appropriate framework, so that all this rests on solid footing. Objects such as Dirac's functions are now called <u>distributions</u>.

The key idea is to define these objects by their actions on "good" functions, keeping in mind that this action should be like an integral. This is a linear operation from the space of functions to numbers, so distributions should be linear functionals on some space of functions. The smaller the space, the larger

the dual space. The standard definition of distributions involves the dual space of smooth functions with compact support. Namely, one considers $C_c^\infty(\mathbb{R})$ endowed with the following topology: The sequence (ϕ_n) converges in $C_c^\infty(\mathbb{R})$ to ϕ if $\phi_n \in C_c^\infty(K)$ for some compact set $K \subset \mathbb{R}$, and

$$\left\| \frac{d^k}{dx^k}\phi_n - \frac{d^k}{dx^k}\phi \right\|_\infty \longrightarrow 0$$

as $n \to \infty$, for any fixed k. The space of distributions is then defined as the dual space of $C_c^\infty(\mathbb{R})$. (Recall that the dual space is the space of all <u>continuous</u> linear functionals.) This definition extends straightforwardly to $\mathbb{R}^d$.

Here are examples of distributions:

- Locally integrable functions (i.e. $f \in L^1_{loc}(\mathbb{R}^d)$ such that $\int_K |f| < \infty$ $\forall$ compact K), such as 1, $e^{\|x\|}$, $\frac{1}{\|x\|}$ (in $d \geq 2$), etc...
- Radon measures (i.e. Borel measures that are finite on compact sets, outer regular on Borel sets, inner regular on open sets). An example is Dirac measure, $\delta(A) = 1$ if $A \ni 0$, $\delta(A) = 0$ if $A \not\ni 0$.
- Let $x_0 \in \mathbb{R}^d$, and α a multi-index. The map $\phi \mapsto \frac{\partial^\alpha}{\partial x^\alpha}\phi(x_0)$ is a distribution (for $\alpha \equiv 0$, this is the Dirac function).

- Cauchy's principle value of $\frac{1}{x}$, denoted $PV\frac{1}{x}$ (in $d=1$):

$$PV\frac{1}{x}(\phi) = \lim_{\varepsilon\to 0+} \int_{|x|>\varepsilon} \frac{1}{x}\phi(x)\,dx .$$

Notice that $\lim_{\varepsilon\to 0+} \int_{\varepsilon}^{\infty} \frac{1}{x}\phi(x)\,dx = \pm\infty$ if $\phi(0)\neq 0$. But

$$\int_{|x|>\varepsilon} \frac{1}{x}\phi(x)\,dx = \int_{|x|>1} \frac{1}{x}\phi(x)\,dx + \underbrace{\int_{\varepsilon<|x|<1} \frac{1}{x}[\phi(x)-\phi(0)]\,dx}_{|\cdot|\leq \sup|\phi'(x)|} + \underbrace{\int_{\varepsilon<|x|<1} \frac{1}{x}\phi(0)\,dx}_{=0},$$

so the convergence as $\varepsilon\to 0+$ is guaranteed.

- Distribution $\frac{1}{x+i0}$, which is another regularisation of $\frac{1}{x}$:

$$\frac{1}{x+i0}(\phi) = \lim_{\varepsilon\to 0+} \int_{-\infty}^{\infty} \frac{1}{x+i\varepsilon}\phi(x)\,dx .$$

Exercise: Show that $\frac{1}{x+i0} = PV\frac{1}{x} - i\pi\delta_0$. (Hint: use the relation $\frac{1}{x+i\varepsilon} = \frac{x}{x^2+\varepsilon^2} - \frac{i\varepsilon}{x^2+\varepsilon^2}$.)

6.2 Tempered distributions

As we shall see, the dual space of smooth functions with compact support is not suitable in Fourier theory. We shall need that the Fourier transform of a test function be a test function. But the Fourier transform of a function with compact support is not compactly supported. This forces us to enlarge a bit the space of test functions (to the Schwartz space), and thus decrease

a bit the space of distributions. Let us recall the definition of the Schwartz space:

Definition: The Schwartz space $S(\mathbb{R}^d)$ of test functions is the linear space of all functions ϕ in $C^\infty(\mathbb{R}^d)$ that satisfy

$$\sup_{x\in\mathbb{R}^d} \|x\|^k \left| \frac{\partial^\alpha}{\partial x^\alpha}\phi(x) \right| < \infty,$$

for any $k\in\mathbb{N}$ and any multi-index $\alpha=(\alpha_1,\dots,\alpha_d)$. The sequence (ϕ_n) converges to ϕ in $S(\mathbb{R}^d)$ iff

$$\sup_{x\in\mathbb{R}^d} \|x\|^k \left| \frac{\partial^\alpha}{\partial x^\alpha}(\phi_n(x)-\phi(x)) \right| \longrightarrow 0$$

as $n\to\infty$, for any k,α.

Definition: A (tempered) distribution is a linear functional on $S(\mathbb{R}^d)$. We denote $S'(\mathbb{R}^d)$ the dual space of $S(\mathbb{R}^d)$.

A distribution is continuous in the sense that $T(\phi_n)\to T(\phi)$ whenever $\phi_n\to\phi$ in the Schwartz space. On S', we have a natural topology, the weak-$*$ topology (pointwise convergence with respect to elements of the original space). That is, the sequence of distributions (T_n) converges to T whenever

$$T_n(\phi) \to T(\phi)$$

for any test function $\phi\in S(\mathbb{R}^d)$.

We have seen that any locally integrable function gives rise to a distribution. This is not quite so with tempered distributions. For instance $\int e^{x^3} \phi(x)\,dx$ is infinite for some test functions, such as $\phi(x) = e^{-x^2}$. But if f is locally integrable, and f does not grow too fast at infinity, then T_f is well-defined:

$$T_f(\phi) = \int f(x)\,\phi(x)\,dx.$$

A major advantage of distribution theory is to extend the notion of derivatives of functions. Consider e.g. $f(x) = |x|$ on $\mathbb{R}$. It is intuitively clear that

$$f'(x) = \operatorname{sgn}(x) = \begin{cases} -1 & \text{if } x < 0 \\ 1 & \text{if } x > 0. \end{cases}$$

The derivative is well-defined everywhere but in one point, and this should not matter when considering Lebesgue spaces. The challenge is to find an appropriate generalisation of the notion of derivatives. Here it is!

__Definition__: Let $\alpha = (\alpha_1, \dots, \alpha_d)$ a multi-index. The __distributional derivative__, or __weak derivative__ of a distribution T is the distribution $\partial^\alpha T$ such that

$$(\partial^\alpha T)(\phi) = (-1)^\alpha\, T\left(\frac{\partial^\alpha}{\partial x^\alpha}\phi\right).$$

Here, $(-1)^\alpha = (-1)^{\alpha_1 + \dots + \alpha_d}$.

In particular, $T'(\phi) = -T(\phi')$ in $d=1$. Observe that $\partial^\alpha T$ is linear and continuous, so it is indeed a distribution. Let us now check that this provides an extension of the notion of derivative, by looking at the distribution T_f that corresponds to a differentiable function f. Integrating by parts,

$$T_f'(\phi) = -T_f(\phi') = -\int_{-\infty}^{\infty} f(x)\,\phi'(x)\,dx = \int_{-\infty}^{\infty} f'(x)\,\phi(x)\,dx = T_{f'}(\phi).$$

Then $T_f' = T_{f'}$ indeed.

__Lemma 6.1__

The map $T \mapsto \partial^\alpha T$ is continuous.

__Proof__: If $T_n \to T$, then $\partial^\alpha T_n(\phi) = (-1)^\alpha T_n(\frac{\partial^\alpha}{\partial x^\alpha}\phi) \to (-1)^\alpha T(\frac{\partial^\alpha}{\partial x^\alpha}\phi) = \partial^\alpha T(\phi)$. $\square$

Examples of weak derivatives:

(a) The weak derivative of $|x|$ is $\operatorname{sgn}(x)$:

$$T_{|x|}'(\phi) = -\int_{-\infty}^{\infty} |x|\,\phi'(x)\,dx = -\int_{-\infty}^{0} \phi(x)\,dx + \int_0^\infty \phi(x)\,dx = T_{\operatorname{sgn}}(\phi).$$

(b) Let H denote Heaviside step function, $H(x) = \begin{cases} 1 & \text{if } x > 0 \\ 0 & \text{if } x < 0 \end{cases}$. Then

$$T_H'(\phi) = -T_H(\phi') = -\int_0^\infty \phi'(x)\,dx = -\phi(x)\Big|_0^\infty = \phi(0).$$

We see that "$H'(x) = \delta(x)$", well-known to physicists!

(c) What about differentiating the Dirac function? Let δ_{x_0} the distribution such that $\delta_{x_0}(\phi) = \phi(x_0)$. Then

$$\delta_{x_0}'(\phi) = -\delta_{x_0}(\phi') = -\phi'(x_0).$$

6.3. Fourier transform of (tempered) distributions

We now extend the Fourier transform from functions to distributions, in the same way as we extended differentiation. Let us first consider the case where the distribution is an L^1 function.

$$\hat{T}_f(\phi) \underset{\substack{\uparrow \\ \text{what we want}}}{=} T_{\hat{f}}(\phi) = \int_{\mathbb{R}^d} \hat{f}(k)\,\phi(k)\,dk$$

$$= \int_{\mathbb{R}^d} \Big[\int_{\mathbb{R}^d} f(x)\, e^{-2\pi i k x}\, dx \Big]\, \phi(k)\, dk$$

$$\overset{\text{Fubini}}{=} \int_{\mathbb{R}^d} \Big[\int_{\mathbb{R}^d} \phi(k)\, e^{-2\pi i k x}\, dk \Big]\, f(x)\, dx$$

$$= T_f(\hat{\phi}).$$

The definition of $\hat{T}$ is now obvious.

Definition: The Fourier transform of the distribution T is the distribution $\hat{T}$ defined by $\hat{T}(\phi) = T(\hat{\phi})$.

For this definition to make sense, we need that $\hat{\phi}$ be a test function for all test functions ϕ. This is why we need the Schwartz space. The map $\hat{T}$ is clearly linear. Let us check that it is continuous:

$$\hat{T}(\phi_n) = T(\hat{\phi}_n) \longrightarrow T(\hat{\phi}) = \hat{T}(\phi).$$

We used the following fact:

Lemma 6.2.
If $\phi_n \to \phi$ in the Schwartz space, then $\hat{\phi}_n \to \hat{\phi}$ in the Schwartz space.

The proof is left as an exercise.

Lemma 6.3
The map $T \longmapsto \hat{T}$ is continuous.

Proof: If $T_n \to T$, then $\hat{T}_n(\phi) = T_n(\hat{\phi}) \longrightarrow T(\hat{\phi}) = \hat{T}(\phi)$. □

Lemma 6.3 is useful. It is sometimes easier to compute the Fourier transform of an approximating sequence and to take the limit. An example is the Coulomb potential $\frac{1}{\|x\|}$ in $d=3$. It is