Corollary 4.8 (Young inequality)

If $f \in L^p(\mathbb{R}^d)$ and $g \in L^q(\mathbb{R}^d)$, then $f * g \in L^r(\mathbb{R}^d)$ where $\frac{1}{r} + 1 = \frac{1}{p} + \frac{1}{q}$, and

$$\| f * g \|_r \leq \| f \|_p \, \| g \|_q .$$

Proof: Fix $f \in L^p$. Using Minkowski inequality (see after Theorem 3.2), we have

$$\begin{aligned} \| f * g \|_p &\leq \Big(\int dx \Big(\int dy \, |g(y)| \, |f(x-y)| \Big)^p \Big)^{1/p} \\ &\leq \int dy \, |g(y)| \Big(\int dx \, |f(x-y)|^p \Big)^{1/p} \\ &= \| f \|_p \, \| g \|_1 . \end{aligned}$$

In addition, we have from Hölder inequality $\| f * g \|_\infty \leq \| f \|_p \, \| g \|_{p'}$ where $\frac{1}{p} + \frac{1}{p'} = 1$. Young inequality then follows from Riesz–Thorin. $\square$

4.5 Heisenberg uncertainty principle

We have seen that $\mathcal{F}\big(\frac{1}{\sqrt{\varepsilon}} e^{-\frac{\pi x^2}{\varepsilon}} \big) = e^{-\pi \varepsilon k^2}$. For small ε, we have $\mathcal{F}(\ ) = \ $. For large ε, $\mathcal{F}(\ ) = \ $. This property holds quite generally. It is closely related to Heisenberg uncertainty principle in physics, which says that it is impossible to "know" precisely the location and the momentum of a quantum particle.

[Quantum mechanics: a particle is represented by $\psi \in L^2(\mathbb{R}^3, \mathbb{C})$ with $\|\psi\|_2 = 1$. $|\psi(x)|^2$ is the probability density to find the particle at x, and $|\hat{\psi}(k)|$ is the probability density that the particle has momentum $k \in \mathbb{R}^3$.
The particle is localised around x_0 if $\int \|x - x_0\|^2 \, |\psi(x)|^2 \, dx$ is small. Then $\hat{\psi}(k)$ cannot be localised around any k.]

Theorem 4.9

Suppose that ψ, $x\psi$, and ψ' belong to $L^2(\mathbb{R})$, and that $\|\psi\|_2 = 1$. Then for any $x_0, k_0 \in \mathbb{R}$,

$$\Big(\int_{-\infty}^{\infty} (x - x_0)^2 \, |\psi(x)|^2 \, dx \Big) \Big(\int_{-\infty}^{\infty} (k - k_0)^2 \, |\hat{\psi}(k)|^2 \, dk \Big) \geq \frac{1}{16 \pi^2},$$

with equality iff $\psi(x) = A \, e^{2\pi i (x - x_0) k_0} \, e^{-B(x - x_0)^2}$ with $B > 0$ and $|A|^2 = \sqrt{\frac{B}{\pi}}$.

Proof: It is enough to prove the claim for $x_0 = k_0 = 0$. The proof relies on integration by parts and Cauchy-Schwarz inequality.

$$1 = \int_{-\infty}^{\infty} |\Psi(x)|^2\,dx = -\int_{-\infty}^{\infty} x \frac{d}{dx} |\Psi(x)|^2\,dx$$

$$= -\int_{-\infty}^{\infty} \left(x\,\Psi'(x)\,\overline{\Psi(x)} + x\,\overline{\Psi'(x)}\,\Psi(x) \right) dx.$$

Then

$$1 \leq 2\int_{-\infty}^{\infty} |x|\,|\Psi(x)|\,|\Psi'(x)|\,dx$$

$$\leq 2\left(\int_{-\infty}^{\infty} |x|^2 |\Psi(x)|^2\,dx\right)^{1/2} \left(\int_{-\infty}^{\infty} |\Psi'(x)|^2\,dx\right)^{1/2}.$$

Finally, we use Plancherel identity:

$$\int_{-\infty}^{\infty} |\Psi'(x)|^2\,dx = \int_{-\infty}^{\infty} |\widehat{\Psi'}(k)|^2\,dk = 4\pi^2 \int_{-\infty}^{\infty} k^2\,|\hat{\Psi}(k)|^2\,dk.$$

The result follows. $\square$

INTERLUDE — Jean Baptiste Joseph Fourier (1768 – 1830)

Joseph Fourier started by studying religion and mathematics, without choosing which direction to follow. In 1793 he became an ardent revolutionary. He was arrested in 1794 during the Terror, but he escaped the guillotine thanks to political changes (Robespierre was guillotined on 28 July 1794).

In 1794 he studies at École Normale with Lagrange, Laplace, Monge. He was arrested and freed again in 1795.

In 1798, he joins Napoléon in his invasion (French: "expedition") of Egypt. This started well, until Nelson destroyed the French fleet in the Battle of the Nile (1 August 1798). Napoléon returned to Paris in 1799, Fourier and the rest of the expeditionary force in 1801.

He resumed his professor position at École Polytechnique, but Napoléon nominated him Prefect of Isère (Grenoble). He worked there on the "Description of Egypt".

1804 – 1807: Fourier writes "On the propagation of heat in solid bodies". This was controversial at the time, because of issues with Fourier series.

When Napoléon marched through Grenoble in "the hundred days", Fourier fled instead of welcoming him. He was nonetheless nominated Prefect of the Rhône. Shortly after this, Waterloo.
In 1822 he published his essay "Théorie analytique de la chaleur", which Lord Kelvin described as "a great mathematical poem".

5. THE WAVE EQUATION

Fourier analysis has many applications in the theory of partial differential equations. Each equation has its own specificities, and several equations have major relevance. We discuss here the wave equation in some details. It gives a good example of the use of Fourier analysis, and it has fascinating (though peculiar) properties. It describes very important physical phenomena, notably sound waves and electromagnetic waves (the latter is a special case of Maxwell equations, that relate electric and magnetic fields).

Here, we simplify the analysis by assuming that initial conditions are smooth. This means that we ignore the dynamics of shock waves, which is a topic of current interest. But we will understand a striking feature of the wave equation, its finite speed of propagation.

We need the Schwartz space of smooth functions with fast decay. This space will be important in the next chapter about distributions.

<u>Definition</u>: The <u>Schwartz space</u> $S(\mathbb{R}^d)$ is the linear space of functions $\phi \in C^\infty(\mathbb{R}^d)$ that satisfy

$$\sup_{x \in \mathbb{R}^d} \|x\|^k \left| \frac{\partial^\alpha}{\partial x^\alpha} \phi(x) \right| < \infty$$

for any $k \in \mathbb{N}$ and any multi-index $\alpha = (\alpha_1, \dots, \alpha_d)$.

The sequence (ϕ_n) converges to ϕ iff

$$\sup_{x \in \mathbb{R}^d} \|x\|^k \left| \frac{\partial^\alpha}{\partial x^\alpha} (\phi_n(x) - \phi(x)) \right| \to 0$$

as $n \to \infty$, for any $k \in \mathbb{N}$ and any multi-index α.

Convergence in the Schwartz space is a very strong property. One can check that the topology induced by the above seminorms turns $S(\mathbb{R}^d)$ into a Fréchet space, i.e. a complete metrisable Hausdorff topological vector space whose topology is defined by a countable family of seminorms. (A space is Hausdorff when the topology separates points.) These technicalities can be safely ignored, though.

A Schwartz function is integrable and its Fourier transform can be defined by the L^1 transform. Exercise: prove that the Fourier transform of a Schwartz function is a Schwartz function. This is a very important property. It follows from Proposition 4.2 that the Fourier transform is a bijection $S(\mathbb{R}^d) \to S(\mathbb{R}^d)$.

Back to the wave equation. We study the equation in d spatial dimensions. The wave equation is

$$\frac{\partial^2 u}{\partial t^2} = \Delta u$$

where Δ is the Laplacian

$$\Delta = \frac{\partial^2}{\partial x_1^2} + \dots + \frac{\partial^2}{\partial x_d^2} .$$

We consider the initial conditions

$$u(x,0) = f(x), \qquad \frac{\partial u}{\partial t}(x,0) = g(x),$$

with $f,g \in S(\mathbb{R}^d)$. We start by performing formal calculations, which will lead to a candidate solution. Then we will check rigorously that the candidate is indeed a solution.

Let $\hat{u}(k,t) = \int_{\mathbb{R}^d} u(x,t)\, e^{-2\pi i k x}\, dx$. The Fourier transform of the equation is

$$\frac{\partial^2 \hat{u}}{\partial t^2}(k,t) = -4\pi \|k\|^2 \hat{u}(k,t) .$$

For fixed k, the solution is

$$\hat{u}(k,t) = A(k) \cos(2\pi \|k\| t) + B(k) \sin(2\pi \|k\| t).$$

The initial conditions are

$$\hat{u}(k,0) = A(k) = \hat{f}(k)$$
$$\frac{\partial}{\partial t} \hat{u}(k,0) = 2\pi B(k) \|k\| = \hat{g}(k)$$

We get $\hat{u}(k,t) = \hat{f}(k) \cos(2\pi\|k\|t) + \hat{g}(k) \dfrac{\sin(2\pi\|k\|t)}{2\pi\|k\|}$.

We now take the inverse Fourier transform, and we hope that the resulting $u(x,t)$ is solution to the wave equation. This actually works!

<u>Theorem 5.1</u>

A solution to the wave equation above, with initial conditions $f,g \in S(\mathbb{R}^d)$, is

$$u(x,t) = \int_{\mathbb{R}^d} \Big[\hat{f}(k) \cos(2\pi\|k\|t) + \hat{g}(k) \frac{\sin(2\pi\|k\|t)}{2\pi\|k\|} \Big] e^{2\pi i k x}\, dk .$$

This is a very nice result, even if the solution looks cumbersome. First, the solution exists. Second, it will be possible to extract some of its properties, without calculating the integral explicitly.

<u>Proof:</u> Because $f,g \in S(\mathbb{R}^d)$, the function $u(x,t)$ above belongs to C^∞. Using dominated convergence, we find

$$\Delta u(x,t) = \int_{\mathbb{R}^d} \Big[\quad \Big] (-4\pi^2 \|k\|^2)\, e^{2\pi i k x}\, dk$$

$$\frac{\partial^2}{\partial t^2} u(x,t) = \int_{\mathbb{R}^d} \Big[-4\pi^2\|k\|^2 \hat{f}(k) \cos(2\pi\|k\|t) - 4\pi^2\|k\|^2 \hat{g}(k) \frac{\sin(2\pi\|k\|t)}{2\pi\|k\|} \Big] e^{2\pi i k x}\, dk$$

Then $u(x,t)$ satisfies the wave equation indeed. As for the initial conditions,

$$u(x,0) = \int \hat{f}(k)\, e^{2\pi i k x}\, dk = f(x)$$
$$\frac{\partial u}{\partial t}(x,0) = \int \hat{g}(k)\, e^{2\pi i k x}\, dk = g(x) .$$

We used Proposition 4.2 (inverse Fourier transform). $\square$