In particular, for $x=0$: $\sum_{n\in\mathbb{Z}} f(n) = \sum_{n\in\mathbb{Z}} \hat{f}(n)$.

Proof: Let $g(x) = \sum_{n\in\mathbb{Z}} f(x+n)$. It follows from the assumptions that $g \in C^1([0,1])$. By Theorems 2.1 or 2.2 (Dini or Jordan criteria), we have

$$g(x) = \sum_{n\in\mathbb{Z}} e^{2\pi i n x} \int_0^1 g(y)\, e^{-2\pi i n y}\, dy .$$

Now we have

$$\int_0^1 g(y)\, e^{-2\pi i n y}\, dy = \int_0^1 \sum_{k\in\mathbb{Z}} f(y+k)\, e^{-2\pi i n y}\, dy$$
$$= \int_{-\infty}^{\infty} f(y)\, e^{-2\pi i n y}\, dy = \hat{f}(n),$$

and the result follows. □

Here are some applications of the Poisson summation formula.

1. Reflection formula for Jacobi's theta function. Let

$$\theta(s) = \sum_{n=-\infty}^{\infty} e^{-\pi n^2 s}, \qquad s>0.$$

Using $\widehat{e^{-\pi x^2 s}}(k) = s^{-\frac{1}{2}} e^{-\pi k^2/s}$, we get

$$\theta(s) = \sum_{n=-\infty}^{\infty} \frac{1}{\sqrt{s}} e^{-\pi \frac{n^2}{s}} = \frac{1}{\sqrt{s}}\, \theta\left(\frac{1}{s}\right).$$

2. Using $\widehat{\frac{1}{x^2+a^2}}(k) = \frac{\pi}{a} e^{-2\pi a |k|}$, where $a>0$, we get

$$\sum_{n=-\infty}^{\infty} \frac{1}{n^2+a^2} = \frac{\pi}{a}\, \frac{1+e^{-2\pi a}}{1-e^{-2\pi a}} = \frac{\pi}{a} \coth \pi a ,$$

Letting $a \searrow 0$, we find $\sum_{n\geq 1} \frac{1}{n^2} = \frac{\pi^2}{6}$ (exercise!).

4.3. Fourier transform of $L^2(\mathbb{R}^d)$ functions

The definition of the $L^2(\mathbb{R}^d)$ Fourier transform is a bit more complicated because we cannot invoke the integral $\int e^{-2\pi i k x} f(x)\, dx$. But it is worth the additional effort, as the L^2 transform has nice specific properties: it is a Hilbert isometry.

We first consider the case where $f \in L^1(\mathbb{R}^d) \cap L^2(\mathbb{R}^d)$. We can use the $L^1(\mathbb{R}^d)$ transform, and the resulting function $\hat{f}(k)$ is in $L^\infty(\mathbb{R}^d)$. More is true:

Theorem 4.4 (Plancherel theorem)
The Fourier transform of $f \in L^1(\mathbb{R}^d) \cap L^2(\mathbb{R}^d)$ is a function $\hat{f}$ that belongs to $L^2(\mathbb{R}^d) \cap L^\infty(\mathbb{R}^d)$. Furthermore, we have Plancherel identity:

$$\|\hat{f}\|_2 = \|f\|_2 .$$

Proof: Let h be the convolution of $\overline{f(-x)}$ with f, i.e.,

$$h(x) = \int_{\mathbb{R}^d} \overline{f(y-x)}\, f(y)\, dy .$$

One easily verifies that $\|h\|_1 \leq \|f\|_1^2$, so that h is an L^1 function and its Fourier transform is given by the formula for L^1 functions. We find $\hat{h}(k) = |\hat{f}(k)|^2$. We check below that h is continuous. By Proposition 4.2, for $x=0$, we get

$$h(0) = \lim_{\varepsilon \to 0+} \int_{\mathbb{R}^d} \hat{g}_\varepsilon(k)\, |\hat{f}(k)|^2\, dk .$$

We have $h(0) = \|f\|_2^2$, and the right side of the equation above converges to $\|\hat{f}\|_2^2$ by monotone convergence. There remains to check that h is continuous.

Let $f_\delta \in C_0^\infty(\mathbb{R}^d)$ be a smooth approximation for f such that $\|f_\delta - f\|_2 < \delta$. Then $h(x) = h_\delta(x) + r_\delta(x)$ with

$$h_\delta(x) = \int_{\mathbb{R}^d} \overline{f_\delta(y-x)}\, f(y)\, dy$$

and

$$|r_\delta(x)| \leq \int_{\mathbb{R}^d} |f(y-x) - f_\delta(y-x)|\, |f(y)|\, dy \leq \delta \|f\|_2 .$$

We used Cauchy-Schwarz. Next,

$$\begin{aligned} |h_\delta(x+\eta) - h_\delta(x)| &\leq \int_{\mathbb{R}^d} |\overline{f_\delta(y-x-\eta)} - \overline{f_\delta(y-x)}|\, |f(y)|\, dy \\ &\leq |\eta|\, \|f\|_1 \sup_x |f_\delta'(x)| . \end{aligned}$$

Then h_δ is continuous. Finally,

$$|h(x+\eta) - h(x)| \leq 2|\eta|\, \|f\|_1 \sup_x |f_\delta'(x)| + |r_\delta(x+\eta)| + |r_\delta(x)| .$$

The right side is as small as we wish by first choosing δ small, then η small.

$\square$

The set $L^1(\mathbb{R}^d) \cap L^2(\mathbb{R}^d)$ is dense in $L^2(\mathbb{R}^d)$ with respect to the $\|\cdot\|_2$ norm (exercise: if $f \in L^2$, show that $f(x)\, \chi_{[-m,m]^d}(x)$ belongs to L^1, and that it converges to f as $m \to \infty$). Let $f \in L^2$, and (f_n) a sequence of functions in $L^1 \cap L^2$ that converges to f. Then

$$\|\hat{f}_m - \hat{f}_n\|_2 \overset{\text{Plancherel}}{=} \|f_m - f_n\|_2 \longrightarrow 0$$

as $m,n \to \infty$. Then $(\hat{f}_n)$ is Cauchy, and it converges to a function $\hat{f} \in L^2(\mathbb{R}^d)$ since L^2 is a complete space. One can check that $\hat{f}$ does not depend on the choice of the approximating sequence, and that it satisfies Plancherel identity $\|\hat{f}\|_2 = \|f\|_2$ (exercise).

Definition: The Fourier transform of $f \in L^2(\mathbb{R}^d)$ is the function $\hat{f} = \lim_{n\to\infty} \hat{f}_n$, where (f_n) is any sequence in $L^1 \cap L^2$ that converges to f.

A big advantage of L^2 is that it is a Hilbert space, that is, there exists an inner product whose induced norm is $\|\cdot\|_2$:

$$(f,g) = \int_{\mathbb{R}^d} \overline{f(x)}\, g(x)\, dx .$$

Corollary 4.5

If $f,g \in L^2(\mathbb{R}^d)$ then $(\hat{f},\hat{g}) = (f,g)$.

Proof: Recall the polarisation identity:

$$(f,g) = \frac{1}{4}\|f+g\|_2^2 - \frac{1}{4}\|f-g\|_2^2 - \frac{i}{4}\|f+ig\|_2^2 + \frac{i}{4}\|f-ig\|_2^2 .$$

The corollary follows from Plancherel identity. $\square$

The inverse Fourier transform of $f \in L^2(\mathbb{R}^d)$ can be defined as

$$f^\vee(x) = \hat{f}(-x).$$

Then $f = (\hat{f})^\vee$, as can be shown by first considering functions in $L^1 \cap L^2$ and by using continuity.

It is instructive to consider the Fourier transform $L^2(\mathbb{R}) \to L^2(\mathbb{R})$ from the perspective of functional analysis. Let $\mathcal{F}$ denote the corresponding operator. It is unitary by Plancherel theorem, so that its spectrum lies on the unit circle. Further, we have

$$\mathcal{F}^4 = \mathrm{Id},$$

so the spectrum consists of ± 1 and $\pm i$. The Gaussian function $g_1(x) = e^{-\pi x^2}$ satisfies $\mathcal{F} g_1 = g_1$, so g_1 is eigenvector of $\mathcal{F}$ with eigenvalue 1. One can check that the other eigenvectors are the Hermite functions

$$h_n(x) = \frac{(-1)^n}{n!} e^{\pi x^2} \frac{d^n}{dx^n} e^{-2\pi x^2} .$$

They indeed satisfy $\mathcal{F} h_n = (-i)^n h_n$. Once normalised, they form an orthonormal basis of $L^2(\mathbb{R})$.

This generalises to higher dimensions, $d \geq 1$, by considering products of d Hermite functions.

4.4. Fourier transform of $L^p(\mathbb{R}^d)$ functions, $1 \le p \le 2$.

We have seen that the Fourier transform maps L^1 functions to L^∞ functions, and L^2 functions to L^2 functions. If $f \in L^p$ with $1 \le p \le 2$, it can be decomposed as $f = f_1 + f_2$ with $f_1 \in L^1$ and $f_2 \in L^2$. Indeed, we can choose

$$f_1(x) = f(x)\, \chi_{|f|\ge 1}(x) \quad \text{and} \quad f_2(x) = f(x)\, \chi_{|f|<1}(x).$$

Then $\hat{f} = \hat{f}_1 + \hat{f}_2 \in L^\infty + L^2$. It turns out that $\hat{f}$ also belong to L^q where $\frac{1}{p} + \frac{1}{q} = 1$. This follows from interpolation.

Theorem 4.6 (Riesz-Thorin interpolation theorem)

Let $p, p_1, p_2, q, q_1, q_2 \in [1,\infty]$ and $\theta \in (0,1)$ such that

$$\frac{1}{p} = \frac{1-\theta}{p_1} + \frac{\theta}{p_2}, \qquad \frac{1}{q} = \frac{1-\theta}{q_1} + \frac{\theta}{q_2}.$$

Assume that T is a linear operator from $L^{p_1} + L^{p_2}$ to $L^{q_1} + L^{q_2}$ that satisfies

$$\| Tf \|_{q_1} \le M_1 \| f \|_{p_1} \quad \forall f \in L^{p_1},$$
$$\| Tf \|_{q_2} \le M_2 \| f \|_{p_2} \quad \forall f \in L^{p_2}.$$

Then
$$\| Tf \|_q \le M_1^{1-\theta} M_2^{\theta} \| f \|_p$$
for every $f \in L^p$.

The proof of this theorem is rather long and it is omitted. See e.g. Folland's Real Analysis. It uses the "three-line" lemma of complex analysis, that says that if f is analytic in $0 < \operatorname{Re} z < 1$, and bounded on $\operatorname{Re} z = 0$, $\operatorname{Re} z = 1$, then it is bounded in the strip by the interpolated values.

We explore two consequences of Riesz-Thorin theorem. The first is the Hausdorff-Young inequality, which implies that the Fourier transform of an L^p function is an L^q function, $\frac{1}{p} + \frac{1}{q} = 1$. The second consequence is Young's inequality about L^p norms of convolutions.

Corollary 4.7 (Hausdorff-Young inequality)

Let $p \in [1,2]$ and q such that $\frac{1}{p} + \frac{1}{q} = 1$. Then
$$\| \hat{f} \|_q \le \| f \|_p.$$

Proof: We have seen that $\| \hat{f} \|_\infty \le \| f \|_1$ (Section 4.1) and $\| \hat{f} \|_2 = \| f \|_2$ (Plancherel). The Hausdorff-Young inequality follows from Riesz-Thorin with $\theta = \frac{2}{q}$. $\square$

Remark: The sharp Hausdorff-Young inequality is
$$\| \hat{f} \|_q \le \left(\frac{p^{1/p}}{q^{1/q}} \right)^{\frac{d}{2}} \| f \|_p,$$
with $p \in [1,2]$ and $\frac{1}{p} + \frac{1}{q} = 1$. The constant is less than 1 for $p \in (1,2)$. The inequality is saturated by certain Gaussian functions.