$$\frac{1}{4\pi^2} - \frac{1}{\pi} A = \sum_{k\in\mathbb{Z}} \Big[k^2 |\hat{x}(k)|^2 + k^2 |\hat{y}(k)|^2 - ik\, \overline{\hat{x}(k)}\, \hat{y}(k) + ik\, \hat{x}(k)\, \overline{\hat{y}(k)} \Big]$$

$$= \frac{1}{2} \sum_{k\in\mathbb{Z}} \Big[|k\hat{x}(k) - i\hat{y}(k)|^2 + |k\hat{y}(k) + i\hat{x}(k)|^2 + (k^2-1)\big(|\hat{x}(k)|^2 + |\hat{y}(k)|^2\big) \Big]$$

$$\geq 0 .$$

There is equality iff $\hat{x}(k) = \hat{y}(k) = 0 \ \forall |k| \geq 2$, and $\hat{x}(1) = i\hat{y}(1)$.
Then $|\hat{x}(1)|^2 = 2\pi^2$ and $x(s) = x(0) + 2\pi \cos(2\pi s)$,
$y(s) = y(0) + 2\pi \sin(2\pi s)$. $\square$

Weyl ergodic theorem

We consider here one of the simplest dynamical systems, i.e. rotations of the circle. Given $\gamma \in \mathbb{R}$, let F_γ be the map $\mathbb{T} \to \mathbb{T}$:

$$F_\gamma(x) = x + \gamma .$$

The key objects of dynamical systems are orbits, or trajectories. The orbit of $x_0 \in \mathbb{T}$ is the sequence $(x_0, F_\gamma(x_0), F_\gamma^2(x_0), \dots)$ obtained by iterating the map F_γ, starting at x_0. If γ is rational, then $F_\gamma^k(x_0) = x_0$ for some finite k, and the orbit contains finitely many distinct points. If γ is irrational, the points do not repeat, and we shall prove that they are uniformly distributed in $\mathbb{T}$.

Let $\langle f \rangle_t$ denote the time average of f: with $x_n = F_\gamma^n(x_0)$,

$$\langle f \rangle_t = \lim_{N\to\infty} \frac{1}{N+1} \sum_{n=0}^{N} f(x_n) .$$

Let $\langle f \rangle_{ph}$ denote the phase-space average of f:

$$\langle f \rangle_{ph} = \int_0^1 f(x)\, dx .$$

Theorem 3.5 (Weyl ergodic theorem)
Suppose that γ is irrational. Then $\langle f \rangle_t = \langle f \rangle_{ph}$ for all $f \in C(\mathbb{T})$, and all $x_0 \in \mathbb{T}$.

(The claim is not true if γ is rational!)

Proof: We first prove the theorem for functions $e^{2\pi i k x}$, $k \in \mathbb{Z}$.
If $k=0$, both averages are equal to 1. If $k \neq 0$, then $\langle e^{2\pi i k x} \rangle_{ph} = 0$, and

$$\langle e^{2\pi i k x} \rangle_t = \lim_{N\to\infty} \frac{1}{N+1} \sum_{n=0}^{N} e^{2\pi i k (x_0 + n\gamma)}$$

$$= \lim_{N\to\infty} \frac{1}{N+1} e^{2\pi i k x_0} \frac{1 - (e^{2\pi i k \gamma})^{N+1}}{1 - e^{2\pi i k \gamma}}$$

$$= 0 .$$

Notice that $e^{2\pi i k \gamma} \neq 1$ if γ is irrational. The property extends to all trigonometric polynomials by linearity. Trigonometric polynomials are dense in $C(\mathbb{T})$ with the sup norm, so for any $f \in C(\mathbb{T})$ and any $\varepsilon > 0$, there exists a trigonometric polyn. g such that $\|f-g\|_\infty < \frac{\varepsilon}{2}$. Then

$$\left| \frac{1}{N+1} \sum_{n=0}^{N} f(x_n) - \int_0^1 f(x)\,dx \right| \leq \left| \frac{1}{N+1} \sum_{n=0}^{N} g(x_n) - \int_0^1 g(x)\,dx \right| + \varepsilon .$$

Then
$$\limsup_{N \to \infty} \left| \frac{1}{N+1} \sum_{n=0}^{N} f(x_n) - \int_0^1 f(x)\,dx \right| \leq \varepsilon .$$

This holds $\forall \varepsilon > 0$. $\square$

Corollary 3.6

If γ is irrational, the points of all orbits are distributed uniformly in $\mathbb{T}$, that is,

$$\lim_{N \to \infty} \frac{\#\{n \in \{0,\dots,N\} : x_n \in I\}}{N+1} = |I|$$

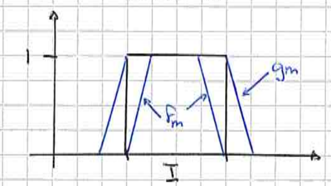
for any interval $I \subset \mathbb{T}$. $|I|$ denotes the length of I.

Proof: Let χ_I the characteristic function of I. The claim follows from Theorem 3.5 by choosing $f = \chi_I$. However, the theorem was only stated and proved for continuous functions. We need to extend it to χ_I.

Let f_m, g_m be continuous functions that satisfy

$$f_m(x) \leq \chi_I(x) \leq g_m(x)$$

$$\lim_{m \to \infty} \int_0^1 f_m(x)\,dx = \lim_{m \to \infty} \int_0^1 g_m(x)\,dx = |I|$$



Then
$$\frac{1}{N+1} \sum_{n=0}^{N} f_m(x_n) \leq \frac{1}{N+1} \sum_{n=0}^{N} \chi_I(x_n) \leq \frac{1}{N+1} \sum_{n=0}^{N} g_m(x_n) .$$

Using Theorem 3.5 for f_m, g_m, we get

$$\int_0^1 f_m(x)\,dx \leq \liminf_{N \to \infty} \sum_{n=0}^{N} \chi_I(x_n) \leq \limsup_{N \to \infty} \sum_{n=0}^{N} \chi_I(x_n) \leq \int_0^1 g_m(x)\,dx .$$

This holds for any m, so we can take $m \to \infty$. $\square$

4. FOURIER TRANSFORMS IN $\mathbb{R}^d$

4.1 Fourier transform of L^1 functions.

The Fourier transform of $f \in L^1(\mathbb{R}^d)$ is the function $\hat{f}(k)$, $k \in \mathbb{R}^d$, given by

$$\hat{f}(k) = \int_{\mathbb{R}^d} e^{-2\pi i kx} f(x)\, dx$$

where $kx = k \cdot x$ denote the usual inner product in $\mathbb{R}^d$.

The following properties are more or less immediate:

- The map $f \mapsto \hat{f}$ is linear.
- $\hat{f}$ is continuous. This follows from $\hat{f}(k+\eta) = \int e^{-2\pi i(k+\eta)x} f(x)\, dx$. The dominated convergence theorem allows to take the limit $\eta \to 0$ under the integral sign. Then $\hat{f}$ is measurable.
- $\hat{f} \in L^\infty(\mathbb{R}^d)$. $\|\hat{f}\|_\infty \leq \|f\|_1$, with equality if f is nonnegative.
- $\lim_{|k| \to \infty} \hat{f}(k) = 0$ (Riemann - Lebesgue lemma). Proof in exercise.
- $\widehat{f * g}(k) = \hat{f}(k)\hat{g}(k)$. This follows from Fubini theorem.
- If $f(x)$ and $xf(x)$ are L^1 functions, then $\hat{f}$ is differentiable and $\frac{d}{dk}\hat{f}(k) = -2\pi i\, \widehat{xf}(k)$.
- If f and f' are L^1 functions, then $\widehat{\frac{d}{dx}f}(k) = 2\pi i k \hat{f}(k)$.

The gaussian function plays a very important rôle. In particular, it is a fixed point of the Fourier transform. With $\lambda > 0$, let

$$g_\lambda(x) = \lambda^{-\frac{d}{2}} e^{-\pi \|x\|^2/\lambda}.$$

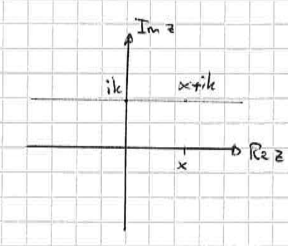
Proposition 4.1

The Fourier transform of a Gaussian is a Gaussian:

$$\hat{g}_\lambda(k) = e^{-\pi \lambda \|k\|^2}.$$

Proof: We can suppose that $d=1$ because everything factorises with respect to dimensions. Also, a change of variables reduces to the case $\lambda = 1$. We have

$$\begin{aligned}\hat{g}_1(k) &= \int_{-\infty}^{\infty} e^{-2\pi i kx} e^{-\pi x^2}\, dx \\ &= e^{-\pi k^2} \int_{-\infty}^{\infty} e^{-\pi(x+ik)^2}\, dx \\ &= e^{-\pi k^2} \int_{-\infty}^{\infty} e^{-\pi x^2}\, dx \\ &= e^{-\pi k^2}.\end{aligned}$$



We used complex analysis to change the contour of integration ($e^{-\pi z^2}$ is analytic in the whole of $\mathbb{C}$). $\square$

The gaussian function can be used to formulate an inverse Fourier transform that holds for any continuous L^1 function. In the case where $\hat{f} \in L^1(\mathbb{R}^d)$, the inverse transform is like the Fourier transform, but with i instead of $-i$.

<u>Proposition 4.2</u>

If $f \in L^1(\mathbb{R}^d)$ is continuous, then

$$f(x) = \lim_{\varepsilon \searrow 0} \int_{\mathbb{R}^d} e^{2\pi i k x} \, \hat{g}_\varepsilon(k) \, \hat{f}(k) \, dk.$$

If, in addition, we have $\hat{f} \in L^1(\mathbb{R}^d)$, then

$$f(x) = \int_{\mathbb{R}^d} e^{2\pi i k x} \, \hat{f}(k) \, dk.$$

The second claim is a straightforward consequence of the first claim, by dominated convergence.

<u>Proof</u>: The right side of the first equation is equal to

$$\lim_{\varepsilon \searrow 0} \int_{\mathbb{R}^d} e^{2\pi i k x} \, \hat{g}_\varepsilon(k) \left[\int_{\mathbb{R}^d} e^{-2\pi i k y} f(y) \, dy \right] dk =$$

$$\underset{\text{Fubini}}{=} \lim_{\varepsilon \searrow 0} \int_{\mathbb{R}^d} f(y) \left[\int_{\mathbb{R}^d} e^{2\pi i k (x-y)} \, \hat{g}_\varepsilon(k) \, dk \right] dy$$

$$\underset{\text{Prop. 4.1}}{=} \lim_{\varepsilon \searrow 0} \int_{\mathbb{R}^d} f(y) \, g_\varepsilon(x-y) \, dy.$$

It remains to verify that g_ε "converges to Dirac" as $\varepsilon \searrow 0$.

Same argument as before.

$$f * g_\varepsilon(x) = \int_{|x-y|<1} f(y) \, g_\varepsilon(x-y) \, dy + \int_{|x-y|>1} f(y) \, g_\varepsilon(x-y) \, dy$$

The second integral is less than

$$\int_{|x-y|>1} |f(y)| \, g_\varepsilon(x-y) \, dy \le \varepsilon^{-\frac{d}{2}} \, e^{-\frac{1}{\varepsilon}} \, \|f\|_1 \to 0$$

as $\varepsilon \to 0$. After the substitution $x-y = \sqrt{\varepsilon}\, u$, the first integral is equal to

$$\int_{|u| < \frac{1}{\sqrt{\varepsilon}}} f(x - \sqrt{\varepsilon}\, u) \, g_1(u) \, du \longrightarrow f(x)$$

by dominated convergence (since f is continuous, $f(x-\sqrt{\varepsilon} u)$ is bounded uniformly in $|u| < \frac{1}{\sqrt{\varepsilon}}$). □

4.2. <u>Poisson summation formula</u>

This is a curious and useful identity, that mixes Fourier series and Fourier transform.

<u>Theorem 4.3</u>

Assume that $f \in C^1(\mathbb{R})$ and that there exist $C < \infty$, $\varepsilon > 0$, such that

$$(1+x^2)^{\frac{1}{2}+\varepsilon} |f(x)| \le C, \qquad (1+x^2)^{\frac{1}{2}+\varepsilon} |f'(x)| \le C$$

for all $x \in \mathbb{R}$. Then $\displaystyle\sum_{n \in \mathbb{Z}} f(x+n) = \sum_{n \in \mathbb{Z}} e^{2\pi i n x} \hat{f}(n).$