A useful property of solutions to the wave equation is the conservation of energy. Given a function $u(x,t)$ (C^1 in time, Schwartz in space), let

$$E(t) = \int_{\mathbb{R}^d} \left(\left|\frac{\partial u}{\partial t}\right|^2 + \left|\frac{\partial u}{\partial x_1}\right|^2 + \dots + \left|\frac{\partial u}{\partial x_d}\right|^2 \right) dx.$$

<u>Theorem 5.2</u>

For the solution $u(x,t)$ of Theorem 5.1, we have $E(t) = E(0)$ for all $t \in \mathbb{R}$.

(Yes, it also applies to $t < 0$ because the wave equation can be run backwards in time, unlike e.g. the heat equation.) For the proof, we use the following identity:

$$|a\cos\alpha + b\sin\alpha|^2 + |-a\sin\alpha + b\cos\alpha|^2 = |a|^2 + |b|^2,$$

$\forall a,b \in \mathbb{C}$, $\alpha \in \mathbb{R}$. It can be seen as the identity $|z|^2 = |z\cdot e_1|^2 + |z\cdot e_2|^2$ where $z = (a,b) \in \mathbb{C}^2$ and $e_1 = (\cos\alpha, \sin\alpha)$, $e_2 = (-\sin\alpha, \cos\alpha)$ are orthonormal vectors in $\mathbb{C}^2$.

<u>Proof of Theorem 5.2:</u> By Plancherel and dominated convergence,

$$\int \left|\frac{\partial u}{\partial t}\right|^2 = \int \left|\frac{\partial \hat u}{\partial t}\right|^2 = \int_{\mathbb{R}^d} \left| -2\pi\|k\| \hat f(k) \sin(2\pi\|k\|t) + \hat g(k)\cos(2\pi\|k\|t)\right|^2 dk$$

$$\int \sum_{i=1}^d \left|\frac{\partial u}{\partial x_i}\right|^2 = \int \sum_{i=1}^d |2\pi i k_i \hat u(k,t)|^2 dk$$

$$= \int |2\pi\|k\|\hat f(k)\cos(2\pi\|k\|t) + \hat g(k)\sin(2\pi\|k\|t)|^2 dk.$$

Using the identity above, we get

$$\int \left|\frac{\partial u}{\partial t}\right|^2 + \sum^d \left|\frac{\partial u}{\partial x_i}\right|^2 = \int_{\mathbb{R}^d} \left(2\pi\|k\|\,|\hat f(k)|^2 + |\hat g(k)|^2 \right) dk.$$

The right side does not depend on t. $\square$

It is possible to formulate Theorem 5.2 more generally: If $u(x,t)$ is sufficiently regular, and if it solves the wave equation, then its energy $E(t)$ is conserved. This gives constraints on possible solutions.

In the case $d=1$, we can rewrite the solution of Theorem 5.1 in the form of d'Alembert solution: Observe that

$$\cos(2\pi k t) = \tfrac{1}{2}\left(e^{2\pi i k t} + e^{-2\pi i k t}\right)$$

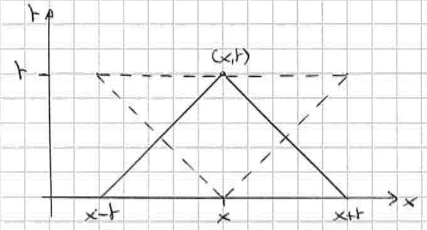
$$\frac{\sin(2\pi k t)}{2\pi k} = \frac{1}{4\pi i k}\left(e^{2\pi i k t} - e^{-2\pi i k t}\right).$$

Then

$$u(x,t) = \frac{1}{2}\int_{-\infty}^{\infty} \hat f(k)\left[e^{2\pi i k(x+t)} + e^{2\pi i k(x-t)}\right] dk + \int_{-\infty}^{\infty} \frac{1}{4\pi i k}\hat g(k)\left[e^{2\pi i k(x+t)} - e^{2\pi i k(x-t)}\right] dk$$

$$= \underbrace{\frac{1}{2}\left(f(x+t) + f(x-t)\right)}_{\text{average of } f \text{ over endpoints of the interval } [x-t,\,x+t]} + \underbrace{\frac{1}{2}\int_{x-t}^{x+t} g(y)\,dy}_{\text{average of } g \text{ over } [x-t,\,x+t]}$$

This brings the notion of backward light cone and Huygens principles: $u(x,t)$ depends on $u(y,0)$ for $y \in [x-t, x+t]$ only. In addition, there is the notion of forward light cone: the situation at $(x,0)$ influences the situation in the interval $[x-t, x+t]$ only. This is the finite speed of propagation. Here, the speed is 1.



What happens in higher dimensions, $d > 1$? As it turns out, some features are the same (finite speed of propagation) but there are important differences too. $d=3$ is especially striking (pun intended). We start with $d=3$, and curiously enough, we will extend the solution from $d=3$ to $d=2$!

The following expression plays an important rôle:

$$M_t f(x) = \frac{1}{4\pi} \int_{S^2} f(x - t\gamma)\, d\sigma(\gamma),$$

where the integral is over the sphere S^2, with the Lebesgue measure.

Lemma 5.3

If $f \in S(\mathbb{R}^3)$, then $M_t f \in S(\mathbb{R}^3)$ for every fixed t. Moreover, $\frac{\partial^k}{\partial t^k} M_t f \in S(\mathbb{R}^3)$ for every $k \in \mathbb{N}$, every fixed t.

The proof is left as an exercise.

Lemma 5.4

$$\frac{1}{4\pi} \int_{S^2} e^{-2\pi i k\cdot\gamma}\, d\sigma(\gamma) = \frac{\sin(2\pi \|k\|)}{2\pi \|k\|}.$$

Proof: The measure $d\sigma(\gamma)$ is rotation invariant, so the Fourier transform is rotation invariant. We can calculate it by choosing $k = (0,0,r)$. We have

$$\frac{1}{4\pi} \int_{S^2} e^{-2\pi i k\cdot\gamma}\, d\sigma(\gamma) = \frac{1}{4\pi} \int_0^{\pi} \sin\theta\, d\theta \int_0^{2\pi} d\varphi\; e^{-2\pi i r \cos\theta}$$

$$= \frac{1}{2} \frac{1}{2\pi i r} e^{-2\pi i r\cos\theta}\Big|_0^{\pi} = \frac{\sin(2\pi r)}{2\pi r}. \qquad \square$$

Next, we observe that

$$\widehat{M_t f}(k) = \hat{f}(k) \frac{\sin(2\pi \|k\| t)}{2\pi \|k\| t}.$$

Indeed, $M_t f$ is like a convolution of f and $d\sigma$. Let us check this explicitly, though.

$$\widehat{M_t f}(k) = \int_{\mathbb{R}^3} dx\; e^{-2\pi i k x} \frac{1}{4\pi} \int_{S^2} d\sigma(y)\; f(x - ty)$$

$$\overset{\text{Fubini}}{=} \frac{1}{4\pi} \int d\sigma(y)\; e^{-2\pi i t k y} \int_{\mathbb{R}^3} dx\; f(x-ty)\; e^{-2\pi i k (x-ty)}$$

$$= \frac{\sin(2\pi \|k\| t)}{2\pi \|k\| t}\; \hat{f}(k).$$

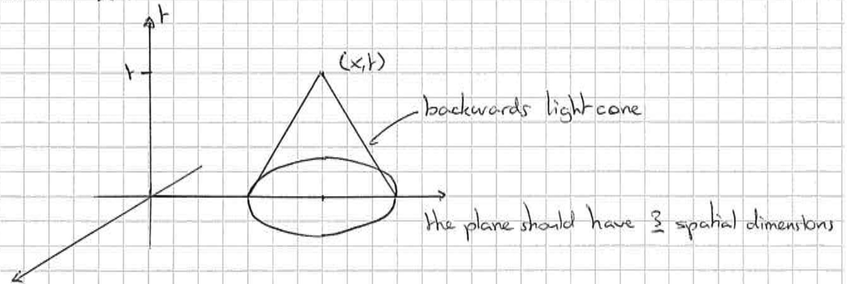
We can now state the analogue of d'Alembert formula in $d=3$.

Theorem 5.5

In $d=3$, a solution to the wave equation $\frac{\partial^2 u}{\partial t^2} = \Delta u$, with initial conditions $u(x,0) = f(x)$ and $\frac{\partial u}{\partial t}(x,0) = g(x)$, where $f, g \in S(\mathbb{R}^3)$, is given by

$$u(x,t) = \frac{\partial}{\partial t}(t M_t f)(x) + t M_t g(x).$$

The striking feature of this expression is that $u(x,t)$ depends on the situation at $t=0$ on the edge of the circle of radius t centred at x.



This explains why flashes of light are possible. Also, the sound comes suddenly and vanishes fast, unless it causes the environment to vibrate and generate more sound. We shall see that the situation is different in $d=2$; the value at (x,t) depends on the interior of the cone as well. This is consistent with the observation of a stone falling in a pond: it creates several circular waves!

Proof of Theorem 5.5: By Theorem 5.1, a solution is $u = u_1 + u_2$ with

$$u_1(x,t) = \int_{\mathbb{R}^3} \hat{g}(k) \frac{\sin(2\pi\|k\|t)}{2\pi\|k\|} e^{2\pi i k x}\, dk$$

$$u_2(x,t) = \int_{\mathbb{R}^3} \hat{f}(k) \cos(2\pi\|k\|t)\, e^{2\pi i k x}\, dk.$$

It follows from the inversion formula that $u_1(x,t) = t M_t g(x)$. For u_2, observe that

$$\int \hat{f}(k) \cos(2\pi\|k\|t)\, e^{2\pi k i x}\, dk = \frac{\partial}{\partial t}\Big(t \int \hat{f}(k) \frac{\sin(2\pi\|k\|t)}{2\pi\|k\|t} e^{2\pi i k x}\, dk \Big),$$

and use the inversion formula. $\square$

The case $d=2$ turns out to be more difficult, but one can extend the result from $d=3$ to $d=2$. Let us define

$$\tilde{M}_t f(x) = \frac{1}{2\pi} \int_{\|y\|<1} f(x-ty) \frac{dy}{\sqrt{1-\|y\|^2}}.$$

Theorem 5.6

A solution to the wave equation in $d=2$, with initial conditions $u(x,0) = f(x)$, $\frac{\partial u}{\partial t}(x,0) = g(x)$, where $f,g \in S(\mathbb{R}^2)$, is

$$u(x,t) = \frac{\partial}{\partial t}\left(t \tilde{M}_t f\right)(x) + t \tilde{M}_t g(x).$$

This is quite similar to the solution in $d=3$! The reason is that the problem in $d=2$ can be viewed as a problem in $d=3$ where all functions are constant in the 3rd dimension.

Proof of Theorem 5.6: Fix $T>0$, and let $\eta(x_3) \in S(\mathbb{R})$ such that $\eta(x_3) = 1$ for all $x_3 \in [-3T, 3T]$. Let

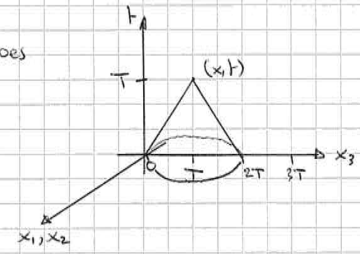
$$\tilde{f}(x_1,x_2,x_3) = f(x_1,x_2)\,\eta(x_3),$$
$$\tilde{g}(x_1,x_2,x_3) = g(x_1,x_2)\,\eta(x_3).$$

Let $\tilde{u}$ be the solution of the $d=3$ equation with initial conditions $\tilde{u}(x,0) = \tilde{f}(x)$, $\frac{\partial \tilde{u}}{\partial t}(x,0) = \tilde{g}(x)$. By Theorem 5.5,

$$\tilde{u}(x,t) = \frac{\partial}{\partial t}\left(t M_t \tilde{f}\right)(x) + t M_t \tilde{g}(x).$$

For $|x_3| < T$ and $|t| < T$, $\tilde{u}$ does not depend on x_3. Let

$$u(x_1,x_2,t) = \tilde{u}(x_1,x_2,0,t)$$



Then

$$\left(\frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2}\right) u = \left(\frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2} + \frac{\partial^2}{\partial x_3^2}\right) \tilde{u}(x_1,x_2,0,t)$$
$$= \frac{\partial^2}{\partial t^2}\tilde{u}(x_1,x_2,0,t) = \frac{\partial^2}{\partial t^2} u(x_1,x_2,t).$$

Then u is solution to the $d=2$ equation. There remains to see that u has the form given in the theorem. We show that $M_t \tilde{f} = \tilde{M}_t f$.

$$M_t \tilde{f}(x_1,x_2,0) = \frac{1}{4\pi}\int_{S^2} \tilde{f}(x - t\gamma)\, d\sigma(\gamma)$$
$$= \frac{1}{4\pi}\int_0^{2\pi} d\varphi \int_0^{\pi} \sin\theta\, d\theta\, \underbrace{\tilde{f}(x_1 - t\sin\theta\cos\varphi,\ x_2 - t\sin\theta\sin\varphi,\ -t\cos\theta)}_{= f(x_1 - t\sin\theta\cos\varphi,\ x_2 - t\sin\theta\sin\varphi)}$$
$$= \frac{1}{2\pi}\int_0^{2\pi} d\varphi \int_0^{\frac{\pi}{2}} \sin\theta\, d\theta\, f(x_1 - t\sin\theta\cos\varphi,\ x_2 - t\sin\theta\sin\varphi)$$
$$\overset{r=\sin\theta}{=} \frac{1}{2\pi}\int_0^{2\pi} d\varphi \int_0^1 \frac{r\,dr}{\sqrt{1-r^2}}\, f(x_1 - tr\cos\varphi,\ x_2 - tr\sin\varphi)$$
$$= \frac{1}{2\pi}\int_{\|y\|<1} \frac{dy}{\sqrt{1-\|y\|^2}}\, f(x - ty)$$
$$= \tilde{M}_t f(x).$$

The same identity also applies to g, and we get the expression of the theorem. $\square$